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# ON THE LEVEL SETS OF LIPSCHITZ FUNCTIONS

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*Dedicated to the memory of Tibor Neubrunn*

ABSTRACT. The paper gives the negative answer to the question of D. Preiss, whether for any Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ and for almost all real numbers $u$ the level set $f^{-1}(u)$ can be covered by a countable number of rectifiable curves.

D. Preiss wondered whether for any Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$ and for almost all real numbers $u$ the level set $f^{-1}(u)$ can be covered by a countable number of rectifiable curves. Let us note that for almost all $u$ this is true up to a remainder of one-dimensional Hausdorff measure zero [1, 3.2.15]. Nevertheless, the answer to the Preiss question is negative, i.e., this zero set cannot be replaced by the empty set.

**Theorem 1.** *There exists a continuously differentiable function $f\colon [-1,1]^2 \to [-1,1]$ such that for any $u \in [-1,1]$ the set $f^{-1}(u)$ cannot be covered by a countable number of rectifiable curves.*

By $f'$ we denote the Fréchet derivative of a function $f$ and by $B(x,r)$ the closed ball of radius $r$ about point $x$. Throughout the following the letters $c_1, c_2, \dots$ denote positive constants.

Let $\psi(s) = 1$ for $s \in [0,\frac{1}{2}]$, $\psi(s) = \sin^2 \pi s$ for $s \in [\frac{1}{2},1]$, $\psi(s) = 0$ for $s \ge 1$, $\psi(s) = \psi(-s)$ for $s < 0$. If $x = (s,t) \in \mathbb{R}^2$, then we set $\phi(x) = \psi(s)\psi(t)$, and for $a > 0$ $S(x,a)$ denotes the square $[s-a, s+a] \times [t-a, t+a]$. Let $f$ be a continuous function defined on a subset of the plane containing a square $S = S(y,a)$. We denote

$$\Phi[f;S](x) = \left(1 - \phi\Big(\frac{x-y}{a}\Big)\right) f(x) + \phi\Big(\frac{x-y}{a}\Big) \min\{f(z)\colon z \in S(y,a)\}.$$

Then $\Phi[f;S]$ coincides with $f$ outside the square $S$, equals to minimum of $f$ on this square for $x \in S(y,a/2)$, and the values of $\Phi$ on $S$ depend only

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on the behaviour of $f$ on it. From these properties it follows that for any finite collection $\Sigma$ of mutually nonoverlapping squares $S_1, \dots, S_n$ the function $\Phi[f;\Sigma] = \Phi[\Phi \dots \Phi[f;S_1]; \dots ; S_n]$ does not depend on an order of sets in $\Sigma$.

**LEMMA 1.** *For any function $f$ continuously differentiable on a square $S$ and for any $x \in S$ the following inequalities are satisfied:*

$$\min\{f(z) \colon z \in S\} \le \Phi[f;S](x) \le \max\{f(x) \colon z \in S\};$$
$$\|\Phi[f;S]'(x)\| \le 14 \max\{\|f'(z)\| \colon z \in S\}.$$

Proof. Let $S = S(y,a)$. Since $0 \le \phi \le 1$ everywhere, for any $x \in S$ we have: $\min\{f(z) \colon z \in S\} \le \Phi[f;S](x) \le f(x) \le \max\{f(z) \colon z \in S\}$. To prove the second assertion, set $m = \min\{f(z) \colon z \in S\}$, $g = f - m$. Then

$$\Phi[f;S]'(x) = \Phi[g;S]'(x) = \left(1 - \phi\left(\frac{x-y}{a}\right)\right) g'(x) - \frac{g(x)}{a}\phi'\left(\frac{x-y}{a}\right),$$

and hence

$$\|\Phi[f;S]'(x)\| \le \|g'(x)\| + \frac{g(x)}{a}\left\|\phi'\left(\frac{x-y}{a}\right)\right\|. \tag{1}$$

Note that

$$\|g'(x)\| = \|f'(x)\| \le \max\{\|f'(z)\| \colon z \in S\},$$
$$\frac{g(x)}{a} \le 2\sqrt{2}\max\{\|f'(z)\| \colon z \in S\},$$

and if $\dfrac{x-y}{a} = (s,t)$, then

$$\left\|\phi'\left(\frac{x-y}{a}\right)\right\| \le \left((\psi'(s)\,\psi(t))^2 + (\psi(s)\,\psi'(t))^2\right)^{1/2} \le (\pi^2+\pi^2)^{1/2} = \sqrt{2}\pi.$$

Substituting this inequality into (1), we obtain

$$\|\Phi[f;S]'(x)\| \le (1+4\pi)\max\{\|f'(z)\| \colon z \in S\},$$

as required. The proof of Lemma 1 is complete. □

For a finite plane set $E$ containing at least two points we denote

$$d(E) = \sum_{x \in E} p(x,E), \quad \text{where} \quad \rho(x,E) = \min\{\|x-y\| \colon y \in E \setminus \{x\}\}.$$

Note that the length of any rectifiable curve covering $E$ is $d(E)/2$ at least.

**LEMMA 2.** *There exists a continuously differentiable function* $g\colon \mathbb{R}^2 \to [0,+\infty)$ *such that* $g(x) = 0$ *if* $x$ *is outside the square* $[-1,1]^2$ *and for any* $u \in (0,c_1]$ *there exists a finite set* $E = E_u \subset g^{-1}(u)$ *for which* $g'(x) \neq 0$ *on* $E$ *and* $d(E) \geq c_2 u^{-1}(-\ln u)^{-3/4}$.

Proof. Let us take $c_3$ for which the condition

$$c_3^2\pi\left(1+4\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{4/3}}\right) < 1 \tag{2}$$

is satisfied. We set $r_n = c_3 n^{-1/2}(\ln n)^{-2/3}$, $R_n = c_3 n^{-1/2}$, $B_n^1 = B(x_n, r_n)$, $B_n^2 = B(x_n, 2r_n)$, $B_n^3 = B(x_n, R_n)$, where $n = 2,3,\dots$ and $x_n$ are points from the square $S = \left[-\frac{1}{2},\frac{1}{2}\right]^2$. By induction on $m$, one can choose $x_m$ such that

$$x_m \notin B_n^2 \quad \text{for} \quad n < m \tag{3}$$

and

$$x_m \notin B_n^3 \quad \text{for} \quad n < m \leq 2n. \tag{4}$$

In fact, if $x_4,\dots,x_{m-1}$ are chosen, then the balls $B_n^2(n < m)$ and $B_n^3(n < m \leq 2n)$ do not cover $S$ because the sum of the areas of these balls does not exceed

$$\pi\left(\sum_{n=2}^{m-1}(2r_n)^2 + \sum_{n<m\leq 2n} R_n^2\right) \leq \pi\left(4\sum_{n=2}^{\infty} r_n^2 + c_3^2 \sum_{n<m\leq 2n}\frac{2}{m}\right)$$

and by (2) it is less than the area of $S$. So there exists a point $x_m$ satisfying (3) and (4).

By (3) and decreasing of $\{r_n\}$, the balls $B_n^1$ are disjoint. Let $g(x) = \psi(\|x - x_n\|/r_n)r_n\lambda_n$, if $x \in B_n^1$ for some $n$, where $\lambda_n = (\ln n)^{-1/12}$, and $g(x) = 0$ otherwise. Obviously, $g$ is continuously differentiable inside every ball $B_n^1$, and further, for any $y \in B_n^1$ we have $|g(y)| \leq c_4\lambda_n(r_n - \|y-x_n\|)/r_n$ and $\|g'(y)\| \leq c_4\lambda_n(r_n - \|y - x_n\|)/r_n$. If $x$ is not an interior point of $B_n^1$, then, using the inequality $\lambda_n(r_n - \|y - x_n\|) \leq \lambda_n \min(r_n, \|x-y\|) \leq \min(\lambda_n r_n, 2\|x-y\|)$, we obtain

$$|g(y)| \leq c_4\|x-y\|\min(\lambda_n, 2\|x-y\|/r_n),$$
$$\|g'(y)\| \leq c_4 \min(\lambda_n, 2\|x-y\|/r_n).$$

If $\|x-y\| \leq \lambda_m r_m/2$ for some $m$, then $\min(\lambda_n, 2\|x-y\|/r)n) \leq \lambda_m$ for any $n$ and we have $|g(y)| \leq c_4\lambda_m\|x-y\|$, $\|g'(y)\|c_4\lambda_m$. Consequently, if $x$ is not an interior point of any $B_n^1$, then the values $|g(y)|/\|x-y\|$, $\|g'(y)\|$ converge to 0 if $y \to x$. This means that $g'(x) = 0$ and $g'$ is continuous at $x$. Therefore, the function $g$ has a continuous derivative over the whole plane.

Let $u > 0$. The set $g^{-1}(u) \cap \{x\colon g'(x) \neq 0\}$ and the ball $B_n^1$ intersect if and only if $u < \max\{g(x)\colon x \in B_n^1\} = r_n\lambda_n$. For $u < \frac{1}{2}$ this inequality holds if $2 \leq n \leq m = [c_5 u^{-2}(-\ln u)^{-3/2}]$. In this case $g^{-1}(u) \cap \{x\colon g'(x) \neq 0\}$ and any $B_2^1, \dots, B_m^1$ intersect. For sufficiently small $u$ we have $m \geq 6$ and the number of balls $B_n^1$ for which $n \leq m \leq 2n$ is $\geq (m+1)/2 \geq 2$. In each of these balls we take a point belonging to $g^{-1}(u) \cap \{x\colon g'(x) \neq 0\}$. $E$ is defined as the collection of such points. By (4), the distance between any two points of $E$ is not less than $c_6 m^{-1/2}$. Therefore,

$$d(E) \geq \sum_{x \in E} c_6 m^{-1/2} \geq c_6 m^{-1/2}(m+1)/2,$$

from which, using the definition of $m$, we easily obtain the conclusion of the lemma. □

**Lemma 3.** *For any $M > 0$ there exist a continuously differentiable function $g_M\colon \mathbb{R}^2 \to \mathbb{R}$ and positive number $\delta_M$ such that $g_M(x) = 0$ if $x$ is outside the square $[-1,1]^2$, $\|g_M'(x)\| \leq 1$ everywhere, and for any $u \in [0, 0.1]$ there exists a finite set $E = E_u \subset g_M^{-1}(u)$, $E \subset [-1,1]^2$ satisfying the following conditions:*

a) $d(E) \geq M$;
b) $\|g_M'(x)\| \leq 0.01$ *for any* $x \in E$;
c) $\|x - y\| \geq \delta_M$ *for any distinct elements* $x, y$ *of* $E$.

Proof. Let $n$ be a sufficiently large positive integer depending on $M$. Set $\xi(s) = s(2 - |s|)$,

$$\xi_n(s) = \begin{cases} \frac{n-k-1}{6n} & \text{if } s \in \left[\frac{k}{n} - \frac{1}{4n}, \frac{k}{n} + \frac{1}{4n}\right],\ 0 \leq k < n, \\ \frac{n-k-1}{6n} + \frac{1}{12n} - \frac{\xi(4nx-4k+2)}{12n}, & \text{if } s \in \left[\frac{k}{n} - \frac{3}{4n}, \frac{k}{n} - \frac{1}{4n}\right],\ 1 \leq k < n, \\ 0 & \text{if } s \geq 1 - \frac{1}{n}, \\ \xi_n(-s), & \text{if } s < 0. \end{cases}$$

It is easy to verify that $\xi_n$ is a continuously differentiable function and $\|\xi_n'(s)\| \leq \frac{2}{3}$ everywhere. Let us consider the function $\mu$ on $\mathbb{R}^2$: $\mu(s,t) = \xi_n(s) - \xi_n(t)/n$. Then $\mu$ is a continuously differentiable function and $\|\mu'(s,t)\| \leq \|\xi_n'(s)\| + \|\xi_n'(t)\|/n \leq \frac{2}{3} + \frac{2}{3n} \leq 1$ everywhere. Note that on each square $S_{k,\ell} = \left[\frac{k}{n} - \frac{1}{4n}, \frac{k}{n} + \frac{1}{4n}\right] \times \left[\frac{\ell}{n} - \frac{1}{4n}, \frac{\ell}{n} + \frac{1}{4n}\right]$ $\mu$ has constant value equal to $v_{k,\ell} = \frac{1}{6} - \frac{2}{6n} +$

$\frac{1}{6n^2} - \frac{kn-\ell}{6n^2}$ $(0 \le k < n,\, 0 \le \ell < n)$. To construct the required function $g_M$ we will correct $\mu$ on these squares. Let $g$ be a function satisfying Lemma 2, $c_7 = \left(\sup\{\|g'(x)\| \colon x \in \mathbb{R}^2\}\right)^{-1}$, and

$$\tilde{g}_M(x) = \mu(x) + c_7(500n)^{-1} \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-1} g\big(4nx - (4k, 4\ell)\big).$$

The function $\tilde{g}_M$ vanishes outside $[-1,1]^2$, and has a continuous derivative on the plane. Furthermore, if $x$ belongs to some square $S_{k,\ell}$, then $\tilde{g}'_M(x) = (c_7/125)\, \tilde{g}'\big(4nx - (4k, 4\ell)\big)$ and $\|\tilde{g}'_M(x)\| \le 0.008$, otherwise $\tilde{g}'_M(x) = \mu'(x)$. In both cases $\|\tilde{g}'_M(x)\| \le 1$. For arbitrary $u \in [c_1c_7/500n, 0.11]$ we consider $v = \min\{u - (\frac{1}{6} - \frac{2}{6n} + \frac{1}{6n^2} - \frac{kn-\ell}{6n^2}) \colon u - (\frac{1}{6} - \frac{2}{6n} + \frac{1}{6n^2} - \frac{kn-\ell}{6n^2}) > 0,\ 0 \le k < n,\, 0 \le \ell < n\}$. Then for $0 \le j \le j_0 = [0.012c_1c_7n] - 1$ there exists $0 \le k < n$, $0 \le \ell < n$ for which $u - (\frac{1}{6} - \frac{2}{6n} + \frac{1}{6n^2} - \frac{kn-\ell}{6n^2}) = v + \frac{j}{6n^2}$ and, using the inequality $(v + \frac{j}{6n^2}) \le c_1c_7(500n)^{-1}$, we obtain, by Lemma 2, the existence of set $E_j \subset S_{k,\ell} \cap \tilde{g}_M^{-1}(n)$ such that

$$\begin{aligned} d(E_j) &\ge \frac{c_2}{4n}\left(500n\, c_7^{-1}\Big(v + \frac{j}{6n^2}\Big)\right)^{-1}\left(-\ln\left(500n\, c_7^{-1}\Big(v + \frac{j}{6n^2}\Big)\right)\right)^{-3/4} \ge \\ &\ge \frac{c_2}{4n}(0.012c_7n)(j+1)^{-1}\ln\big(0.012c_7n(j+1)^{-1}\big)^{-3/4} \end{aligned} \tag{5}$$

and

$$\tilde{g}'_M(x) \ne 0 \quad \text{for} \quad x \in E_j\,. \tag{6}$$

Let $E = \bigcup_{j=0}^{j_0} E_j$. Then $E \subset \bigcup_{k=0}^{n-1}\bigcup_{\ell=0}^{n-1} S_{k,\ell}$ and, consequently, $\|\tilde{g}'_M(x)\| \le 0.008$ for $x \in E$. Since $\rho(x,E) \ge \rho(x,E_j)/\sqrt{2}$ for any $x \in E_j$, (5) implies that

$$d(E) \ge 0.003c_2c_7 \sum_{j=0}^{j_0} (j+1)^{-1}\ln\big(0.012c_7n(j+1)^{-1}\big)^{-3/4}.$$

For large $n$ the last expression is greater than $M$. Note that, by (6), for any $x \in E$ and for any $v$ close to $u$ one can find $y$ close to $x$ such that $g_M(y) = v$. Therefore, there exist a neighbourhood $U(u)$ of $u$, a number $\delta(u) > 0$, and the sets $E_{u,v}$ for all $v \in U(u)$ such that $E_{u,v} \subset g_M^{-1}(u)$, $d(E_{u,v}) \ge M$, $\|g'_M(x)\| \le 0.01$, $\|x - y\| \ge \delta(u)$ for any $v \in U(u)$, $x \in E_{u,v}$, $y \in E_{u,v}$. By compactness of the segment $[c_1c_7/500n, 0.11]$, it can be covered by a finite system of sets $U(u)$, $u \in Y$. Setting $\delta_M = \min\{\delta(u) \colon u \in Y\}$ and $g_M = h \circ \tilde{g}_M$, where $h$ is a suitable $C^1$-contraction with $h(t) = 0$ for $t \le c_1c_7/500n$ and $h(t) = 0.1$ for $t \ge 0.11$, we finish the proof of Lemma 3. □

**LEMMA 4.** *There exist sequences of continuously differentiable functions* $f_n$, *numbers* $\varepsilon_n$, *where* $n = 1, 2, \dots$, *such that a finite set* $E_n = E_n(u)$ *corresponds to every* $n \in \mathbb{N}$, $u \in [0, 0.1]$ *and the following conditions are satisfied:*

a) $f_n(x) = 0$ *for* $n \geq 1,\ x \notin [-1,1]^2$;

b) $\|f_n'(x)\| \leq 0.01 \cdot 0.8^{n-1}$ *for* $n \geq 1,\ x \in E_n$;

c) $\|f_n'(x) - f_{n-1}'(x)\| \leq 0.8^{n-1}$ *for* $n \geq 2$;

d) $0 < \varepsilon \leq \varepsilon_{n-1}/20$ *for* $n \geq 2$;

e) $\|x - y\| \geq \varepsilon_n$ *for* $n \geq 1,\ x \in E_n,\ y \in E_n,\ x \neq y$;

f) $E_n \subset f_n^{-1}(u)$;

g) $E_n = \bigcup\limits_{y \in E_{n-1}} E_n^y$, $d(E_n^y) \geq 1$, *and* $\|x - y\| \leq 0.1\,\varepsilon_{n-1}$ *for* $n \geq 2$, $y \in E_{n-1},\ x \in E_n^y$.

Proof. We use induction on $n$. Let us take $f_1 = g_1$, $\varepsilon_1 = \delta_1$, and $E_1(u) = E_u$ $(u \in [0, 0.1])$, where $g_1, \delta_1$, and $E_u$ were constructed in Lemma 3. Then the conditions a), b), e) are satisfied for $n = 1$. Suppose that $n \geq 2$ and the lemma holds for $n-1$. Let $N$ be such a large positive integer that

$$\|f_{n-1}'(x) - f_{n-1}'(y)\| \leq 0.001 \cdot 0.8^{n-2} \quad \text{if} \quad \|x - y\| \leq 2/N, \tag{7}$$

$$2/N \leq 0.1\varepsilon_{n-1}. \tag{8}$$

Denote $S_{k,\ell} = \left[\frac{k-1}{N}, \frac{k}{N}\right] \times \left[\frac{\ell-1}{N}, \frac{\ell}{N}\right]$, $x_{k,\ell} = \left(\frac{2k-1}{2N}, \frac{2\ell-1}{2N}\right)$, where $-N < k \leq N$, $-N < \ell \leq N$. Then $S_{k,\ell} = S\left(x_{k,\ell}, \frac{1}{2N}\right)$. Let $\Sigma = \{S_{k,\ell} \colon \|f_{n-1}'(x_{k,\ell})\| \leq 0.011 \cdot 0.8^{n-2}\}$, $D = y \colon S\left(y, \frac{1}{2N}\right) \in \Sigma\}$. We set

$$f_n(x) = \Phi[f_{n-1}; \Sigma] + \frac{0.17}{N} \cdot 0.8^{n-2} \sum_{y \in D} g_{4N}\big(4N(x - y)\big),$$

where $g_{4N}$ is the function constructed in Lemma 3.

The property a) is evident. Let us verify the condition c). If $x$ is not a point of any square $S \in \Sigma$, then $f_n'(x) = f_{n-1}'(x)$. If $S \in \Sigma$, $S = S\left(y, \frac{1}{2N}\right)$, $x \in S$, then, in virtue of the choice of $\Sigma$ and (7),

$$\begin{aligned} \|f_{n-1}'(x)\| &\leq \|f_{n-1}'(y)\| + \|f_{n-1}'(x) - f_{n-1}'(y)\| \leq \\ &\leq 0.011 \cdot 0.8^{n-2} + 0.001 \cdot 0.8^{n-2} \leq 0.012 \cdot 0.8^{n-2}. \end{aligned} \tag{9}$$

It follows from Lemma 1, that $\|\Phi[f_{n-1}; \Sigma)]'(x)\| \leq 0.168 \cdot 0.8^{n-2}$. Therefore, if $x \notin S\left(y, \frac{1}{4N}\right)$, then

$$\begin{aligned} \|f_n'(x) - f_{n-1}'(x)\| &\leq \|f_{n-1}'(x)\| + \|f_n'(x)\| \leq \|f_{n-1}'(x)\| + \|\Phi[f_{n-1}; \Sigma]'(x)\| \leq \\ &\leq 0.012 \cdot 0.8^{n-2} + 0.168 \cdot 0.8^{n-2} = 0.18 \cdot 0.8^{n-2}. \end{aligned}$$

If $x \in S\left(y, \frac{1}{4N}\right)$, then $f_n'(x) = \frac{0.17}{N} \cdot 0.8^{n-2} \cdot 4N g_{4N}'\left(4N(x - y)\right)$ and, by Lemma 3, $\|f_n'(x)\| \le \frac{0.17}{N} \cdot 0.8^{n-2} \cdot 4N \le 0.68 \cdot 0.8^{n-2}$ and

$$\|f_n'(x) - f_{n-1}'(x)\| \le \|f_{n-1}'(x)\| + \|f_n'(x)\| \le$$
$$\le 0.012 \cdot 0.8^{n-2} + 0.68 \cdot 0.8^{n-2} < 0.7 \cdot 0.8^{n-2}.$$

In all cases the condition c) is satisfied.

Fix $u \in [0, 0.1]$ and consider arbitrary $y \in E_{n-1}$. Then

$$f_{n-1}(y) = u \tag{10}$$

and, using b) for $n-1$ instead of $n$, we have

$$\|f_{n-1}'(y)\| \le 0.01 \cdot 0.8^{n-2}. \tag{11}$$

For some $k, \ell$ $y \in S_{k,\ell}$. From (11) and the choice of $N$ it follows that $\|f_{n-1}'(x_{k,\ell})\| \le 0.011 \cdot 0.8^{n-2}$, i.e. $S_{k,\ell} \in \Sigma$. Let us denote $v = \min\{f_{n-1}(z) \colon z \in S_{k,\ell}\}$, $w = \left(\frac{0.17}{N} \cdot 0.8^{n-2}\right)^{-1}(u - v)$. Then

$$f_n(x) = v + \frac{0.17}{N} \cdot 0.8^{n-2} g_{4N}\left(4N(x - x_{k,\ell})\right) \quad \text{for} \quad x \in S\left(x_{k,\ell}, \tfrac{1}{4N}\right). \tag{12}$$

Using (9) for $x \in S$ and (10), we have

$$0 \le u - v \le 0.012 \cdot 0.8^{n-2} \cdot \sqrt{2}/N < \frac{0.017}{N} \cdot 0.8^{n-2},$$

i.e. $0 \le w \le 0.1$. We take $E_w$ by Lemma 3 and set $E_n^y = x_{k,\ell} + \frac{1}{4N} E_w$, $\varepsilon_n = \frac{\delta_{4N}}{4N}$. It follows from (8) that $\varepsilon \le \delta_{4N}\varepsilon_{n-1}/80$, and, consequently, d) holds. Furthermore, the assertions of Lemma 3 imply that:

1) $f_n(x) = u$ for any $x \in E_n^y$, and f) holds;
2) $d(E_n^y) \ge d(E_w)/(4N) \ge 1$;
3) $E_n^y \subset S_{k,\ell}$, and, by (8), g) holds;
4) if $x_1 \in E_n^y$, $x_2 \in E_n^y$, then $\|x_1 - x_2\| \ge \frac{\delta_{4N}}{4N} = \varepsilon_n$; if $x_1 \in E_n^y$, $x_2 \in E_n^z$, $y \ne z$, then $\|x_1 - x_2\| \ge \|y - z\| - \|x_1 - y\| - \|z - x_2\| \ge \varepsilon_{n-1} - 0.1\varepsilon_{n-1} - 0.1\varepsilon_{n-1} \ge \varepsilon_n$, by e) for $n-1$ instead of $n$, d) and f); so in both cases e) holds;
5) if $x \in E_n^y$, then, by (12), $\|f_n'(x)\| \le \frac{0.17}{N} \cdot 0.8^{n-2}(4N)0.01 < 0.8^{n-1}$, and b) also holds.

The lemma is proved. □

We now proceed directly to the proof of the theorem. We use the notation of Lemma 4. It follows from c) and equality $f_n(1,1)=0$ that $\{f_n\}$ uniformly converges to a continuously differentiable function $f_0$. Let us fix arbitrary $u\in[0,0.1]$ and prove that $f_0^{-1}(u)\cap[-1,1]^2$ cannot be covered by a countable number of rectifiable curves.

Let $E$ be the set of points $x$ for which there exists a sequence $\{x_n\}$ convergent to $x$ such that $x_n\in E_n$ $(n=1,2,\ldots)$. Then $E$ is a closed set and

$$E\subset f_0^{-1}(u)\cap[-1,1]^2. \tag{13}$$

For $n\in\mathbb{N}$ and $y\in E_n$ we denote

$\eta_n=0.1\sum\limits_{i=n}^{\infty}\varepsilon_i$, $\quad E^n(y)=\{x\in E\colon \|x-y\|\le\eta_n\}$. Note that, by d),

$$\eta_n\le 0.2\,\varepsilon_n. \tag{14}$$

For any $y=x_n\in E_n$ one can construct a sequence $\{x_{n+1}\in E_{n+1}(x_n),$ $x_{n+2}\in E_{n+2}(x_n+1),\ldots,\}$. Then, by g), we have for $n\le k<\ell$

$$\|x_\ell-x_k\|\le\sum_{i=k}^{\ell-1}\|x_{i+1}-x_i\|\le\sum_{i=k}^{\ell-1}0.1\,\varepsilon_i=\eta_k-\eta_\ell, \tag{15}$$

i.e. $\{x_i\}$ converges to some $x\in E$ and $\|x-y\|\le\eta_n$. Therefore,

$$E^n(y)\ne\emptyset\quad\text{for}\quad y\in E_n. \tag{16}$$

Let us take arbitrary $x\in E$, $n\in\mathbb{N}$, and $\varepsilon>0$. There exists $x_\ell\in E_\ell$ such that $\|x_\ell-x\|\varepsilon$ and $\ell\ge n$. Then $x_\ell\in E_\ell(x_{\ell-1})$ for some $x_{\ell-1}\in E_{\ell-1},\ldots,$ $x_{n+1}\in E_{n+1}(x_n)$ for some $x_n\in E_n$. Using (15), we have

$$\|x-x_n\|\le\varepsilon+\|x_\ell-x_n\|\le\varepsilon+\eta_n-\eta_\ell\le\varepsilon+\eta_n.$$

Since the number $\varepsilon>0$ is an arbitrary one and the set $E_n$ is finite, $\|x-y\|\le\eta_n$ for some $y\in E_n$, i.e. $x\in E^n(y)$. We obtain that

$$E\subset\bigcup_{y\in E_n}E^n(y). \tag{17}$$

Suppose that our assertion is not true. Then, by (13), $E$ can be covered by a countable number of curves whose lengths do not exceed 0.2. The intersection of at least one of these curves, say $\chi$, with $E$ is not a nowhere dense set in $E$. It means that for some $x_0 \in \chi$ and $n \in \mathbb{N}$.

$$E \cap B(x_0, 2\eta_{n-1}) \subset \chi. \tag{18}$$

From (17) there follows the existence of such $y \in E_{n-1}$ that

$$\|x_0 - y\| \leq \eta_{n-1}. \tag{19}$$

By (16), we can correlate an element $\tau(x) \in E^n(x)$ to any $x \in E_n$. Denote $F = \{\tau(x) \colon x \in E_n^y\}$. Then, using g), we obtain for any $z \in F$:

$$\|z - y\| \leq \|z - x\| + \|x - y\| \leq \eta_n + 0.1\,\varepsilon_{n-1} = \eta_{n-1}.$$

From this, (18), and (19) it follows that $F \subset \chi$.

Using (14) and e), we have for distinct $x_1 \in E_n^y$, $x_2 \in E_n^y$

$$\begin{aligned} \|\tau(x_1) - \tau(x_2)\| &\geq \|x_1 - x_2\| - \|x_1 - \tau(x_1)\| - \|x_2 - \tau(x_2)\| \geq \\ &\geq \|x_1 - x_2\| - 0.2\eta_n - 0.2\eta_n \geq 0.6\|x_1 - x_2\|. \end{aligned}$$

Consequently, $\rho\big(\tau(x), F\big) \geq 0.6\,\rho\,(x, E_n^y)$ and $d(F) \geq 0.6\,d(E_n^y) \geq 0.6$ by g). Since $F \subset \chi$, the above implies that the length of $\chi$ is 0.3 at least. This contradicts our supposition.

Therefore, we have the continuously differentiable function $f_0 \colon [-1,1]^2 \to \mathbb{R}$ such that for any $u \in [0, 0.1]$ the set $f_0^{-1}(u)$ cannot be covered by a countable number of rectifiable curves. To complete the proof of the theorem, it is sufficient to set $f(x) = \cos\big(10\pi f_0(x)\big)$. □

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REFERENCE

[1] FEDERER, H.: *Geometric Measure Theory*, Springer–Verlag, 1969.

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