Tatra Mountains Math. Publ., 1 (1992), 125–134

TATRA MOUNTAINS
Mathematical Publications

STATES ON SOFT FUZZY ALGEBRAS — FINITE AND COUNTABLE ADDITIVITY

MIRKO NAVARA – PAVEL PTÁK

ABSTRACT. We bring a summary of recent results on states on soft fuzzy algebras (s. f. algebras) and point out some of their explicit consequences. We are mainly interested in the characterization of state spaces, the enlargements of s. f. algebras related to states, extensions of states, etc. A special attention is paid to the comparison of results for finitely additive states and for countable additive states. The basic technical tool for our investigation is a Boolean representation of s. f. algebras, which, we believe, might shed light on many questions of fuzzy logics, too (see [1], [2], [4], [6], [16], [21], etc.).

1 Basic definitions

Let us first recall the basic definitions we shall deal with in the sequel (see [5, 7, 19]).

DEFINITION 1.1. Let X be a non-empty set. A *soft fuzzy algebra* on X (resp. a *soft fuzzy σ-algebra*) is a set $F \subset [0,1]^X$ satisfying the following conditions:

(1) the constant zero function belongs to F,
(2) if $a \in F$, then $a' = 1 - a \in F$,
(3) if $\{a_j\}_{j\in J}$ is a finite (resp. countable) subset of F, then $\bigvee_{j\in J} a_j \in F$ (the symbol $\bigvee$ means here the pointwise supremum of functions),
(4) the constant function $1/2$ does not belong to F.

We call X the *domain* of F.

It can be seen easily that an s. f. algebra is a distributive de Morgan lattice with a least element, 0, and a greatest element, 1. The mapping $'$, a *complementation*, is an order antiisomorphism. The function $a \wedge a'$, for any $a \in F$, does not exceed $1/2$ (obviously, it does not have to be 0 in general).

In what follows, we shall sometimes deal with Boolean representations of s. f. algebras and therefore combine the s. f. algebra notions with Boolean notions. For the sake of unification, let us formulate next definitions (and Def. 6.1 later on) in a structure containing both s. f. algebras and Boolean algebras. A suitable structure for this purpose seems to be a distributive de Morgan lattice. (The definition of a distributive de Morgan lattice is self-explanatory.)

Let F, G be distributive de Morgan lattices (resp. distributive de Morgan σ-lattices). A mapping $h\colon F \to G$ is called a *homomorphism* (resp. σ*-homomorphism*) if

(1) for each $a \in F$, $h(a') = h(a)'$,

(2) for each finite (resp. countable) set $\{a_j\}_{j\in J} \subset F$, $h(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} h(a_j)$.

Observe that h preserves the partial ordering of F and G. Moreover, if we write $W_0(F) = \{a \wedge a'\colon a \in F\}$ (weak zeros) and $W_1(F) = \{a \vee a'\colon a \in F\}$ (weak units), then h maps $W_0(F)$ into $W_0(G)$ and $W_1(F)$ into $W_1(G)$. Obviously, if both F and G are Boolean algebras, then our definition of a homomorphism coincides with the standard Boolean one.

DEFINITION 1.2. Let F be a distributive de Morgan lattice (resp. σ-lattice). A *state* (resp. σ*-state*) on F is a mapping $s\colon F \to [0,1]$ such that

(1) if $a \in W_1(F)$, then $s(a) = 1$,

(2) if $\{a_j\}_{j\in J}$ is a finite (resp. countable) pairwise orthogonal subset of F (i.e., if $a_i \le a_j'$ for $i \ne j$), then $s(\bigvee_{j\in J} a_j) = \sum_{j\in J} s(a_j)$.

Let us denote by $S(F)$ (resp. $S_\sigma(F)$) the space of states (resp. σ-states) on F.

Thus, a state is a generalization of the Boolean notion of a probability measure. It is believed (see [7], [12], [19], etc.) that states might find an application in the quantum axiomatics or in "soft" sciences. For the later use, let us recall that F is said to be *unital* (resp. σ*-unital*) if for each $a \in F \setminus W_0(F)$ there is a state (resp. σ-state) s such that $s(a) = 1$.

2 Boolean representations of s. f. algebras and state spaces properties

In this section we establish certain analogy of s. f. algebras and Boolean algebras.

DEFINITION 2.1. A Boolean algebra (resp. Boolean σ-algebra) B is a *Boolean representation (resp. Boolean* σ*-representation)* of an s. f. algebra (resp. s. f. σ-algebra) F if there is a homomorphism (resp. σ-homomorphism) $h\colon F \xrightarrow{\text{onto}} B$

such that for each state (resp. σ-state) s on F there is a state (resp. σ-state) t on B satisfying $s = t \circ h$.

It turns out that the states on an s. f. algebra and the states on its Boolean representation are in a one-to-one correspondence [17]. (We allow here also the *degenerate* Boolean algebra with $0 = 1$.) However, the representation mapping h is seen not to be injective if F is not a Boolean algebra (in fact, $W_0(F)$ always belongs to the kernel of h). Let us first take up the σ-additive case.

A. Boolean σ-representations of s. f. σ-algebras.

The next result — the existence and characterization of Boolean σ-representations — was proved in [15]:

THEOREM 2.2. *Every s. f. σ-algebra F has a Boolean σ-representation. Moreover, there is a minimal and a maximal Boolean σ-representation, $B_\sigma^{\min}$ and $B_\sigma^{\max}$, respectively. A Boolean σ-algebra B is a Boolean σ-representation of F if and only if there are σ-homomorphisms $h_1\colon B_\sigma^{\max} \xrightarrow{\text{onto}} B$ and $h_2\colon B \xrightarrow{\text{onto}} B_\sigma^{\min}$.*

The maximal and the minimal Boolean σ-representations are unique up to Boolean isomorphisms. The maximal Boolean σ-representation is a reflection of F in the category of Boolean σ-algebras (see e.g. [14]). The kernel of the maximal Boolean σ-representation is the minimal σ-ideal $I_\sigma = \{a \in F\colon (\exists b \in W_1(F)\colon a \wedge b \in W_0(F))\}$ from [7, Th. 3.4]; in fact, the maximal Boolean σ-representation has already been constructed in [7] (however, the consequences for the σ-state spaces have not been derived there). The kernel of the minimal Boolean σ-representation is $\bigcap_{s \in S_\sigma(F)} s^{-1}(0)$. Thus, $\bigcap_{t \in S_\sigma(B^{min})} t^{-1}(0) = \{0_{B^{min}}\}$. Moreover, we have the following result.

PROPOSITION 2.3 [15, Prop. 3.4, Th. 4.5]. *A Boolean σ-representation B of an s. f. σ-algebra F is minimal if and only if B is σ-unital.*

It is a natural question to ask whether all Boolean σ-algebras can be obtained as Boolean σ-representations of s. f. σ-algebras. The answer to the latter question is given in the following theorem (a Loomis-Sikorski theorem for s. f. σ-algebras).

THEOREM 2.4 ([15, Th. 3.5], see also [17, Th. 2.6]). *Every Boolean σ-algebra is a maximal Boolean σ-representation of an s. f. σ-algebra. Every σ-unital Boolean σ-algebra is a minimal Boolean σ-representation of an s. f. σ-algebra.*

R e m a r k 2.5. In [7], the authors prove that each Boolean σ-algebra is a σ-homomorphic image of an s. f. σ-algebra. However, their result does not lead to a Boolean σ-representation in our sense.

As there are Boolean σ-algebras admitting no nonzero σ-additive measures (see e.g. [22]), Th. 2.4 has the following consequence which answers the question posed in [5, 7].

COROLLARY 2.6 [17, Cor. 2.7]. *There is an s. f. σ-algebra admitting no σ-state.*

If an s. f. σ-algebra admits no σ-state then its minimal Boolean σ-representation is degenerate. The maximal Boolean σ-representation is never degenerate [7].

B. Boolean representations of s. f. algebras.

In the case of finitely additive structures Boolean representations are much simpler. Preceding results on Boolean σ-representations (Th. 2.2, Prop. 2.3 and Th. 2.4) remain valid. However, as all Boolean algebras are unital, the maximal and the minimal Boolean representations coincide.

THEOREM 2.7. *Every s. f. algebra F has a Boolean representation. This Boolean representation is unique (up to a Boolean isomorphism).*

The Boolean representation is the reflection of F in the class of Boolean algebras. Its kernel is the minimal ideal $I = \{a \in F\colon (\exists b \in W_1(F)\colon a \wedge b \in W_0(F))\} = \bigcap_{s\in S(F)} s^{-1}(0)$. As a consequence, all s. f. algebras are unital. By an analogy with Th. 2.4, we obtain the following result.

PROPOSITION 2.8. *Every non-degenerate Boolean algebra is a Boolean representation of an s. f. algebra.*

Boolean representations are always non-degenerate. The degenerate Boolean algebra is the Boolean representation of a fuzzy algebra which is not soft, i.e. which contains the element $1/2$ [19]. As every non-degenerate Boolean algebra admits a (two-valued) state, so does every s. f. algebra.

PROPOSITION 2.9 ([7, Th. 3.7] see also [18, Cor. 1.6]). *Every s. f. algebra admits a two-valued state.*

In view of Th. 2.4 and Prop. 2.8, neither minimal nor maximal Boolean representation has to be a complete Boolean algebra. Nevertheless, if the affine dimension of the state space is finite (resp. countable in the σ-additive case), the Boolean representation admits a strictly positive state and it is therefore complete. In this case an alternative approach is possible (see [4]).

3 Direct sums

In this section we find a characterization of states on the direct sum of s. f. algebras — they are exactly convex combinations of the respective "coordinate" states.

DEFINITION 3.1 [18]. Let F_j, $j \in J$, be a collection of s. f. algebras (resp. s. f. σ-algebras) with domains X_j. Let us suppose that the sets $X_j, j \in J$, are mutually disjoint. Put $X = \bigcup_{j \in J} X_j$ and define the set F of all fuzzy subsets a of X such that $a \mid X_j \in F_j$ for all $j \in J$. Then F is an s. f. algebra (resp. s. f. σ-algebra) on X. We call it the *direct sum of soft fuzzy algebras* F_j, $j \in J$.

THEOREM 3.2 [18, Th. 2.2]. *Let F_j, $j \in J$ be a finite (resp. countable) collection of s. f. algebras (resp. s. f. σ-algebras) with domains X_j. Let s_j be a state (resp. σ-state) on F_j $(j \in J)$. Let F be the direct sum of F_j $(j \in J)$. Then for any choice of non-negative reals c_j $(j \in J)$ with $\sum_{j \in J} c_j = 1$, the mapping s defined by the formula $s(a) = \sum_{j \in J} c_j s_j(a \mid X_j)$ is a state (resp. σ-state) on F. Moreover, all states (resp. σ-states) on F can be expressed in the latter form.*

Remark 3.3. The latter theorem for σ-states on s. f. σ-algebras remains valid if J is of the first uncountable cardinality [23]. In that case only countable many of the numbers c_j $(j \in J)$ may be nonzero.

4 Enlargements

Here we discuss the existence of state determined enlargements of s. f. algebras. Prior to stating the results, let us recall a few definitions.

DEFINITION 4.1 [17, 18]. Let F, G be s. f. algebras (resp. s. f. σ-algebras). Then G is said to be an *enlargement* (resp. *σ-enlargement*) of F if there exists an injective homomorphism (resp. σ-homomorphism) $e\colon F \to G$.

Observe that if G is an enlargement of F and if e is the enlargement mapping then $e(F)$ is an s. f. algebra in its own right. Thus, the s. f. algebra F may be viewed as a subset of G. In a dual expression of this, we call F a *soft fuzzy subalgebra* (resp. *soft fuzzy sub-σ-algebra*) of G.

If C_1, C_2 are two convex subsets of R^I (I is a set), then we call C_1, C_2 *affinely homeomorphic* if there is an affine isomorphism of C_1 onto C_2 which is a topological homeomorphism.

DEFINITION 4.2 [17]. An s. f. algebra (resp. s. f. σ-algebra) F is called *state-inflatable* (resp. σ*-state-inflatable*) if it admits enlargements (resp. σ-enlargements) with preassigned spaces of states (resp. σ-states), i.e., for each Boolean algebra (resp. Boolean σ-algebra) B there is an enlargement (resp. σ-enlargement) G of F such that the spaces $S(G), S(B)$ $(resp. S_\sigma(G), S_\sigma(B))$ are affinely homeomorphic.

THEOREM 4.3 [18, Th. 3.1]. *Every s. f. algebra is state-inflatable.*

For the finite-dimensional state spaces we obtain the following corollary.

COROLLARY 4.4 [18, Cor. 3.2]. *Let n be a natural number. Then every s. f. algebra admits such an enlargement G that $S(G)$ is an n-dimensional simplex. In particular, every s. f. algebra admits such an enlargement G that $S(G)$ is a singleton.*

In the case of σ-additive states the situation is somehow different.

THEOREM 4.5 [17, Cor. 2.9]. *An s. f. σ-algebra F is σ-state-inflatable if and only if F possesses a two-valued σ-state.*

5 Extensions of states

In this paragraph we ask if every state on an s. f. algebra can be extended over its enlargement. As we have seen, there are enlargements whose state spaces are singletons. Thus, if we want to make our question meaningful, we have to assume the unitality of the enlargement in question.

THEOREM 5.1 [18, Th. 4.2]. *Let F, G be s. f. algebras and let G be an enlargement of F. Let G be unital. Then every state $s \in S(F)$ can be extended over G.*

In other words, the latter theorem asserts that if $s \in S(F)$, then there is a $t \in S(G)$ such that $t \mid F = s$.

Obviously, the situation is quite different if we require countable additivity of states. In fact, in the σ-additive setup the question does not seem to be well posed (an analogue of Th. 5.1 does not hold even if F is a Boolean σ-algebra).

6 Representation of observables and the uniqueness property

In this section we study observables (generalized random variables). We show that Boolean σ-representations allow a natural representation of observables, too. Further, we introduce the uniqueness property for bounded observables (analogous to the property studied for quantum logics — see [10, 20]). It turns out that exactly the minimal Boolean σ-representation possesses this property.

Observables seem to be meaningful only in the context of the countable additive states. Thus, we restrict our attention to s. f. σ-algebras in this section.

DEFINITION 6.1. An *observable* on a distributive de Morgan σ-lattice F is a σ-homomorphism of the Borel σ-algebra $\mathcal{B}(R)$ into F [7]. An observable x is *bounded* if $x(R) = x((-m,m))$ for some $m \in R$. We denote by $\mathcal{O}(F)$ the set of all observables on F. If $s \in S(F)$, $x \in \mathcal{O}(F)$, then $s \circ x$ is a probability measure on $\mathcal{B}(R)$ — the *distribution* of the observable x in the state s. We denote by $E(s,x)$ its *mean value* (if it exists).

Let F be an s. f. σ-algebra, B its Boolean σ-representation and h the corresponding σ-homomorphism. If x is an observable on F then $y = h \circ x$ is an observable on B. The mapping $h_{\mathcal{O}} : x \mapsto h \circ x$ $(\mathcal{O}(L) \to \mathcal{O}(B))$ is generally not injective, but it is surjective as the following theorem says.

THEOREM 6.2. *Let B be a Boolean σ-representation of an s. f. σ-algebra F and let y be an observable on B. Then $y = h \circ x$ for some $x \in \mathcal{O}(F)$.*

Let us explicitly formulate the "observable" analogy coming into existence in the couple of an s. f. σ-algebra and its Boolean σ-representation.

THEOREM 6.3 [15, Th. 4.4]. *Let F be an s. f. σ-algebra and B its Boolean σ-representation. Then we have the following results:*

(1) *there exists a homomorphism* $h\colon F \xrightarrow{\text{onto}} B$,

(2) *there exists a bijection* $h_S\colon S_\sigma(F) \longrightarrow S_\sigma(B)$ *such that*

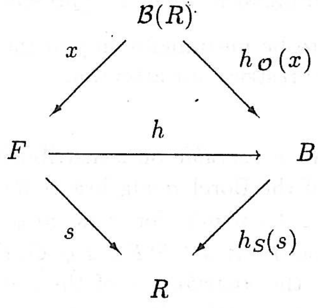
$$\forall s \in S_\sigma(F)\ \forall a \in F\colon s(a) = h_S(s)(h(a))\,,$$

(3) *there exists* $h_{\mathcal{O}} : \mathcal{O}(F) \xrightarrow{\text{onto}} \mathcal{O}(B)$ *such that*

$$\forall x \in \mathcal{O}(F)\ \forall T \in \mathcal{B}(R)\colon h(x(T)) = h_{\mathcal{O}}(x)(T)\,.$$

For all $x \in \mathcal{O}(F)$ and $s \in S_\sigma(F)$ the Borel measures $s \circ x$, $h_S(s) \circ h_{\mathcal{O}}(x)$ are equal and therefore the observables $x, h_{\mathcal{O}}(x)$ have the same distribution in the corresponding states $s, h_S(s)$, resp.

In view of Th. 6.3, the following diagram commutes:



Boolean σ-representations are generally not σ-fields of sets. This makes it more or less meaningless to try to represent observables by measurable functions. On the other hand, we can represent observables as "pointless random variables" (see [3]).

We say that two different bounded observables y, z are *indistinguishable* if their mean values $E(s,y), E(s,z)$ are equal in each σ-state s. A distributive de Morgan σ-lattice is said to have the *uniqueness property* if it admits no indistinguishable observables.

THEOREM 6.4 [15, Th. 4.5]. *An s. f. σ-algebra has the uniqueness property if and only if it is (isomorphic to) a σ-field. A Boolean σ-representation of an s. f. σ-algebra has the uniqueness property if and only if it is minimal.*

The significance of the minimal Boolean σ-representation is that it represents exactly those events and observables (and also related phenomena) which can be distinguished by states. Thus, one can avoid the ambiguity appearing in s. f. σ-algebras.

As applications of Boolean σ-representations, let us mention the s. f. σ-algebra analogues of the central limit theorem and the strong law of large numbers [9], the Radon-Nikodým theorem [8], the existence of a joint distribution [7] and the individual ergodic theorem [11]. All these results may be derived from the existence of a Boolean σ-representation. The latter representation approach allows also to develop a calculus for observables (with the disadvantage that the corresponding observables on s. f. σ-algebras are determined only up to the indistinguishability; this does not appear in the approach of e.g. [9]).

7 Questions related to "concrete" σ-representations

Attempts were made to represent s. f. σ-algebras by σ-fields of sets [5, 6, 13, 19]. This approach has led to promising applications [8, 7, 9, 11]. On the other hand, it has also presented itself with certain flaws. The thing is that the correspondence between the elements of an s. f. σ-algebra and the elements of the representing σ-field is not unique. This imperfection is necessary in the representation by σ-fields as the following result says.

PROPOSITION 7.1 [15, Prop. 2.3]. *There is an s. f. σ-algebra F admitting no σ-homomorphism into a σ-field.*

Moreover, the σ-representations by σ-fields have also the following rather negative features: The correspondence between states is injective, but it is not onto. The observables are represented by measurable functions, but different observables on F may be represented by the same function and different measurable functions may correspond to a single observable. All these inconveniences (safe for the injectivity of the observable correspondence) are avoided in the use of Boolean σ-representations. (Obviously, in the finitely additive case Boolean representation by a field of sets is always possible.)

REFERENCES

[1] CATTANEO, G.—DALLACHIARA, M. L.—GUINTINI, R.: *Empirical semantics for fuzzy-intuitionistic quantum logic*, To appear.

[2] CATTANEO, G., DALLACHIARA, M. L., GUINTINI, R.: *Fuzzy intuitionistic quantum logics*, To appear.

[3] DROSSOS, C. A., MARKAKIS, G.: *Boolean powers and stochastic spaces*, To appear.

[4] DROSSOS, C. A.—SHAKHATREH, M.: *Boolean methods in F-quantum spaces*, Tatra Mountains Math. Publ. **1** (1992).

[5] DVUREČENSKIJ, A.: *On the existence of probability measures on fuzzy measurable spaces*, Fuzzy Sets Syst. **43** (1991), 173–181.

[6] DVUREČENSKIJ, A.: *On a representation of observables in fuzzy measurable spaces*, J. Math. Anal. Appl., (to appear).

[7] DVUREČENSKIJ, A., RIEČAN, B.: *On joint distribution of observables for F-quantum spaces*, Fuzzy Sets Syst. **39** (1991), 65–73.

[8] DVUREČENSKIJ, A.: *The Radon-Nikodým theorem for fuzzy probability spaces*, Fuzzy Sets Syst. (to appear).

[9] DVUREČENSKIJ, A.—TIRPÁKOVÁ, A.: *A note on a sum of observables in F-quantum spaces and its properties*, BUSEFAL **35** (1988), 132–137.

[10] GUDDER, S. P.: Stochastic Methods in Quantum Mechanics, North Holland, New York, 1979.

[11] HARMAN, B., RIEČAN, B.: *On the individual ergodic theorem in F-quantum spaces*, In: Proc. Intern. Sem. on Fuzzy Sets Theory, Poznań, 1987.

[12] LONG, LE BA: *Fuzzy quantum posets and their states*, Acta Math. Univ. Comenianae **58-59** (1991), 231–237.

[13] MESIAR, R.: *On the sum of general fuzzy observables*, In: Proc. 2nd Winter School on Measure Theory, Liptovský Ján, 1990, pp. 151–155.

[14] MITCHELL, B.: Theory of Categories, Academic Press, New York/London, 1965.

[15] NAVARA, M.: *Boolean representations of fuzzy quantum spaces*, To appear.

[16] NAVARA, M.: *Algebraic approach to fuzzy quantum spaces*, To appear.

[17] NAVARA, M.—PTÁK, P.: *P-measures on soft fuzzy σ-algebras*, To appear.

[18] NAVARA, M.—PTÁK, P.: *On the state space of soft fuzzy algebras*, BUSEFAL **48** (1991), 55–63.

[19] PIASECKI, K.: *Probability of fuzzy events defined as denumerable additivity measure*, Fuzzy Sets Syst. **17** (1985), 271–284.

[20] PTÁK, P.—ROGALEWICZ, V.: *Regularly full logics and the uniqueness problem for observables*, Ann. Inst. H. Poincaré **38** (1983), 69–74.

[21] PYKACZ, J.: *Fuzzy set ideas in quantum logics*, In: Proc. Conf. Quantum Logics, Gdańsk, Poland, 1990.

[22] SIKORSKI, R.: Boolean Algebras, Springer-Verlag, Berlin, 1969.

[23] SOLOVAY, R. M.: *Real-valued measurable cardinals*, In: Proc. Symp. Pure Math. (D. Scott, ed.), Vol. 13 (1), Providence, 1971, pp. 397–428.

Department of Mathematics
Faculty of Electrical Engineering
Technical University of Prague
166 27 Prague 6
CZECHO-SLOVAKIA