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A PHYSICAL EXAMPLE OF QUANTUM FUZZY SETS, AND THE CLASSICAL LIMIT

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ABSTRACT. We present an explicit physical example of an experimental situation on a physical entity that gives rise to a fuzzy set. The fuzziness in the example is due to fluctuations of the experimental apparatus, and not to an indetermination about the states of the physical entity, and is described by a varying parameter ε. For zero value of the parameter (no fluctuations), the example reduces to a classical mechanics situation, and the corresponding fuzzy set is a quasi-crisp set. For maximal value (maximal fluctuations), the example gives rise to a quantum fuzzy set, more precisely a spin-model. In between, we have a continuum of fuzzy situations, neither classical, nor quantum. We believe that the example can make us understand the nature of the quantum mechanical fuzziness and probability, and how these are related to the classical situation.

Introduction

After the now rather commonly accepted failure of 'local' hidden variable theories to offer a model that could substitute for quantum mechanics [1, 2, 3, 4, 5, 6, 7], it is often thought that an 'understandable' explanation for the probabilities of quantum mechanics is now impossible, and that only the rather vague view of the presence of ontological probabilities, inherent in nature, remains. As we know, hidden variable theories in general try to explain the quantum probabilities as being due to a lack of knowledge about the complete reality of the state of the entity under study, because this is indeed the way in which classical probabilities appear in classical theories. There is, however, another possibility, namely to explain the presence of the quantum probabilities as being due to a lack of knowledge about the complete reality of the experimental situations that are considered in relation with the entity under study [8, 9, 10]. In this paper we want to elaborate this approach, as it has been put forward in ref. [9], [10], [11] and [12], but now to vary by means of a parameter ε the fluctuations that have been introduced. The reason why we want to study this example with varying fluctuations on the experimental situations, is because we want to investigate whether this approach makes it possible to define the classical limit. If this is true, we have possibly found a way to unify classical and quantum mechanics in a completely new way, in accordance with the proposed explanation of the quantum probabilities due to a lack of knowledge about the complete reality of the experimental situations. We also want to investigate which the fuzzy sets are that can be introduced naturally in this physically well defined situation.

The model

In quantum mechanics the state p of a physical entity S is represented by a unit vector x_p in a Hilbert space H. An experiment e on the physical entity S is represented by a self-adjoint operator A_e on H. If the experiment e has only two possible outcomes, that we call "yes" and "no", the self-adjoint operator that represents it is an orthogonal projector E_e. The probability $P(e = \text{yes} \mid p)$ to get an outcome "yes" for the experiment e is given by $\|E_e(x_p)\|^2$. The presence of this quantum mechanical probability leads in a natural way to the introduction of fuzzy sets [13, 14], related to a quantum mechanical measurement situation. 1) If Σ is the set of states of an entity S, then any experiment e defines a fuzzy set on Σ as follows $\mu_e : \Sigma \to [0, 1]$; $\mu_e(p) = \|E_e(x_p)\|^2$. 2) If Υ is the set of yes-no experiments, then any state p defines a fuzzy set on Υ as follows $\nu_p : \Upsilon \to [0, 1]$; $\nu_p(e) = \|E_e(x_p)\|^2$. Let us present our example. As we mentioned, we introduce the quantum fuzziness as fluctuations on a classical and deterministic measuring apparatus. Our physical entity S is a particle with fixed negative charge q in a state $p_{\theta\phi}$ such that it is located on a sphere

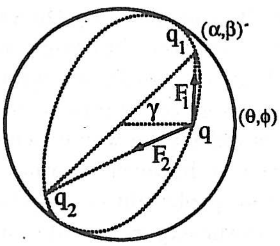


Fig. 1.

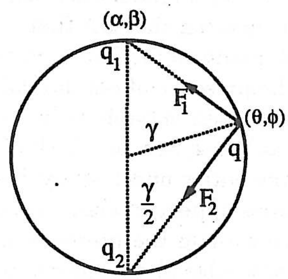


Fig. 2.

of radius r at a point (r, θ, ϕ). The experiment $e_{\alpha\beta}$ consists of the following operation: we choose two particles with positive charges q_1 and q_2 such that

$q_1+q_2=Q$. The charge q_1 is chosen at random between $\frac{1-\varepsilon}{2}Q$ and $\frac{1+\varepsilon}{2}Q$. This choice represents the lack of knowledge about the experimental situation and ε is the parameter describing the amount of lack of knowledge.Once the charges q_1 and q_2 are chosen we put the two particles diametrically on the sphere, such that q_1 is in the point (r,α,β) and q_2 is in the point $(r,\pi-\alpha,\pi+\beta)$. This is the set-up of the experiment $e_{\alpha\beta}$, the entity being in the state $p_{\theta\phi}$. Under the influence of the Coulomb forces F_1 (between q and q_1) and F_2 (between q and q_2), the charge q will move, and, since we suppose that this happens in a viscous medium, under the influence of friction, finally will end up at q_1 or at q_2. We suppose that if $\|F_1\|>\|F_2\|$ ($\|F_2\|>\|F_1\|$), q ends up at q_1 (q_2), we give the outcome e_1 (e_2) to $e_{\alpha\beta}$ and the effect of $e_{\alpha\beta}$ changes the state $p_{\theta\phi}$ in $p_{\alpha\beta}(p_{\pi-\alpha,\pi+\beta})$. If γ is the angle between (r,θ,ϕ) and (r,α,β), then

$$\|F_1\|=\frac{q_1 q}{4\pi\varepsilon_0 r^2\sin^2\gamma/2},$$
$$\|F_2\|=\frac{q_2 q}{4\pi\varepsilon_0 r^2\cos^2\gamma/2}.$$

Let us now evaluate the probability $p(e_{\alpha\beta}=e_1|p_{\theta\phi})$ that we get the outcome e_1 for $e_{\alpha\beta}$ if the particle q is in the state $p_{\theta\phi}$:

$$\begin{aligned}P(e_{\alpha\beta}=e_1|p_{\theta\phi}) &= P\left(\|F_1\|>\|F_2\|\right)=\\ &= P\left(q_1\cos^2\gamma/2>q_2\sin^2\gamma/2\right)=P\left(q_1>Q\sin^2\gamma/2\right).\end{aligned}$$

We can visualize this with the following picture. For the clearness of the figure we represent γ in the interval $[-\pi,+\pi]$ although the angle between space directions is limited to $[0,\pi]$. For simplicity the direction (α,β) of the measuring rod is represented in the middle of the diagram. The state $p_{\theta\phi}$ forms an angle $\gamma=\frac{\pi}{3}$

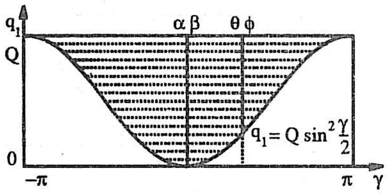


Fig. 3.

with the measuring rod. The condition $q_1>Q\sin^2\gamma/2$ is fulfilled in the dashed zone. So, for any experiment such that (γ, q_1) belongs to the dashed zone we obtain e_1 as outcome. Now, q_1 is submitted to fluctuations, whose magnitude labelled by ε : q_1, is uniformly distributed in the interval $\left[Q\frac{(1-\varepsilon)}{2}, Q\frac{(1+\varepsilon)}{2}\right]$. This allows us to compute $P(q_1 > Q\sin^2\gamma/2)$, which is given by the ratio of the length of the interval for which q_1 fulfills the condition, divided by the length of the total interval, which is $Q\varepsilon$.

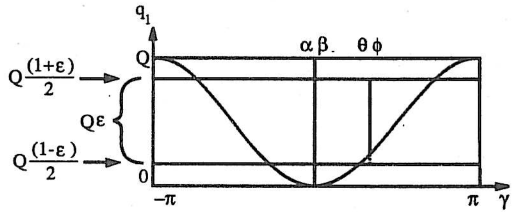


Fig. 4.

We obtain
$$p(e_{\alpha\beta} = e_1 | p_{\theta\phi}) = \frac{\max\left(\sin^2\frac{\gamma}{2}, \frac{1+\varepsilon}{2}\right)}{\varepsilon} - \frac{\max\left(\sin^2\frac{\gamma}{2}, \frac{1-\varepsilon}{2}\right)}{\varepsilon}$$

The limiting cases

$\varepsilon = 1$. The quantum case — maximal fluctuations.

We get $P(e_{\alpha\beta} = e_1 | p_{\theta\phi}) = \frac{1-\sin^2\gamma/2}{1} = \cos^2\gamma/2$. These are the same probabilities as the ones related to the outcomes of a Stern-Gerlach spin measurement on a spin1/2 quantum particle, of which the state (θ, ϕ) is represented by the

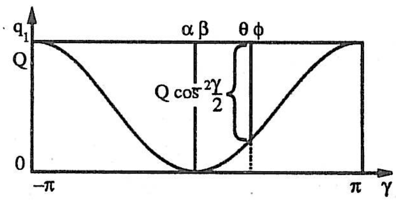


Fig. 5.

vector $\left(e^{-i\phi/2}\cos\frac{\theta}{2}, e^{i\phi/2}\sin\frac{\theta}{2}\right)$, and the experiment $e_{\alpha\beta}$ by the self-adjoint operator $S_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} \cos\alpha & e^{-i\beta}\sin\alpha \\ e^{i\beta}\sin\alpha & -\cos\alpha \end{pmatrix}$ in a two dimensional Hilbert space, which shows the equivalence between our model with $\varepsilon = 1$ and the quantum model tum model of a spin 1/2 particle. The fuzzy set defined by the state $p_{\theta\phi}$ is $\nu_{\theta\phi}\colon \Upsilon \to [0,1]$; $\nu_{\theta\phi}(e_{\alpha\beta}) = P(e_{\alpha\beta} = e_1 | p_{\theta\phi}) = \cos^2 \gamma/2$, where Υ is the set of all yes-no experiments. It is a pure quantum fuzzy set.

$\varepsilon = 0$. The classical case — no fluctuation.

Here, q_1 has no randomness at all, and is fixed once for all with the value $Q/2$. Then $P(e_{\alpha\beta} = e_1 | p_{\theta\phi}) = 1$ if $|\gamma| \in [0, \frac{\pi}{2}[$, $P(e_{\alpha\beta} = e_1 | p_{\theta\phi}) = 0$ if $|\gamma| \in]\frac{\pi}{2}, \pi]$. The situation $\gamma = \pm\frac{\pi}{2}$ corresponds to a classical unstable equilibrium.

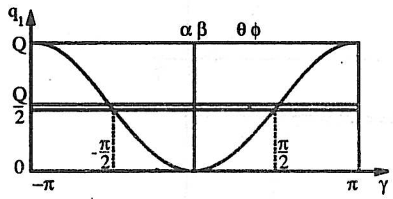


Fig. 6.

A careful calculation gives then probability $\frac{1}{2}$ for e_1, and $\frac{1}{2}$ for e_2. The fuzzy set induced by the state $p_{\theta\phi}$ we shall call a quasi-crisp set, because it takes values 0 and 1, except on a zero-measure subset of Υ. In a forthcoming paper we shall study in detail these quasi-crisp sets, that in our approach represent the classical objects. They are almost crisp sets and can almost be treated alike.

The intermediate case — $\varepsilon = 1/2$.

Here, q_1 is distributed in the interval $\left[\frac{Q}{4}, \frac{3Q}{4}\right]$. For $|\gamma|$ smaller than $\frac{\pi}{3}$ (bigger than $\frac{2\pi}{3}$), we have a deterministic behavior, where the outcome e_1 (e_2) is certain (probability 1), independent of the fluctuations on q_1. Physically,

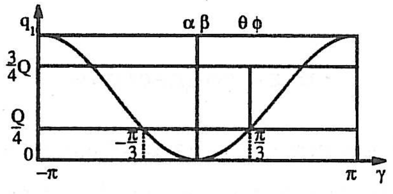
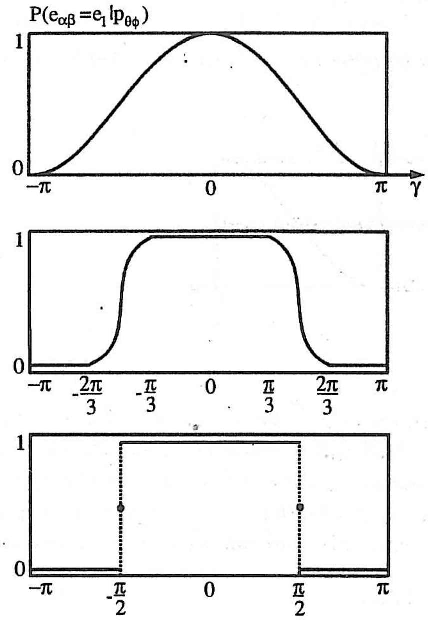


Fig. 7.

this reflects the fact that, when $p_{\theta\phi}$ is very close to $p_{\alpha\beta}$ (to $p_{\pi-\alpha,\pi+\beta}$), the

ε-limited fluctuations on q_1 are not strong enough to cause indeterminism. For $|\gamma|$ belonging to $]\frac{\pi}{3}, \frac{2\pi}{3}[$ we have an indeterministic quantum-like behavior. To summarize this survey we give here the probability law resulting for the three cases:



$\varepsilon = 1$ quantum

$\varepsilon = \frac{1}{2}$ intermediate

$\varepsilon = 1$ classical.

Fig. 8.

It is easy to show that the size of the classical zone γ_{cl} is given by $\cos \gamma_{cl} = \varepsilon$, with $\gamma_{cl} = 0$ in the quantum case, $\frac{\pi}{3}$ for $\varepsilon = \frac{1}{2}$ and $\frac{\pi}{2}$ in the classical case.

Fuzzy connectives

A) Standard and bold definitions — a graphical approach

Two definitions of fuzzy sets are mentioned in reference 14 as being the "standard" definition of Zadeh, and the "bold" definition of Giles. If A and B are fuzzy sets on a set whose elements are denoted by p, and take values $\mu_A(p)$ and $\mu_B(p)$ on them, it is possible to define fuzzy connectives in the following way: $\mu_{A \cup_S B}(p) = \max(\mu_A(p), \mu_B(p))$ and $\mu_{A \cap_S B}(p) = \min(\mu_A(p), \mu_B(p))$ define the standard union and intersection of A and B; $\mu_{A\cup_b B}(p) = \min(\mu_A(p) + \mu_B(p), 1)$ and $\mu_{A\cap_b B}(p) = \max(\mu_A(p) + \mu_B(p) - 1, 0)$ define the bold union and intersection of A and B. We recover equivalent results in our model by making use of a graphical method, as we show here. Let us take for simplicity two directions A and B of the measuring rod, coplanar with the direction $p_{\theta\phi}$ of the state and making angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with $p_{\theta\phi}$. We represent this on figure 9, with p in the middle of the diagram. For any couple of points (γ, q_1) belonging to the vertically dashed zone S_A, the "yes zone" of A, $q_1 > Q\sin^2(\gamma - \gamma_A)$ and, hence, if a state lies along the γ direction, the values of the charge taken in S_A will give the outcome "yes" for the measurement A. Similarly the horizontally dashed zone S_B represents the "yes zone" of B. If now we define the measurements $A \cap B$ and $A \cup B$ by assigning them the yes zones $S_A \cap S_B$ and $S_A \cup S_B$, we get the following results:

$$\mu_{A\cup B}(p) = \frac{h_{S_A \cup S_B}}{Q} = \max\left(\frac{h_{S_A}}{Q}, \frac{h_{S_B}}{Q}\right) = \max(\mu_A(p), \mu_B(p)) = \mu_{A\cup_s B}(p),$$

$$\mu_{A\cap B}(p) = \frac{h_{S_A \cap S_B}}{Q} = \min\left(\frac{h_{S_A}}{Q}, \frac{h_{S_B}}{Q}\right) = \min(\mu_A(p)\mu_B(p)) = \mu_{A\cap_s B}(p),$$

and we recover the standard result. If now, we replace S_B by S_B^c, obtained

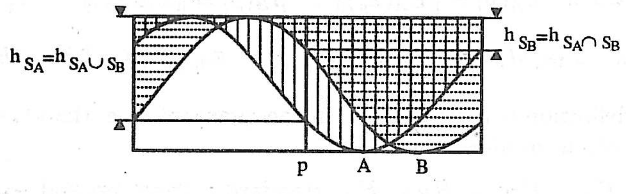


Fig. 9.

from S_B by a permutation of q_1 and $q_2 = Q - q_1$ (see figure 10), we get

$$\mu_{A\cup B^c}(p) = \frac{h_{S_A \cup S_{B^c}}}{Q} = \min(\mu_A(p) + \mu_B(p), 1) = \mu_{A\cup_b B}(p),$$

$$\mu_{A\cap B^c}(p) = \frac{h_{S_A \cap S_{B^c}}}{Q} = \max(\mu_A(p) + \mu_B(p), 1) = \mu_{A\cap_b B}(p),$$

and we recover the bold definition.

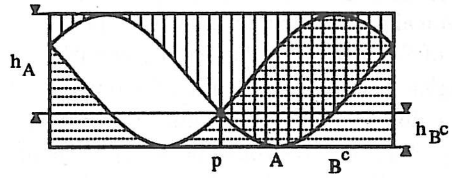


Fig. 10.

B) A quantum induced definition for fuzzy connectives

In quantum mechanics, more specifically in the quantum logic approach [15, 16, 17], whenever two dichotomic experiments A and B are described by projectors E_A and E_B on subspaces H_A and H_B of the Hilbert space H, disjunction of A and B is represented by the projector $E_A \vee E_B$ on $H_A \oplus H_B$ (the direct sum of H_A and H_B) and conjunction of A and B is represented by the projector $E_A \wedge E_B$ on $H_A \cap H_B$. This leads us to propose the following definition for fuzzy connectives (we will call them quantum connectives and label them by the subscript q):

$$\text{when} \quad \mu_A(p) = \|E_A(x_p)\|^2, \quad \mu_B(p) = \|E_B(x_p)\|^2, \quad x_p \in H;$$

$$\text{then} \quad \mu_{A\cap_q B}(p) = \|E_A \wedge E_B(x_p)\|^2, \quad \mu_{A\cup_q B}(p) = \|E_A \vee E_B(x_p)\|^2.$$

This definition is more general, in the quantum case, than the standard and bold definitions as shown here:

1) when $H_A \subset H_B$ or $H_B \subset H_A$: standard connectives and quantum connectives are equivalent,

2) when $H_A \cap H_B = \emptyset$: bold connectives and quantum connectives are equivalent,

3) in any other situation it is possible to find states such that quantum connectives are neither standard nor bold. Moreover, it is sometimes impossible to express q connectives $\mu_{A\cup_q B}(p)$ and $\mu_{A\cap_q B}(p)$ as pointwise functions of $\mu_A(p)$ and $\mu_B(p)$. For instance, take H, a four dimensional space, with orthonormal base $\{e_1, e_2, e_3, e_4\}$, E_A the projector on $\text{lin}(e_1, e_2)$ and E_B the projector on $\text{lin}(e_2, e_3)$. If $x_p = p_1 e_1 + p_2 e_2 + p_3 e_3 + p_4 e_4$, then

$\mu_A(p) = \|p_1\|^2 + \|p_2\|^2$, $\mu_B(p) = \|p_2\|^2 + \|p_3\|^2, \mu_{A\cup_q B}(p) = \|p_1\|^2 + \|p_2\|^2 + \|p_3\|^2, \mu_{A\cap_q B}(p) = \|p_2\|^2$.

Nevertheless, for compatible measurements A and B, the following expression remains true: $\mu_{A\cup B}(p) + \mu_{A\cap B}(p) = \mu_A(p) + \mu_B(p)$ whenever we apply the bold, standard or quantum definition of fuzzy connectives.

Conclusion

The ε-charge-model that we have presented allows to define the 'classical limit' inside this approach of 'lack of knowledge about or fluctuations on the experimental situation'. We are now investigating whether this classical limit can be used to paste quantum theories with classical theories. Theoretical as well as experimental questions have still to be answered. We wonder for example whether a large molecule (smaller fluctuations) of spin 1/2 will deviate from the quantum predictions along with our ε-model. We also want to remark that the ε-charge-model that we present here is a realization of a dynamical system with an explicit quantum-like structure. It is obvious that we can build along the same lines dynamical quantum systems, by introducing a description of the effects of experiments and fluctuations on these effects. Such an approach can perhaps elucidate the relation between quantum-like indeterminism and irreversibility. Indeed an essential feature of the construction of our model is the irreversible motion of the charge.

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