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JOINT CONTINUITY OF CHRISTENSEN MEASURABLE MULTIADDITIVE MAPS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. We discuss the question of continuity of both separately and jointly Christensen measurable multiadditive mappings; a similar result concerning polynomial functions is given.

1. Introduction

Let $(G,+)$ be an abelian Polish topological group and let $\mathcal{M}_G$ stand for the σ-algebra of all universally measurable subsets of G. By $\mathcal{H}_0(G)$ we denote the collection of all Haar zero sets in G, i.e. the subfamily of $\mathcal{M}_G$ consisting of all sets admitting a probability measure $\mu\colon \mathcal{M}_G \to [0,1]$ such that $\mu(A+x)=0$ for every $x \in G$. Following P. Fischer and Z. Słodkowski [5] denote by $\mathcal{C}_0(G)$ the family of all subsets of the members of $\mathcal{H}_0(G)$ and write

$$\mathcal{C}(G)\colon = \{B \cup C\colon B \in \mathcal{M}_G \quad \text{and} \quad C \in \mathcal{C}_0(G)\}.$$

Any set from $\mathcal{C}_0(G)$ is termed a *Christensen zero set* whereas the phrase "$D \subset G$ is Christensen measurable" will stand for "$D \in \mathcal{C}(G)$"; a probability measure in question while considering a Haar zero set A will be called a *testing measure* for A. The family $\mathcal{C}_0(G)$ forms a σ-ideal invariant under translations with respect to zero whereas $\mathcal{C}_0(G)$ yields a σ-algebra containing all universally measurable sets.

Finally, a mapping f from G into a topological space Y is called *Christensen measurable* provided that $f^{-1}(U) \in \mathcal{C}(G)$ for every open set $U \in Y$.

The notion of Christensen measurability of both sets and mappings proved to be very good analogues of the classical Haar measurability to which they

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reduce in the case where the topological group considered happens to be locally compact.

For further details and proofs the reader is referred to J. P. R. Christensen's monograph [4].

Extending a well-known result stating that every Haar measurable homomorphism of a locally compact group into a separable group is continuous (see e.g. E. Hewitt & K. A. Ross [10], P. Fischer and Z. Słodkowski have shown in [6] that the same remains valid for Christensen measurability. In 1988 J. P. R. Christensen and P. Fischer [5] proved that any Christensen measurable biadditive mapping on the product of two abelian Polish groups into another abelian Polish group has to be continuous. However, their proof methods do not carry over automatically neither to multiadditive mappings nor to polynomial functions (see Section 4. below). Therefore, in the present work we will discuss the question of continuity of both separately and jointly Christensen measurable multiadditive mappings; a similar result concerning polynomial functions is given in Section 4.

It should be emphasized here that, in contrast to the case of Haar measurability on LCA-groups, in general, Fubini type theorems fail to hold for the σ-algebras $\mathcal{C}(G_1 \times G_2)$ and $\mathcal{C}(G_1)$, $\mathcal{C}(G_2)$ of Christensen measurable sets as well as for the σ-ideals $\mathcal{C}_0(G_1 \times G_2)$ and $\mathcal{C}_0(G_1)$, $\mathcal{C}_0(G_2)$ of Christensen zero sets (see J. P. R. Christensen's paper [3]). Therefore, in particular, the $\mathcal{C}(G_1 \times G_2)$-measurability of a set (of a mapping) does not imply the $\mathcal{C}(G_1)$-measurability of $\mathcal{C}_0(G_2)$-almost all horizontal sections of that set (that mapping). Some partial positive results in connection with that (essential) difficulty are given in [9].

2. Separate measurability

In what follows, we shall permanently be assuming that a positive integer k and topological groups $(G_1,+),\dots,(G_k,+)$, $(\Gamma,+)$ (not necessarily abelian) are given. A map $A_k\colon G_1\times\dots\times G_k \to \Gamma$ is termed k-additive provided that for each $i \in \{1,\dots,k\}$ and all choices of points $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)$ from $G_1\times\dots\times G_{i-1}\times G_{i+1}\times\dots\times G_k$ the i-th section $A_k(x_1,\dots,\ x_{i-1},x_{i+1},\dots,x_k)$ of A_k yields a homomorphism of $(G_i,+)$ into $(\Gamma,+)$.

We begin with the following

Lemma 1. *Assume that $(G_1,+),\dots,(G_k,+)$ are Polish groups and $(\Gamma,+)$ is a metrizable topological group. Then every k-additive separately continuous mapping from $G_1\times\dots\times G_k$ into Γ is jointly continuous.*

Proof. For $k=1$ the assertion is trivial. Assume the validity of the lemma for a positive integer $k-1$ and fix a k-additive mapping $A_k\colon G_1\times\dots\times G_k \to \Gamma$.

By induction hypothesis, for every $i \in \{1, \dots, k\}$ and for an arbitrarily fixed point $a^i \in G_i$ the section

$$A_{ik,a^i} := A_k(\cdot, \dots, \cdot, a^i, \cdot, \dots, \cdot) \quad \text{is jointly continuous.} \tag{1}$$

Now we are going to show that the map A_k has a point of joint continuity. This will be done with the aid of the following result due to J. Calbrix and J. P. Troallic [1]:

Let X and Y be two topological spaces and let Z be a metric space. Suppose that a map $\varphi\colon X \times Y \to Z$ is separately continuous. If Y satisfies the axiom of second countability, then there exists a residual set $A \subset X$ such that φ is jointly continuous at each point of $A \times Y$.

Since the mapping $\varphi\colon G_1 \times (G_2 \times \cdots \times G_k) \to \Gamma$ given by the formula

$$\varphi(s,t) := A_k(s,t), \quad s \in G_1, \quad t \in G_2 \times \cdots \times G_k,$$

is separately continuous, there exists a residual set $A \subset G_1$ such that φ is jointly continuous at each point of $A \times (G_2 \times \cdots \times G_k)$. By the classical Baire category theorem $A \neq \emptyset$ and therefore φ actually possesses a joint continuity point $(x_0^1, (x_0^2, \dots, x_0^k))$. Plainly, $x_0 := (x_0^1, x_0^2, \dots, x_0^k)$ yields a joint continuity point of A_k.

Now, fix arbitrarily a point $x = (x^1, x^2, \dots x^k)$ and a sequence $x_n = (x_n^1, x_n^2, \dots x_n^k)$, $n \in \mathbb{N}$, of elements of $G_1 \times \cdots \times G_k$, tending to x as $n \to \infty$. Then, setting $a_n := (x_n^1 - x^1, x_n^2, \dots, x_n^k)$, $n \in \mathbb{N}$, one gets

$$A_k(x_n) = A_k(a_n) + A_{k,x^1}(x_n^2, \dots, x_n^k), \quad n \in \mathbb{N}.$$

In view of the joint continuity of A_{k,x^1}, to prove the continuity of A_k at x one has to show that $A_k(a_n) \to \hat{0}$ as $n \to \infty$, where $\hat{0}$ stands for the neutral element of Γ. To this aim, denote by 0^1 the neutral element of G_1 and observe that for any sequence $b_n = (b_n^1, b_n^2, \dots, b_n^k) \to (0^1, x_0^2, \dots, x_0^k)$ one has

$$A_k(b_n) \to \hat{0} \quad \text{as} \quad n \to \infty. \tag{2}$$

In fact, this results immediately from the joint continuity of A_{k,x_0^1}, the continuity of A_k at x_0 and from the equalities

$$A_k(b_n) = -A_{k,x_0^1}(b_n^2, \dots, b_n^k) + A_k(x_0^1 + b_n^1, b_n^2, \dots, b_n^k), \quad n \in \mathbb{N}.$$

Put $b_n^1 := x_n^1 - x^1$ and $b_n^i := x_n^i - x^i + x_0^i$, $i = 2, \dots, k$, $n \in \mathbb{N}$. Then

$$b_n := (b_n^1, b_n^2, \dots, b_n^k) \to (0^1, x_0^2, \dots, x_0^k) \quad \text{as} \quad n \to \infty,$$

whence

$$\begin{aligned} A_k(a_n) &= A_k(b_n^1, x_n^2, \ldots, x_n^k) \\ &= A_k(b_n^1, b_n^2, x_n^3, \ldots, x_n^k) + A_k(b_n^1, -x_0^2 + x^2, x_n^3, \ldots, x_n^k) \\ &= A_k(b_n^1, b_n^2, x_n^3, \ldots, x_n^k) + 0(1) \quad \text{as} \quad n \to \infty , \end{aligned}$$

in view of (1). Proceeding similarly with the sequences $(x_n^3)_{n \in \mathrm{N}}, \ldots, (x_n^k)_{n \in \mathrm{N}}$ one gets finally

$$A_k(a_n) = A_k(b_n) + o(1) \quad \text{as} \quad n \to \infty .$$

This completes the proof because of (2). □

Now we are in a position to prove

THEOREM 1. *Let $(G_1, +), \ldots, (G_k, +)$ and $(\Gamma, +)$ be abelian Polish topological groups and let $A_k \colon G_1 \times \cdots \times G_k \to \Gamma$ be a k-additive mapping whose i-th sections $A_k(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_k)$ are $\mathcal{C}(G_i)$-measurable for all choices of points $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ from $G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_k$ and for each $i \in \{1, \ldots, k\}$. Then A_k is jointly continuous.*

P r o o f. Each i-th section of A_k yields a Christensen measurable homomorphism from G_i into Γ, $i = 1, \ldots, k$. By means of the result of Fischer-Słodkowski [6] quoted above all these sections are continuous. Thus A_k is separately continuous; an appeal to Lemma 1 finishes the proof. □

3. Joint measurability

Now, we are going to generalize a Christensen-Fischer result [5, Theorem 3.] on the continuity of Christensen (jointly) measurable biadditive mappings to the multiadditive case.

THEOREM 2. *Let $(G_1, +), \ldots, (G_k, +)$ and $(\Gamma, +)$ be abelian Polish groups. Then any k-additive $\mathcal{C}(G_1 \times \cdots \times G_k)$-measurable mapping from $G_1 \times \cdots \times G_k$ into Γ is continuous.*

P r o o f. Let $G := G_1 \times \cdots \times G_k$ and let $A_k \colon G \to \Gamma$ be a k-additive $\mathcal{C}(G)$-measurable mapping. In view of the Lemma it suffices to prove that A_k is separately continuous (actually, in view of Theorem 1, Christensen measurability of all the i-th sections would be sufficient but in the present case such an observation does not simplify the matter).

We shall show that for every $(u^1, \ldots, u^{k-1} \in G_1 \times \cdots \times G_{k-1}$ the k-th section $A_k(u^1, \ldots, u_0^{k-1}, \cdot)$ is continuous at zero. Suppose the contrary; then there exists a $u_0 = (u_0^1, \dots, u_0^{k-1} \in G_1 \times \cdots \times G_{k-1}$ and a sequence $(h_n)_{n \in \mathbb{N}}$ of elements of G_k, convergent to the neutral element $0^k \in G_k$, such that for some $\varepsilon_0 > 0$ one has

$$\rho(h_n, 0^k) < \frac{1}{2^n} \qquad \text{and} \qquad \hat{\rho}\big(A_k(u_0, h_n), \hat{0}\big) \geq \varepsilon_0 > 0$$

for all $n \in \mathbb{N}$ (here ρ and $\hat{\rho}$ stand for complete translation invariant metrics generating the topologies of G_k and Γ, respectively, whereas $\hat{0}$ denotes the neutral element of the group $(\Gamma, +)$).

Following the idea of the proof of Theorem 3 from [5] we introduce the compact abelian group $H = \{0, 1\}$ with the Haar measure μ_H determined by the equalities

$$\mu_H(\{0\}) = \mu_H(\{1\}) = \frac{1}{2}$$

and form the product group $K := H^{\mathbb{N}}$ equipped with the completed product Haar measure μ_K. Now, let $\Theta_1 \colon H^{k-1} \to G_1 \times \cdots \times G_{k-1}$ and $\Theta_2 \colon K \to G_k$ be defined by the formulas

$$\Theta_1(x) \colon (x^1 u_0^1, \dots, x^{k-1} u_0^{k-1}) \quad \text{for all} \quad x = (x^1, \dots, x^{k-1}) \in H^{k-1}$$

and

$$\Theta_2(y) := \sum_{n - \mathbb{N}} y(n) h_n \quad \text{for all} \quad y \in K.$$

Obviously, both Θ_1 and Θ_2 are continuous and hence so is the function $\Theta \colon H^{k-1} \times K \to G$ given by the formula $\Theta(x, y) := \big(\Theta_1(x), \Theta_2(y)\big)$, $x \in H^{k-1}$, $y \in K$. Proceeding like in [5] we deduce the existence of a pair $(a, b) \in (G_1 \times \cdots \times G_{k-1}) \times G_k$ such that the mapping $D \colon H^{k-1} \times K \to \Gamma$ given by

$$D(x, y) := A_k\big(\Theta_1(x) + a, \Theta_2(y) + b\big), \quad x \in H^{k-1}, \quad y \in K,$$

is jointly Haar measurable.

Since the Fubini theorem holds true for the Haar measurability we infer that for $\mu_{H^{k-1}}$-almost all $(x^1, \dots, x^{k-1}) \in H^{k-1}$ the sections $D(x^1, \dots, x^{k-1}, \cdot)$ are Haar measurable on K. However, the empty set is the only null-set in H^{k-1}; consequently, all of these sections are Haar measurable on K.

Let $\{\gamma_n \colon n \in \mathbb{N}\}$ be a dense subset of Γ and let $\varepsilon \in \big(0, \frac{1}{4^k}\varepsilon_0\big)$ be fixed. Since Γ is obviously covered by the union of all balls $B(\gamma_n, \varepsilon)$ centered at $\gamma_n, n \in \mathbb{N}$, and having the radius ε we infer that for every sequence $(x^1, \dots, x^{k-1}) \in H^{k-1}$ there exists a positive integer $n_{x^1, \dots, x^{k-1}}$ such that the measurable set

$$F_{x^1, \dots, x^{k-1}} := D(x^1, \dots, x^{k-1}, \cdot)^{-1}\big(B(\gamma_{n_{x^1, \dots, x^{k-1}}}, \varepsilon)\big) \subset K$$

is of positive Haar measure in K. By means of the Pettis theorem (see e. g. E. Hewit & K. A. Ross [10] each of the sets $F_{x^1,\dots,x^{k-1}} - F_{x^1,\dots,x^{k-1}}$ contains a neighbourhood of the neutral element 0_K in K; this implies the existence of a neighbourhood U_0 of 0_K such that

$$U_0 \subset \bigcap \{F_{x^1,\dots,x^{k-1}} - F_{x^1,\dots,x^{k-1}} \colon (x^1,\dots,x^{k-1}) \in H^{k-1}\}.$$

Obviously, almost all sequences $(\delta_{n,p})_{n\in\mathbb{N}}$, where $\delta_{n,p}$ stands for the Kronecker symbol, are in U_0; say $(\delta_{n,p})_{n\in\mathbb{N}} \in U_0$ for all $p \geq p_0$. In other words, for each sequence $(x^1,\dots,x^{k-1})$ from H^{k-1} and every $p \geq p_0$ one may find a $y_{x^1,\dots,x^{k-1},p}$ from K such that both $y_{x^1,\dots,x^{k-1},p}$ and $y_{x^1,\dots,x^{k-1},p} - (\delta_{n,p})_{n\in\mathbb{N}}$ belong to $F_{x^1,\dots,x^{k-1}}$; this means that both $D(x^1,\dots,x^{k-1},\ y_{x^1,\dots,x^{k-1},p})$ and $D(x^1,\dots,x^{k-1},\ y_{x^1,\dots,x^{k-1},p} - (\delta_{n,p})_{n\in\mathbb{N}})$ are in the ball $B(\gamma_{n_{x^1,\dots,x^{k-1}}},\varepsilon)$ whence

$$\hat{\rho}\big(D(x^1,\dots,x^{k-1},\ y_{x^1,\dots,x^{k-1},p}),\ D(x^1,\dots,x^{k-1},\ y_{x^1,\dots,x^{k-1},p} - (\delta_{n,p})_{n\in\mathbb{N}})\big) < \\ < 2\varepsilon \quad (3)$$

for each $(x^1,\dots,x^{k-1}) \in H^{k-1}$ and every $p \geq p_0$. Since $\Theta_2(y) - \Theta_2\big(y - (\delta_{n,p})_{n\in\mathbb{N}}\big) = h_p$ for all $p \in \mathbb{N}$ and $y \in K$, relation (3) jointly with the definition of D implies that

$$\hat{\rho}\big(A_k(x^1u_0^1 + a^1,\dots,x^{k-1}u_0^{k-1} + a^{k-1}, h_p),\hat{0}\big) < 2\varepsilon \tag{4}$$

for each $(x_1,\dots,x^{k-1}) \in H^{k-1}$ and every $p \geq p_0$, where $(a^1,\dots,a^{k-1}) = a$.

Fix a $p \geq p_0$; with the aid of these 2^{k-1} inequalities (corresponding to all possible choices of sequences $(x^1,\dots,x^{k-1}) \in H^{k-1}$ it is not hard to show that estimation

$$\hat{\rho}\big(A_k(c^1,\dots,c^{k-1},h_p),\hat{0}\big) < 2^k\varepsilon \tag{5}$$

holds true for every member of the family $\mathcal{I}$ of all sequences $(c^1,\dots,c^{k-1})$ with $c^i \in \{u_0^i, a^i\}$, $i = 1,\dots,k-1$, except for u_0. Indeed, applying (4) for the vectors $(0,\dots,0)$ and $(0,\dots,0,1,0,\dots,0)$ (1 at i-th place, $i = 1,\dots,k-1$) we get

$$\hat{\rho}\big(A_k(a^1,\dots,a^{k-1},h_p),\hat{0}\big) < 2\varepsilon$$

and

$$\hat{\rho}\big(A_k(a^1,\dots,a^{i-1},u_0^i + a^i,a^{i+1},\dots,a^{k-1},h_p),\hat{0}\big) < 2\varepsilon,$$

respectively, whence

$$\hat{\rho}\big(A_k(a^1,\dots,a^{i-1},u_0^i,\ a^{i+1},\dots a^{k-1},h_p),\hat{0}\big) < 4\varepsilon \tag{6}$$

for all $i \in \{1, \ldots, k-1\}$. Next, we extract similarly the distances of the form

$$\hat{\rho}\big(A_k(a^1, \ldots, a^{i-1},\, u_0^i,\, a^{i+1}, \ldots, a^{j-1},\, u_0^j,\, a^{j+1}, \ldots, a^{k-1}, \hat{0}\big)\,;$$

all of them are less than 8ε. To visualize this, let us take $i = 1$ and $j = 2$, for example. We have from (4):

$$\hat{\rho}\big(A_k(u_0^1 + a^1, u_0^2 + a^2, a^3, \ldots, a^{k-1}, h_p), \hat{0}\big) < 2\varepsilon$$

and

$$\hat{\rho}\big(A_k(u_0^1 + a^1, a^2,\, a^3, \ldots, a^{k-1}, h_p), \hat{0}\big) < 2\varepsilon\,,$$

whence

$$\hat{\rho}\big(A_k(u_0^1 + a^1, u_0^2, a^3, \ldots, a^{k-1}, h_p), \hat{0}\big) < 4\varepsilon$$

and, in view of (6),

$$\hat{\rho}\big(A_k(u_0^1, u_0^2, a^3, \ldots, a^{k-1}, h_p), \hat{0}\big) < 8\varepsilon\,.$$

Proceeding in this way we shall finally arrive at (5).

Since

$$A_k(u_0, h_p) = A_k(u_0 + a, h_p) - \sum_{\mathcal{I}} A_k(c^1, \ldots, c^{k-1}, h_p)\,,$$

we get

$$\hat{\rho}\big(A_k(u_0, h_p), \hat{0}\big) \le \hat{\rho}\big(A_k(u_0 + a, h_p), \hat{0}\big) + \sum_{\mathcal{I}} \hat{\rho}\big(A_k(c^1, \ldots, c^{k-1}, h_p), \hat{0}\big)\,.$$

Consequently, be means of (4) and (5), we obtain

$$\hat{\rho}\big(A_k(u_0, h_p), \hat{0}\big) \le 2\varepsilon + (2^{k-1} - 1)2^k\varepsilon < 4^k\varepsilon < \varepsilon_0$$

for all $p \in \mathbb{N}$, $p \ge p_0$. This contradicts the definition of the sequence $(h_n)_{n\in\mathbb{N}}$ and finishes the proof. $\square$

4. Polynomial functions

Assume that we are given a commutative semigroup $(S,+)$, a commutative group $(G,+)$ and a positive integer n. Recall that a map $f\colon S \to G$ is called a polynomial function of at most n-th degree if and only if f is a solution to the functional equation of Fréchet

$$\Delta_y^{n+1} f(x) = 0, \quad x, y \in S, \tag{7}$$

here Δ_u^p stands for the p-th iterate of the difference operator $\Delta_y f(x) := f(x+y) - f(x)$, $x, y \in S$, $p \in \mathbb{N}$.

It is well known (see e. g. M. Kuczma [11]) that any Lebesgue measurable polynomial function from $\mathbb{R}$ into $\mathbb{R}$ has to be continuous (and hence a standard polynomial on $\mathbb{R}$). More generally, a result due to Z. Gajda [8] (see also Z. Gajda [7]) states that any Christensen measurable polynomial function of at most n-th degree from an abelian Polish group into an abelian topological group with a countable base and with uniquely performable division by $n!$ is necessarily continuous. However, this division is (tacitly) assumed to be continuous. Under the above circumstances this need not be the case; J. Chmieliński exhibits in [2] an example of an abelian metrizable separable group with discontinuous but uniquely performable division by 2.

In what follows, using totally different methods (in comparison with the above mentioned Gajda's papers), we will prove the continuity of Christensen measurable polynomial functions between abelian Polish groups. This will show up as an application of the result established in the previous section.

We begin with two lemmas.

Lemma 2. *Let $(G,+)$ be an abelian Polish group and let T be a homeomorphic automorphism of G. Then, every Christensen measurable set $M \subset G$ has Christensen measurable image $T(M)$ in G.*

Proof. Since $T(M)$ is universally measurable provided so is M itself, it is enough to show that $T(M) \in \mathcal{C}_0(G)$ whenever $M \in \mathcal{C}_0(G)$. To this aim, assume that $M \subset H \in \mathcal{H}_0(G)$ and μ is a testing measure for H. Obviously, we have $T(M) \subset T(H) \in \mathcal{M}_G$ and since $T^{-1}(A) \in \mathcal{M}_G$ provided that $A \in \mathcal{M}_G$, the formula $\nu := \mu \circ T^{-1}$ correctly defines a probability measure on $\mathcal{M}_G$, which turns out to be a testing measure for the set $T(H)$. Indeed, for any $x \in G$ one has

$$\nu\big(T(H)+x\big) = \nu\big(T(H+T^{-1}(x))\big) = \mu\big(H+T^{-1}(x)\big) = 0,$$

which ends the proof. □

LEMMA 3. *Let $(G,+)$ be an abelian Polish group uniquely divisible by a positive integer m. Then the multiplication by m yields a homeomorphism on G.*

P r o o f. We have to show that a function $\varphi\colon G \to G$ given by the formula $\varphi(x) := \frac{1}{m}x, x \in G$, is continuous. To this end, fix an open set $U \subset G$ and note that $B := \varphi^{-1}(U) = mU$. It is well known that Borel subsets of a Polish space are Souslin; hence B is a Souslin set as a continuous image of the open set U. Analogously, $G \setminus B = m(G \setminus U)$ is a Souslin set, too. By a classical theorem of Souslin, B has to be a Borel set. Consequently, φ turns out to be a Borel function; in particular, φ yields a Christensen measurable homomorphism. On account of the Fischer-Słodkowski theorem [6], φ is continuous.

There are numerous results establishing the openness of a homomorphism between various special topological groups (see e. g. A. W i l a n s k y [13]). Some of them actually cover Lemma 2 but the proof presented here seems to be particularly simple and short. □

THEOREM 3. *Let $(G,+)$ and $(\Gamma,+)$ be two abelian Polish groups and let $f\colon G \to \Gamma$ be a Christensen measurable polynomial function of at most n-th degree. If the division by $n!$ is uniquely performable in Γ, then f is continuous.*

P r o o f. Denote by 0 the neutral elements of the groups considered and put

$$A_n(x_1,\dots,x_n)\colon = \frac{1}{n!}\Delta_{x_1}\circ\dots\circ\Delta_{x_n}f(0) \tag{8}$$

for all n-tuples $(x_1,\dots,x_n) \in G^n$. It is well known that A_n yields an n-additive and symmetric map from G^n into Γ (see e. g. L. S z é k e l y h i d i [12]). On the other hand, an easy induction gives

$$\Delta_{x_1}\circ\dots\circ\Delta_{x_n}f(0) = \sum_{\varepsilon_1,\dots,\varepsilon_n=0}^{1}(-1)^{n-(\varepsilon_1+\dots+\varepsilon_n)}f(\varepsilon_1x_1+\dots+\varepsilon_nx_n) \tag{9}$$

for all $(x_1,\dots,x_n) \in G^n$. We are going to show that the map A_n is $\mathcal{C}(G^n)$-measurable. To this end, fix arbitrarily numbers $\varepsilon_1,\dots,\varepsilon_n \in \{0,1\}$ and consider a map $\varphi\colon G^n \to \Gamma$ given by the formula

$$\varphi(x_1,\dots,x_n)\colon = f(\varepsilon_1x_1+\dots+\varepsilon_nx_n), \quad (x_1,\dots,x_n) \in G^n.$$

This is a $\mathcal{C}(G^n)$-measurable map. In fact, let $1 \le i_1 < i_2 < \dots < i_k \le n$ be such that $\varepsilon_{i_j} = 1$ for $j \in \{1,\dots,k\}$ and $\varepsilon_p = 0$ for $p \in \{1,\dots,k\} \setminus \{i_1,\dots,i_k\}$. Then

$$\varphi(x_1,\dots,x_n) = f(x_{i_1}+\dots+x_{i_k})$$

for all $(x_1,\dots,x_n)\in G^n$, and for any open set $U\subset\Gamma$ one has $M:=f^{-1}(U)\in\mathcal{C}(G)$ whence

$$\begin{aligned}\varphi^{-1}(U)&=\big\{(x_1,\dots,x_n)\in G^n\colon x_{i_1}+\dots+x_{i_k}\in M\big\}=\\&=\big\{T(x_1,\dots,x_n)\in G^n\colon x_{i_k}\in M\big\}\end{aligned}$$

where

$$T(x_1,\dots,x_n):=(x_1,\dots,x_{i_k-1},x_{i_k}-(x_{i_1}+\dots+x_{i_{k-1}}),x_{i_k+1},\dots,x_n)$$

for $(x_1,\dots,x_n)\in G^n$. One can easily check that the map $T\colon G^n\to G^n$ is a homeomorphic automorphism of the group G^n. Therefore, by means of Lemma 2, in view of the $\mathcal{C}(G^n)$-measurability of the set $M(i_k):=G^{i_k-1}\times M\times G^{n-i_k}$ (see [9, Theorem 2], for instance) we infer that

$$\varphi^{-1}(U)=T\big(M(i_k)\big)\in\mathcal{C}(G^n),$$

as claimed.

Consequently, on account of [9], [8] and Lemma 3, the map A_n is $\mathcal{C}(G^n)$-measurable. An appeal to Theorem 2 gives now the joint continuity of A_n. Thus, the diagonalization $A^n\colon G\to\Gamma$ given by

$$A^n(x):=A_n(x,\dots,x),\quad x\in G,$$

is continuous as well. Therefore, the function $g\colon G\to\Gamma$ defined by the formula

$$g(x)=f(x)-A^n(x),\quad x\in G, \tag{10}$$

is $\mathcal{C}(G)$-measurable. Since $\Delta_y^nA^n(x)=(n!)A^n(y)$ for all $x,y\in G$ (see L. Székelyhidi [12] or M. Kuczma [11], for example), it follows from (10) and (8) that

$$\Delta_y^ng(x)=\Delta_y^nf(x)-(n!)A^n(y)=\Delta_n^nf(x)-\Delta_y^nf(0)=0$$

for all $x,y\in G$; the latter equality results from the fact that the map $G\ni x\mapsto\Delta_y^nf(x)\in\Gamma$ is constant (see [12] or [11], again).

Consequently, g is a Christensen measurable solution to [7] with $n+1$ replaced by n and so, induction completes the proof. □

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