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PSEUDO–ARITHMETICAL OPERATIONS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. An axiomatic approach to the pseudo-addition and the pseudo-multiplication is presented (to be a basis for further study of pseudo-additive measures and integrals). Under additional requirements we get different types of these operations, e.g. Weber's or Pap's ones. An extension to the operations of pseudo-subtraction and pseudo-division is shown.

1. Introduction

The axiomatic concepts of pseudo-addition and pseudo-multiplication on $[0,+\infty]$ suggested by Sugeno and Murofushi [5] are the extensions of well-known concepts of t-norm and t-conorm. They used them for the sake of generalization of fuzzy integral introduced by Weber [6]. Other modifications of these concepts (non-axiomatic ones) can be found in the paper by Ichihashi, Tanaka and Asai [1]. E. Pap [3, 4] applied a special pseudo-addition and pseudo-multiplication to the building up of g-calculus and described some interesting applications of it to the solution of some differential equations.

In this paper we will present a pseudo-addition and a pseudo-multiplication based on a suitable system of axioms leading to the concepts used in [1, 3, 4]. Using some results of Sugeno and Murofushi [5] we show that these conditions are equivalent to the existence of a function g generating these operations. We extend both pseudo-addition and pseudo-multiplication to the whole extended real line $[-\infty,+\infty]$. Moreover we introduce the operations of pseudo-subtraction and pseudo-division. Our results will serve as a basis for the further investigation of fuzzy measure theory.

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2. Pseudo-addition

We begin with the operation of pseudo-addition on $[0,+\infty]$.

DEFINITION 2.1. *Pseudo-addition* $\oplus$ on $[0,+\infty]$ is a binary operation on $[0,+\infty]$ satisfying the following axioms:

(A1) $x\oplus 0 = 0\oplus x = x$, $\forall x\in[0,+\infty]$.

(A2) $(x\oplus y)\oplus z = x\oplus(y\oplus z)$, $\forall x,y,z\in[0,+\infty]$.

(A3) If $x\le x'$ and $y\le y'$ then $x\oplus y\le x'\oplus y'$ for every $x,y,x',y'\in[0,+\infty]$.

(A4) If $x_n\to x$ and $y_n\to y$ then $x_n\oplus y_n\to x\oplus y$.

(A5) If $x>0$ and $y\in[0,+\infty)$ then there exists $n\in\mathbb{N}$ such that $\underbrace{x\oplus x\oplus\cdots\oplus x}_{n\text{-times}}\ge y$.

(A6) If $x<+\infty$ and $y<+\infty$ then $x\oplus y<+\infty$.

The axioms (A1)–(A4) are identical to axioms (P1)–(P4) in [5]. The fifth axiom (A5) guarantees that pseudo-addition has the Archimedean property and the last of axioms (A6) is the axiom of finiteness.

Using Sugeno and Murofushi [5] we obtain the following result.

THEOREM 2.2. *A binary operation $\oplus$ on $[0,+\infty]$ fulfills axioms (A1)–(A4) iff it has a representation $\{\langle(\alpha_k,\beta_k),g_k\rangle;\ k\in K\}$.*

DEFINITION 2.3. Let $\{(\alpha_k,\beta_k);\ k\in K\}$ be a family of disjoint open intervals in $[0,+\infty]$ indexed by a countable set K and let for each $k\in K$ exist a continuous and strictly increasing function

$$g_k\colon[\alpha_k,\beta_k]\to[0,+\infty],\quad g_k(\alpha_k)=0.$$

We say that a binary operation $\oplus$ *has a representation*

$$\{\langle(\alpha_k,\beta_k),g_k\rangle; k\in K\}\quad\text{iff}$$

$$x\oplus y=\begin{cases} g_k^*\big(g_k(x)+g_k(y)\big), & x,y\in[\alpha_k,\beta_k],\ \text{for some } k\in K,\\ \max\{x,y\}, & \text{otherwise,}\end{cases}$$

where g_k^* is the pseudo-inverse of g_k defined by

$$g_k^*(x)=\sup\{c\in[\alpha_k,\beta_k],\ g_k(c)<x\}=g_k^{-1}\big(\min\{x,g_k(\beta_k)\}\big).$$

THEOREM 2.4. *A binary operation* $\oplus$ *on* $[0,+\infty]$ *is a pseudo-addition iff there exists a continuous and strictly increasing function*

$$g\colon [0,+\infty] \to [0,+\infty], \quad g(0)=0, \quad g(+\infty)=+\infty,$$

such that

$$x \oplus y = g^{-1}\big(g(x)+g(y)\big) \quad \textit{for every} \quad x,y \in [0,+\infty]. \tag{1}$$

P r o o f . Let $\oplus$ be a pseudo-addition, i.e. it satisfies the axioms (A1)–(A6). By Theorem 2.2 it has a representation $\{\langle(\alpha_k,\beta_k),g_k\rangle;\ k\in K\}$. When $K=\emptyset$ then

$$x \oplus y = \max\{x,y\} \quad \text{for every } x,y \in [0,+\infty],$$

and axiom (A5) cannot be fulfilled. Thus $K \neq \emptyset$. The fifth axiom (A5) together with the continuity (A4) are equivalent to the statement $x \oplus x > x$ for every $x \in (0,+\infty)$. Since all points α_k, β_k, $k \in K$ from representation of binary operation $\oplus$ have the property $y \oplus y = y$ (it's easy to see from Definition 2.3) so the only way to choose them is $\alpha_1 = 0$ and $\beta_1 = +\infty$. Hence we obtain this representation in the form $\{\langle(0,+\infty),g\rangle\}$. Then it holds:

$$x \oplus y = g^*\big(g(x)+g(y)\big) \quad \text{for every } x,y \in [0,+\infty],$$

where $g^*(x) = g^{-1}(\min\{g(+\infty),x\})$.

If $g(+\infty) = M$, M is a finite number then it follows from the continuity of function g that there exits a number $u < +\infty$ such that $g(u) > \frac{M}{2}$. Hence we have

$$u \oplus u = g^*\big(g(u)+g(u)\big) = +\infty$$

since $g(u)+g(u) > M = g(+\infty)$.
This is in contradiction with the finiteness axiom (A6). Therefore $g(+\infty) = +\infty$ and the pseudo-inverse function g^* is identical with the inverse function g^{-1}. Thus we have proved that

$$x \oplus y = g^{-1}\big(g(x)+g(y)\big)$$

holds for every $x,y \in [0,+\infty]$.

It is easy to prove that a binary operation $\oplus$ defined by (1) satisfies axioms (A1)–(A6). □

R e m a r k 2.5. The function g generating pseudo-addition $\oplus$ is not determined uniquely. For arbitrary positive constant $c \in (0,+\infty)$ the function $c \cdot g$ generates the same pseudo-addition $\oplus$. Adding the condition $g(1) = 1$ we have unique relation between pseudo-addition $\oplus$ and its generator g.

EXAMPLE 2.6. Let $g_r(x) = x^r$, $x \in [0,+\infty]$, $r > 0$.

Then this function g_r generates pseudo-addition $\underset{r}{\oplus}$ where

$$x \underset{r}{\oplus} y = \sqrt[r]{(x^r + y^r)}, \quad x, y \in [0,+\infty].$$

3. Pseudo-multiplication

Now we define a pseudo-multiplication $\odot$ corresponding to the given pseudo-addition $\oplus$.

DEFINITION 3.1. A binary operation $\odot$ on interval $[0,+\infty]$ is said to be a *pseudo-multiplication corresponding to the pseudo-addition* $\oplus$ iff it satisfies the following axioms:

(M1) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$, $\forall a, x, y \in [0,+\infty]$.

(M2) If $a \le b$ then $a \odot x \le b \odot x$ for every $x \in [0,+\infty]$.

(M3) $a \odot x = 0$ iff $a = 0$ or $x = 0$.

(M4) There exists a left unit, i.e. an element $e \in [0,+\infty]$ so that $e \odot x = x$ for every $x \in [0,+\infty]$.

(M5) If $a_n \to a \in (0,+\infty)$ and $x_n \to x$ then $a_n \odot x_n \to a \odot x$, and $(+\infty) \odot x = \lim_{a \to +\infty} (a \odot x)$.

This system of axioms was formulated by Sugeno and Murofushi [5]. Using Theorem 2.4 and Theorem 5.1 from [5] we obtain:

THEOREM 3.2. *A binary operation $\odot$ is a pseudo-multiplication corresponding to the pseudo-addition $\oplus$ generated by generator g iff there exists a continuous and strictly increasing function*

$$h\colon [0,+\infty] \to [0,+\infty], \quad h(0) = 0, \quad h(+\infty) = +\infty$$

such that for every $a, x \in [0,+\infty]$ it holds:

$$a \odot x = g^{-1}\big(h(a) \cdot g(x)\big).$$

A pseudo-multiplication $\odot$ corresponding to a pseudo-addition $\oplus$ need not be commutative.

THEOREM 3.3. *A pseudo-multiplication $\odot$ corresponding to the pseudo-addition $\oplus$ generated by a generator g is commutative, i.e. satisfies the axiom*

(M6) $a \odot x = x \odot a\,, \ \forall a, x \in [0, +\infty]$

iff the function h generating the operation $\odot$ has the form $h = c \cdot g$, where $c \in (0, +\infty)$ is a constant.

P r o o f. i) Let $h = c \cdot g$. Then

$$a \odot x = g^{-1}\big(c \cdot g(a) \cdot g(x)\big) \qquad \text{and} \qquad x \odot a = g^{-1}\big(c \cdot g(x) \cdot g(a)\big).$$

It is evident that $x \odot a = a \odot x$.

ii) Let pseudo-multiplication $\odot$ be commutative, i.e. $x \odot a = a \odot x$. Then $g^{-1}\big(h(a) \cdot g(x)\big) = g^{-1}\big(h(x) \cdot g(a)\big)$, hence $h(a) \cdot g(x) = h(x) \cdot g(a)$ for every $a, x \in [0, +\infty]$. For a $= 1$ we obtain

$$h(x) = \frac{h(1) \cdot g(x)}{g(1)} = h(1) \cdot g(x) = c \cdot g(x)\,, \quad \text{where} \quad c = h(1) \in (0, +\infty)\,.$$

□

Moreover if we require that the unit of a pseudo-multiplication is just number 1, we obtain the following corollary.

COROLLARY 3.4. *A pseudo-multiplication $\odot$ corresponding to the pseudo-addition $\oplus$ generated by the generator g is commutative and has the unit $e = 1$ iff it holds:*

$$a \odot x = g^{-1}\big(g(a) \cdot g(x)\big) \quad \textit{for every} \quad a, x \in [0, +\infty]\,. \tag{2}$$

DEFINITION 3.5. A pseudo-multiplication $\odot$ corresponding to the pseudo-addition $\oplus$ given by the formula (2), where g generates the operation $\oplus$, is said to be *consistent with the pseudo-addition* $\oplus$.

EXAMPLE 3.6. Let $\oplus = \underset{r}{\oplus}$, i.e. the pseudo-addition $\oplus$ is generated by the function $g_r(x) = x^r$, $r > 0$. Then the pseudo-multiplication $\underset{r}{\odot}$ consistent with the pseudo-addition $\underset{r}{\oplus}$ is the nothing but the common multiplication, because of

$$a \underset{r}{\odot} x = \sqrt[r]{a^r \cdot x^r} = a \cdot x\,.$$

In the following theorem we show that the distributivity from the left is equivalent to the commutativity of the operation $\odot$.

THEOREM 3.7. *Let a pseudo-multiplication $\odot$ corresponding to the pseudo-addition $\oplus$ generated by the generator g be distributive from the left, i.e. for every $a,b,x \in [0,+\infty]$*

$$(a \oplus b) \odot x = (a \odot x) \oplus (b \odot x)\,.$$

Then the operation $\odot$ is a commutative pseudo-multiplication.

Proof. By Theorem 3.2 we obtain

$$(a \oplus b) \odot x = g^{-1}\big(h(a \oplus b) \cdot g(x)\big)$$

and

$$(a \odot x) \oplus (b \odot x) = g^{-1}\big(h(a) \cdot g(x) + h(b) \cdot g(x)\big)\,.$$

Using the distributivity from the left it means that

$$h(a \oplus b) = h(a) + h(b) \qquad \text{and} \qquad a \oplus b = h^{-1}\big(h(a) + h(b)\big)\,.$$

Then the function h generates the same operation $\oplus$ and by Remark 2.5 it has the form $h = c \cdot g$, where $c \in (0,+\infty)$ is a constant. Theorem 3.3 implies that the pseudo-multiplication $\odot$ is commutative. □

Note that the axiom (M1), i.e. the distributivity from the right, together with the commutativity imply the distributivity from the left.

Remark 3.8. A pseudo-multiplication $\odot$ consistent with a pseudo-addition $\oplus$ is associative, i.e.

$$(a \odot b) \odot c = a \odot (b \odot c) \qquad \text{for every } a,b,c \in [0,+\infty]\,.$$

THEOREM 3.9. *A pseudo-multiplication $\odot$ corresponding to the pseudo-addition $\oplus$ (where $\oplus$ is generated by g) satisfies the axioms:*

(M7) $(a+b) \odot x = (a \odot x) \oplus (b \odot x)$, $\forall a,b,x \in [0,+\infty]$,

(M8) $(a \cdot b) \odot x = a \odot (b \odot x)$, $\forall a,b,x \in [0,+\infty]$

iff $h(a) = a$ and hence $a \odot x = g^{-1}(a \cdot g(x))$.

Proof. Applying the formula (1) and Theorem 3.2 to the axiom (M7) we obtain the equality $h(a+b) = h(a) + h(b)$. The only continuous non-negative solution is $h(a) = c \cdot a$, where $c \in (0,+\infty)$ is a constant. By the same manner we get the equality $h(a \cdot b) = h(a) \cdot h(b)$ from the axiom (M8). Then it is $h(a) = a^d$, where d is a positive constant. Thus $h(a) = a$ can be the only solution of this problem. The opposite implication is trivial. $\square$

R e m a r k 3.10. Pseudo-multiplications satisfying the axioms of Theorem 3.9 have the following property: $x \oplus x = 2 \odot x$. Weber's theory of integration based on the pseudo-additive measures uses just these pseudo-multiplications. If we also require the commutativity then the only possibility is $h = g = i$, what is the case of the common addition and multiplication.

EXAMPLE 3.11. Let the pseudo-addition $\oplus$ be from Example 2.6 for $r = 2$, i.e.

$$a \oplus b = \sqrt{(a^2 + b^2)}, \quad a, b \in [0, +\infty].$$

Then the pseudo-multiplication $\odot$ satisfying the conditions from Theorem 3.9 is given by the relation

$$a \odot x = \sqrt{a} \cdot x, \quad a, x \in [0, +\infty].$$

and the pseudo-multiplication consistent with the pseudo-addition $\oplus$ is the common multiplication.

R e m a r k 3.12. The extension of the above mentioned operations on interval $[-\infty, +\infty]$ may be realized immediately. It is sufficient to extend the function g generating the pseudo-addition $\oplus$ on the odd function putting $g(x) = -g(-x)$ for every $x \in [-\infty, 0)$.

4. Pseudo-subtraction and pseudo-division

Now we introduce the operations of the pseudo-subtraction $\ominus$ and pseudo-division $\oslash$.

DEFINITION 4.1. Let a function g be a generator of a pseudo-addition $\oplus$ on interval $[-\infty, +\infty]$. Binary operations $\ominus$ and $\oslash$ on $[-\infty, +\infty]$ defined by the formulas

$$x \ominus y = g^{-1}\big(g(x) - g(y)\big) \quad \text{and} \tag{3}$$

$$x \oslash y = g^{-1}\left(\frac{g(x)}{g(y)}\right) \tag{4}$$

if expressions $g(x) - g(y)$ and $\frac{g(x)}{g(y)}$ have sense are said to be the pseudo-subtraction and pseudo-division consistent with the pseudo-addition $\oplus$.

DEFINITION 4.2. Let $g\colon [-\infty,+\infty] \to [-\infty,+\infty]$ be a continuous, strictly increasing and odd function such that $g(0) = 0$, $g(1) = 1$, $g(+\infty) = +\infty$. The system of pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ generated by this function is said to be the consistent system.

EXAMPLE 4.3. Let $g\colon g(x) = x^r$ where r is a positive and odd number. Then the system of pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ such that

$$a \oplus b = \sqrt[r]{a^r + b^r}\,, \quad a \ominus b = \sqrt[r]{a^r - b^r}\,,$$

$a \odot b$ and $a \oslash b$ are the common multiplication and division creates the consistent system.

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