Tatra Mountains Math. Publ., 2 (1993), 167–174

TATRA MOUNTAINS
Mathematical Publications

THE FUNDAMENTAL THEOREM OF CALCULUS IN AN ABSTRACT SETTING

ÁRPÁD SZÁZ

Dedicated to the memory of Tibor Neubrunn

ABSTRACT. We show that a strong version of the fundamental theorem of calculus for the generalized Riemann integral of Kurzweil and Henstock can be naturally extended to a more general net integral of Gy. Szabó and the present author.

Introduction

The fundamental theorem of calculus for the ordinary Riemann integral in its standard form [11, p. 115] states that if f and F are functions on an interval $[r,s]$ such that $F'(t)=f(t)$ for all $t\in[r,s]$ and f is integrable on $[r,s]$, then $\int_r^s f(t)\,dt = F(s)-F(r)$.

And it is well known that, in spite of the fact that the differentiability condition can be weakened in various ways, the integrability condition cannot be omitted even if the Riemann integral is replaced by the much more general Lebesgue one [12, p. 175].

Therefore, a considerable amount of effort has been made to define an even more general integral in order that every derivative may become integrable. The most striking achievement of these efforts is certainly the generalized Riemann integral [8] of Kurzweil and Henstock.

The fundamental theorem of calculus for the generalized Riemann integral in its generalized form [8, p. 27] states that if f and F are functions on $[r,s]$ such that there exists a countable subset W of $[r,s]$ for which F is continuous at each $t\in W$ and $F'(t)=f(t)$ for all $t\in[r,s]\setminus W$, then $\int_r^s f(t)\,dt = F(s)-F(r)$.

AMS Subject Classification (1991): 28A15, 28B05.
Key words: generalized derivatives and integrals.

Because of the existence of non-constant singular functions [12, p. 179], it is clear that the countable set W cannot be replaced here by an arbitrary set of measure zero. However, since the generalized Riemann integral is not confined to the real line, it is evident that the above theorem can be extended to more general settings.

Therefore, the main purpose of this note is to establish an abstract form of the fundamental theorem of McLeod [8, p. 27]. To this end, we shall first lay out some strange, but reasonable extensions of the generalized Riemann integral and the usual derivative. Here, we mainly follow the ideas of our former papers [14] and [15].

1. A generalized integral

Let Ω be a topological space, $\mathcal{R}$ be the collection of all relations $R \subset \Omega^2$ such that $t \in R(t)^\circ$ for all $t \in \Omega$, and $\mathcal{S}$ be a nonvoid collection of subsets of Ω, which will, to some extent, be specified later.

Denote by Γ the collection of all triples (σ, τ, R) such that

$$\sigma = (\sigma_i)_{i\in I} \qquad \text{and} \qquad \tau = (\tau_i)_{i\in I}$$

are finite families in $\mathcal{S}$ and Ω, respectively, and $R \in \mathcal{R}$, such that σ is disjoint and

$$\tau_i \in \overline{\sigma}_i \qquad \text{and} \qquad \sigma_i \subset R(\tau_i)$$

for all $i \in I$.

Moreover, for any

$$(\sigma, \tau, R) \in \Gamma \qquad \text{and} \qquad (\rho, \nu, S) \in \Gamma,$$

define

$$(\sigma, \tau, R) \leq (\rho, \nu, S)$$

if each member σ is a union of some members of ρ and $S \subset R$. Then, Γ is, in general, only a nonvoid preordered set.

Therefore, we assume that $\mathcal{S}$ has the following two basic properties:

(1) for each $A \in \mathcal{S}$ and $B \in \mathcal{S}$, there exists a finite disjoint family σ in $\mathcal{S}$ such that $A \setminus B = \cup\sigma$;

(2) for each $A \in \mathcal{S}$ and $R \in \mathcal{R}$, there exist finite families σ and τ in $\mathcal{S}$ and Ω, respectively, such that $(\sigma, \tau, R) \in \Gamma$ and $\cup\sigma = A$.

Namely, it is not very hard to see that Γ is directed if and only if $\mathcal{S}$ has the above two properties. Moreover, that the collection of all bounded right-open intervals in $\mathbb{R}$, which is our prime example for $\mathcal{S}$, has the above properties [14].

Suppose that X, Y and Z are normed spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ which are equipped with a bilinear map

$$(x, y) \longmapsto xy$$

from $X \times Y$ into Z such that

$$|xy| \leq |x||y|$$

for all $x \in X$ and $y \in Y$.

Let f and μ be functions from Ω and $\mathcal{S}$ into X and Y, respectively. And for each

$$\alpha = (\sigma, \tau, R) \in \Gamma,$$

with $\sigma = (\sigma_i)_{i \in I}$ and $\tau = (\tau_i)_{i \in I}$, define

$$S_\alpha(f, \mu) = \sum_{i \in I} f(\tau_i)\mu(\sigma_i).$$

Then $\left(S_\alpha(f, \mu)\right)_{\alpha \in \Gamma}$ is a net in Z, and we may introduce an integral of f on Ω with respect to μ by

$$\int_\Omega f d\mu = \lim_\alpha S_\alpha(f, \mu),$$

whenever this limit exists.

Moreover, for $A \in \mathcal{S}$, we can also define

$$\int_A f d\mu = \int_\Omega f d\mu_A,$$

where μ_A is the function from $\mathcal{S}$ into Y such that for each $B \in \mathcal{S}$

$$\mu_A(B) = \mu(B) \quad \text{if} \quad B \subset A \qquad \text{and} \qquad \mu_A(B) = 0 \quad \text{if} \quad B \not\subset A.$$

Now, a function ν from $\mathcal{S}$ into Z may be called the indefinite integral of f with respect to μ if

$$\nu(A) = \int_A f d\mu$$

for all $A \in \mathcal{S}$.

And we may naturally look for some sufficient conditions on f, μ and ν in order that ν should be the indefinite integral of f with respect to μ.

2. A generalized derivative

DEFINITION 2.1. Let μ be a function from $\mathcal{S}$ into Y and $k > 0$. For each finite disjoint family $\sigma = (\sigma_i)_{i\in I}$ in $\mathcal{S}$, define

$$v_\sigma^k(\mu) = \sum_{i\in I} |\mu(\sigma_i)|^k .$$

Moreover, let

$$V_\Omega^k(\mu) = \sup_\sigma v_\sigma^k(\mu) .$$

Then $V_\Omega^k(\mu)$ will be called the *total k-variation of μ on Ω*. Moreover, if $V_\Omega^k(\mu) < +\infty$, then μ will be said to be of *bounded k-variation on Ω*.

Remark 2.2. Now, we can also define

$$V_A^k(\mu) = V_\Omega^k(\mu_A)$$

for any $A \in \mathcal{S}$.

Then, it is not hard to see that

$$\sum_{i\in I} V_{\sigma_i}^k(\mu) \le V_A^k(\mu)$$

for any finite disjoint family $\sigma = (\sigma_i)_{i\in I}$ in $\mathcal{S}$ with $\cup\sigma \subset A$.

Moreover, for the convenience of our subsequent treatment, it seems also desirable to introduce the following additional terminology.

DEFINITION 2.3. If μ is a function from $\mathcal{S}$ into Y and $k > 0$, then we say that:

(1) μ is of *partially bounded k-variation on Ω* if $V_A^k(\mu) < +\infty$ for all $A \in \mathcal{S}$;

(2) μ is of *locally bounded k-variation on Ω* if for each $t \in \Omega$ there exists an $R \in \mathcal{R}$ such that $V_A^k(\mu) < +\infty$ for all $A \in \mathcal{S}$ with $t \in \bar{A}$ and $A \subset R(t)$.

Remark 2.4. It is not very hard to see that if in particular μ is finitely additive in the usual sense that

$$\mu(\cup\sigma) = \sum_{i\in I} \mu(\sigma_i)$$

for any finite disjoint family $\sigma = (\sigma_i)_{i\in I}$ in $\mathcal{S}$ with $\cup\sigma \in \mathcal{S}$, then μ is of partially bounded 1-variation if and only if μ is of locally bounded 1-variation.

Moreover that if in addition $\mathcal{S}$ is closed under finite intersections, then a similar assertion holds for $k > 1$.

For a concise formulation of our subsequent definition of variational differentiability, it seems also convenient to write $V_A^0(\mu) = 1$ for any $A \in \mathcal{S}$.

DEFINITION 2.5. Let $k \geq 0$, and μ and ν be functions from $\mathcal{S}$ into Y and Z, respectively, such that μ is of locally bounded k-variation on Ω if $k > 0$, and $\tau \in \Omega$.

Denote by $D_\mu^k \nu(\tau)$ the set of all vectors $u \in X$ such that for each $\varepsilon > 0$ there exists an $R \in \mathcal{R}$ such that

$$|\nu(A) - u\mu(A)| \leq \varepsilon V_A^k(\mu)$$

for all $A \in \mathcal{S}$ with $\tau \in \bar{A}$ and $A \subset R(\tau)$.

Then $D_\mu^k \nu(\tau)$ will be called the *k-variational derivative of ν at τ with respect to μ*. Moreover, if $D_\mu^k \nu(\tau) \neq \emptyset$, then ν will be said to be *k-variationally differentiable at τ with respect to μ*.

Remark 2.6. Note that if in particular Ω is an interval in $\mathbb{R}$ and $\mathcal{S}$ is the collection of all bounded right-open subintervals of Ω, $(X, Y, Z) = (X, \mathbb{K}, X)$ and $xy = yx$ for all $x \in X$ and $y \in Y$, moreover F is a function from $\overline{\Omega}$ into Z and

$$\mu([r, s[) = s - r \qquad \text{and} \qquad \nu([r, s[) = F(s) - F(r)$$

for all $[r, s[\in \mathcal{S}$, then for a vector $u \in X$ we have

(1) $u \in D_\mu^0 \nu(\tau)$ if and only if F is continuous at τ;

(2) $u \in D_\mu^1 \nu(\tau)$ if and only if F is differentiable at τ and $F'(\tau) = u$.

3. A fundamental theorem

THEOREM 3.1. *Suppose that $f \colon \Omega \to X$, and*

$$\mu \colon \mathcal{S} \to Y \qquad \textit{and} \qquad \nu \colon \mathcal{S} \to Z$$

such that μ is of partially bounded k-variation for some $k > 0$ and ν is finitely additive, and moreover W is a countable subset of Ω such that

(1) $f(t) \in D_\mu^0 \nu(t)$ for all $t \in W$;

(2) $f(t) \in D_\mu^k \nu(t)$ for all $t \in \Omega \setminus W$.

Then ν is the indefinite integral of f with respect to μ in the sense that

$$\nu(A) = \int_A f d\mu$$

for all $A \in \mathcal{S}$.

P r o o f. Let $A_0 \in \mathcal{S}$ and $\varepsilon > 0$. Then, because of the countability of W, for each $t \in W$ there exists an $\varepsilon_t > 0$ such that

$$\sum_{t\in W} \varepsilon_t < \varepsilon\, 2^{-1}.$$

Moreover, by the assumption (1), for each $t \in W$, there exists an $R_t \in \mathcal{R}$ such that

$$|\nu(A) - f(t)\mu(A)| \leq \varepsilon_t$$

for all $A \in \mathcal{S}$ with $t \in \bar{A}$ and $A \subset R_t(t)$.

On the other hand, since μ is of partially bounded k-variation,

$$V = V^k_{A_0}(\mu) < +\infty\,.$$

Moreover, by the assumption (2), for each $t \in \Omega \setminus W$, there exists an $R_t \in \mathcal{R}$ such that

$$|\mu(A) - f(t)\mu(A)| \leq \varepsilon\big(2(V+1)\big)^{-1} V^k_A(\mu)$$

for all $A \in \mathcal{S}$ with $t \in \bar{A}$ and $A \subset R_t(t)$.

Define $R_0 \subset \Omega^2$ such that

$$R_0(t) = R_t(t)$$

for all $t \in \Omega$. Then, by the definition of $\mathcal{R}$, it is clear that $R_0 \in \mathcal{R}$. Moreover, because of the property (2) of $\mathcal{S}$, there exist finite families σ_0 and τ_0 in $\mathcal{S}$ and Ω, respectively, such that

$$(\sigma_0, \tau_0, R_0) \in \Gamma \qquad \text{and} \qquad \cup\, \sigma_0 = A_0\,.$$

Assume now that

$$\alpha = (\sigma, \tau, R) \geq \alpha_0 = (\sigma_0, \tau_0, R_0)$$

and

$$\sigma = (\sigma_i)_{i\in I} \qquad \text{and} \qquad \tau = (\tau_i)_{i\in I}\,.$$

Moreover, define

$$I_0 = \big\{ i \in I : \sigma_i \subset A_0 \big\},$$

and

$$I_1 = \big\{ i \in I_0 : \tau_i \in W \big\} \qquad \text{and} \qquad I_2 = \big\{ i \in I_0 : \tau_i \in \Omega \setminus W \big\}.$$

Then, because of our former assumptions and definitions, it is clear that

$$\begin{aligned}|\nu(A_0) - S_\alpha(f, \mu_{A_0})| &= |\sum_{i\in I_0} \nu(\sigma_i) - \sum_{i\in I_0} f(\tau_i)\mu(\sigma_i)| \leq \\ &\leq \sum_{i\in I_1} |\nu(\sigma_i) - f(\tau_i)\mu(\sigma_i)| + \sum_{i\in I_2} |\nu(\sigma_i) - f(\tau_i)\mu(\sigma_i)| \leq \\ &\leq \sum_{i\in I_1} \varepsilon_{\tau_i} + \varepsilon\big(2(V+1)\big)^{-1} \sum_{i\in I_2} V^k_{\sigma_i}(\mu) < \varepsilon .\end{aligned}$$

Consequently, we have

$$\nu(A_0) = \lim_\alpha S_\alpha(f, \mu_{A_0}) ,$$

and this proves

$$\nu(A_0) = \int_\Omega f d\mu_{A_0} = \int_{A_0} f d\mu .$$

□

Remark 3.2. By Remark 2.6, it is clear that this theorem includes the fundamental theorem of McLeod [8, p. 27] as a particular case.

On the other hand, it is also worth noticing that this theorem is, to some extent, also related to the Radon–Nikodym theorem [12, p. 130] too.

Namely, if ν is the indefinite integral of f with respect to μ, then f may be called a Radon–Nikodym derivative of ν with respect to μ.

Acknowledgement. The author is deeply indebted to Professor Beloslav Riečan for pointing out a mistake in the earlier form of Remark 2.4 and for his other valuable comments.

Moreover, the author is also indebted to several other colleagues in Bratislava who showing interest in his former and present works provided encouragement to survive the numerous rejections and ignorances.

REFERENCES

[1] AHMED, S. I.—PFEFFER, W. F.: *A Riemann integral in a locally compact Hausdorff space*, J. Austral. Math. Soc. **41** (1986), 115–137.

[2] DAVIES, R. O.—SCHUSS, Z.: *A proof that Henstock's integral includes Lebesgue's*, J. London Math. Soc. **2** (1970), 561–562.

[3] GROSS, G.—SHISHA, O.: *A new approach to integration*, J. Math. Anal. Appl. **114** (1986), 289–294.

[4] GULYÁS, GY.: *A particular net integral on the real line*, A competition work written in Hungarian under the supervision of the author at Lajos Kossuth University, Debrecen, 1986.

[5] HENSTOCK, R.: *A Riemann-type integral of Lebesgue power*, Canad. J. Math. **20** (1968), 79–87.

[6] HILDEBRANDT, T. H.: *Theory of Integration*, Academic Press, New York, 1963.

[7] KURZWEIL, J.: *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J. **7** (1957), 418–449.

[8] McLEOD, R. M.: *The Generalized Riemann integral*, Carus Math. Monographs 20, Math. Assoc. Amer., 1980.

[9] McSHANE, E. J.: *A unified theory of integration*, Amer. Math. Monthly **30** (1973), 349–359.

[10] RIEČAN, B.: *On the Kurzweil integral in compact topological spaces*, Rad. Mat. **2** (1986), 151–163.

[11] RUDIN, W.: *Principles of Mathematical Analysis*, McGraw–Hill, New York, 1964.

[12] RUDIN, W.: *Real and Complex Analysis*, McGraw–Hill, New York, 1974.

[13] SZABÓ, GY.—SZÁZ, Á.: *The net integral and a convergence theorem*, Math. Nachr. **135** (1988), 53–65.

[14] SZABÓ, GY.—SZÁZ, Á.: *Defining nets for integration*, Publ. Math. Debrecen **36** (1989), 237–252.

[15] SZÁZ, Á.: *Basic tools and mild continuities in relator spaces*, Acta Math. Hung. **50** (1987), 177–201.

Received September 3, 1992

Department of Mathematics
Lajos Kossuth University
H-4010 Debrecen
HUNGARY