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ON SYSTEMS OF SEQUENCES OF REALS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. The paper is concerned with the squeezing property for systems of sequences of reals which was studied by D. E. Peek.

This paper deals with some conditions for systems of sequences of reals; these conditions have been studied in [2].

The preliminaries are as follows. Let $\mathbb{R}$ be the set of all reals and let $\mathcal{S}$ be the set of all sequences of reals. The elements of $\mathcal{S}$ will be denoted by $X, Y, Z, \dots$, or by $(x_n), (y_n), (z_n), \dots$. If $k \in \mathbb{R}$ and if X and Y are elements of $\mathcal{S}$, then the symbols kX and $X+Y$ have obvious meanings.

Assume that $\equiv$ is a binary relation on $\mathcal{S}$ and let A be a nonempty subset of $\mathcal{S}$. Consider the following conditions for the pair $(A, \equiv)$:

A1. If $X \in A$ and $k \in \mathbb{R}$, then $kX \in A$.

A2. If $X, Y, Z, W \in A$ and $X+Z, Y+W \in A$, $X \equiv Y$, $Z \equiv W$, then $Y+W \equiv X+Z$.

A3. If $X, Y \in A$, $Z \in \mathcal{S}$, $X \equiv Y$ and if for each n the relation $x_n \leqq z_n \leqq y_n$ is valid, then $Z \in A$ and $Z \equiv X$.

A4. If $X \in A$ and Y, Z are subsequences of A, then $Y, Z \in A$ and $Y \equiv Z$.

A5. If $x_n = (-1)^n$ for each n, then $(x_n) \notin A$.

A6. If $X \notin A$ and X is bounded, then X has two subsequences Y and Z such that $Y, Z \in A$ and $Y \not\equiv Z$.

The property expressed in A3 is called the *squeezing property* in [1].

Let $a \in \mathbb{R}$ and $(x_n) \in \mathcal{S}$. If (x_n) converges to a (in the usual sense), then we write $\lim x_n = a$; the sequence (x_n) is said to be convergent. Let $\mathcal{C}$ be the set of all convergent sequences of reals.

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In [2] the following results have been proved:

(α) If the pair $(A, \equiv)$ satisfies the conditions A1 – A6, then $A = \mathcal{C}$ and for $X, Y \in \mathcal{S}$ the relation $X \equiv Y$ means that both X and Y are convergent and have the same limit.

(β) If $i \in \{1, 2, 4, 5, 6\}$, then the condition Ai is independent from the system of remaining conditions.

It is remarked in [1] that the question of the independence of the condition A3 from the other five conditions remains unsolved.

For each $a \in \mathbb{R}$ with $a > 1$ we shall construct a system $A(a)$ of sequences of reals and a binary relation $\equiv_a$ on $\mathcal{S}$ such that $(A(a), \equiv_a)$ satisfies all the above considered properties except A3. Hence A3 is independent from the remaining properties. Moreover, it will be shown that $(A(a), \equiv_a)$ fulfils a certain minimality condition.

If $(x_n) \in \mathcal{S}$, $x \in \mathbb{R}$ and $x_n = x$ for each n, then we write $(x_n) = \text{const } x$. In view of (α), if $(A, \equiv)$ satisfies the conditions A1 – A6, then also the following two conditions are valid:

A7. If $X \in A$ and $X \equiv \text{const } 0$, then $mX \equiv \text{const } 0$ for each $m \in \mathbb{R}$.

A8. The relation $\equiv$ is transitive.

We denote by $\mathcal{A}$ the collection of all pairs $(A_i, \equiv_i)$ such that $A_i \subseteq \mathcal{S}$, $\equiv_i$ is a binary relation on $\mathcal{S}$, $(A_i, \equiv_i)$ satisfies the conditions A1, A2, A4, A5, A6 and it does not satisfy the condition A3.

Next, let $\mathcal{A}_1$ be the set of all elements $(A_i, \equiv_i)$ of $\mathcal{A}$ which satisfy the conditions A7, A8 and the condition $\mathcal{C} \subseteq A_i$.

For $(A_1, \equiv_1)$ and $(A_2, \equiv_2)$ in $\mathcal{A}_1$ we put $(A_1, \equiv_1) \leqq (A_2, \equiv_2)$ if $A_1 \subseteq A_2$ and $\equiv_1 \subseteq \equiv_2$. Then $\mathcal{A}_1$ is a partially ordered set.

Let $(x_n) \in \mathcal{S}$ and $(A_0, \equiv_0) \in \mathcal{A}_1$. Let $\mathcal{A}_1[x_n]$ be the set of all elements $(A_i, \equiv_i)$ of $\mathcal{A}_1$ such that $(x_n) \in A_i$ and $(x_n) \equiv_0 \text{const } 0$. If $\mathcal{A}_1[x_n]$ is nonempty and $(A_0, \equiv_0)$ is its least element, then $(A_0, \equiv_0)$ is said to be generated by (x_n).

Let $a \in \mathbb{R}$, $a > 1$. In [1] there was investigated the compatibility of some conditions concerning sequences in $\mathbb{R}$ with defining (a^n) as to have a "limit" equal to 0.

Now let us consider the sequence (a^n) in the context of the above conditions. It will be shown that there exists $(A_0, \equiv_0)$ in $\mathcal{A}_1$ which is generated by (a^n); we give a constructive description of $(A_0, \equiv_0)$.

Hence, in particular, both $\mathcal{A}$ and $\mathcal{A}_1$ are nonempty.

A sequence (x_n) with $x_n = n$ for each n will be denoted by N. Let $a \in \mathbb{R}$, $a > 1$. We define a subset $A_1(a)$ of $\mathcal{S}$ as follows. A sequence $(y_n) \in \mathcal{S}$ belongs to $A_1(a)$ iff there are $k \in \mathbb{R}$ and a subsequence (t_n) of N such that $y_n = ka^{t_n}$ for each n.

LEMMA 1. *Let* $X, Y \in A_1(a)$, $Z = (z_n) \in \mathcal{C}$, $X + Y = Z$. *Then* $\lim z_n = 0$.

Proof. Let $X = (x_n)$, $Y = (y_n)$,

$$x_n = ka^{p_n}, \quad y_n = la^{q_n} \quad \text{for each } n \in N,$$

where $k, l \in \mathbb{R}$ and $(p_n), (q_n)$ are subsequences of N.

If $k = l = 0$, then $z_n = 0$ for each $n \in N$. It is obvious that the case $k = 0$, $l \neq 0$ is impossible; similarly, the case $k \neq 0$, $l = 0$ is impossible as well. Thus it suffices to assume that $k \neq 0$ and $l \neq 0$.

Next, if k and l have the same sign, then either $x_n + y_n \longrightarrow +\infty$ or $x_n + y_n \longrightarrow -\infty$, which is a contradiction. Hence without loss of generality we can assume that $k > 0$ and $l < 0$. Put $l_0 = -l$.

a) Assume that there exists a subsequence $(n(1))$ of N such that $p_{n(1)} > q_{n(1)}$ for each $n(1)$. Then

$$z_{n(1)} = ka^{p_{n(1)}} - l_0 a^{q_{n(1)}} = \frac{k^2 a^{2p_{n(1)}} - l_0^2 a^{2q_{n(1)}}}{ka^{p_{n(1)}} + l_0 a^{q_{n(1)}}} =$$
$$= \frac{k^2 - l_0^2 a^{2q_{n(1)} - 2p_{n(1)}}}{ka^{-p_{n(1)}} + l_0 a^{q_{n(1)} - 2p_{n(1)}}} \longrightarrow \infty,$$

which is a contradiction.

b) Suppose that there exists a subsequence $(n(1))$ of N such that $p_{n(1)} = q_{n(1)}$ for each $n(1)$. If $k^2 > l_0^2$, then by the same method as in a) we obtain that $z_{n(1)} \longrightarrow \infty$, which is a contradiction. If $k^2 < l_0^2$, then we would have $z_{n(1)} \longrightarrow -\infty$, which is impossible. If $k^2 = l_0^2$, then $k = l_0$, whence $z_{n(1)} = 0$ for each $n(1)$. This implies that $\lim z_n = 0$.

c) Now suppose that neither the assumption from a) nor the assumption from b) is valid. Then there is $m \in N$ such that $p_n < q_n$ for each $n > m$. We have

$$z_n = ka^{p_n} - l_0 a^{q_n} = a^{p_n}(k - l_0 a^{q_n - p_n}).$$

At first suppose that there exists a subsequence $(n(1))$ of N such that $k - l_0 a^{q_{n(1)} - p_{n(1)}} \neq 0$ for each $n(1)$. Then there is a subsequence $(n(2))$ of $(n(1))$ such that either

(1) $k - l_0 a^{q_{n(2)} - p_{n(2)}} > 0$ for each $n(2)$,

or

(2) $k - l_0 a^{q_{n(2)} - p_{n(2)}} < 0$ for each $n(2)$.

If (1) holds, then $z_{n(2)} \longrightarrow \infty$. If (2) is valid, then $z_{n(2)} \longrightarrow -\infty$. Both these cases are impossible. Hence there is $m_1 \in N$ with $m_1 \geqq m$ such that $k - l_0 a^{q_n - p_n} = 0$ for each $n \geqq m_1$. Thus $z_n = 0$ for each $n \geqq m_1$ and therefore $\lim z_n = 0$. □

Put $A(a) = \mathcal{C} \cup A_1(a)$. Next, for $X, Y \in \mathcal{S}$ we set $X \equiv_a Y$ if some of the following conditions is satisfied:

(i) $X, Y \in \mathcal{C}$ and $\lim x_n = \lim y_n$;
(ii) $X, Y \in A_1(a)$;
(iii) $X \in A_1(a)$, $Y \in \mathcal{C}$ and $\lim y_n = 0$;
(iv) $Y \in A_1(a)$, $X \in \mathcal{C}$ and $\lim x_n = 0$.

Lemma 2. *Let $a \in \mathbb{R}$, $a > 1$. The pair $(A(a), \equiv_a)$ satisfies the conditions* A1, A2, A4, A5 *and* A6. *It does not satisfy the condition* A3.

Proof. It is obvious that the conditions A1, A4, A5 and A6 are valid. Let us deal with the condition A2.

Let $X, Y, Z, W \in A(a)$. Suppose that $X + Z, Y + W \in A(a)$ and that $X \equiv_a Y$, $Z \equiv_a W$.

a) If all X, Y, Z and W belong to $\mathcal{C}$, then we obviously have $Y + W \equiv_a X + Z$.

b) If the assumption from a) fails to be valid then without loss of generality we can suppose that $X \notin \mathcal{C}$. Then from $X \equiv_a Y$ we infer that either $Y \notin \mathcal{C}$ or $\lim y_n = 0$.

b$_1$) Suppose that $Y \notin \mathcal{C}$. We apply the condition that $Y + W \in A(a)$. If $W \in \mathcal{C}$, then $Y + W$ cannot belong to $\mathcal{C}$ (since $Y + W$ fails to be bounded). If $W \notin \mathcal{C}$, then in view of the Lemma 1, either $\lim(y_n + w_n) = 0$ or $Y + W \notin \mathcal{C}$. Similarly we can verify that either $\lim(x_n + z_n) = 0$ or $X + Z \notin \mathcal{C}$. In all these cases the relation $Y + W \equiv_a X + Z$ is valid.

b$_2$) Suppose that $Y \in \mathcal{C}$. Then from $X \equiv_a Y$ we get that $\lim y_n = 0$. Consider the relation $X + Z \in A(a)$. If $Z \in \mathcal{C}$, then $X + Z$ does not belong to $\mathcal{C}$ and thus in view of the Lemma 1 we obtain $\lim z_n = 0$. Thus for $X + Z$ we have either $X + Z \notin \mathcal{C}$ or $\lim(x_n + z_n) = 0$.

We already verified that either $\lim z_n = 0$ or $Z \notin \mathcal{C}$. Hence from $Z \equiv_a W$ we obtain that either $\lim w_n = 0$ or $W \notin \mathcal{C}$. Thus either $\lim(y_n + w_n) = 0$ or $Y + W \notin \mathcal{C}$ (cf. Lemma 1). We conclude that $Y + W \equiv_a X + Z$.

Therefore the condition A2 is satisfied.

Put $x_n = -a^n$, $y_n = a^n$ and $z_n = 1$ for each $n \in N$. Then $X, Y \in A(a)$, $X \equiv_a Y$ and $x_n \leqq z_n \leqq y_n$ for each n. We have $Z \in A(a)$, but $Z \not\equiv_a X$. Thus the condition A3 does not hold. □

PROPOSITION. *Let* $a \in \mathbb{R}$, $a > 1$ *and let* $(A(a), \equiv_a)$ *be as above. Then* $(A(a), \equiv_a)$ *is the element of* $\mathcal{A}_1$ *which is generated by the sequence* (a^n).

Proof. a) In view of Lemma 2, $(A(a), \equiv_a)$ belongs to $\mathcal{A}$. According to the definition of $A(a)$ and $\equiv_a$ the conditions A7 and A8 are satisfied and $\mathcal{C} \subseteq A(a)$. Hence $(A(a), \equiv_a)$ belongs to $\mathcal{A}_1$. Also, by the definition of $\equiv_a$, the relation $(a^n) \equiv_a$ const 0 is valid.

b) Let $(A, \equiv) \in \mathcal{A}_1[a^n]$. Then $\mathcal{C} \subseteq A$. According to A1 and A5 we have $A_1(a) \subseteq A$. Hence $A(a) \subseteq A$. From the definition of $\equiv_a$ and from A7, A8 we get that $\equiv_a \subseteq \equiv$. Clearly $(A(a), \equiv_a) \in \mathcal{A}_1[a^n]$. Therefore $(A(a), \equiv_a)$ is the least element of $\mathcal{A}_1[a^n]$. □

Another approach to the question proposed in [2] was applied by R. Frič (without considering any minimality condition); his result will be submitted for publication.

The following questions remain open:

1. What are the cardinalities of the collections $\mathcal{A}$ and $\mathcal{A}_1$?
2. Does there exist an element $(a_i, \equiv_i)$ of $\mathcal{A}$ such that $\mathcal{C} \not\subseteq A_i$?

REFERENCES

[1] J. JAKUBÍK: *On the convergence in linear spaces,* (in Slovak), Matem. fyz. časopis **6** (1956), 57–67.

[2] D. E. PEEK: *Limits without epsilons*, Real Anal. Exchange **17** (1991–92), 751–754.

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