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g, p–FUZZIFICATION OF ARITHMETIC OPERATIONS

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ABSTRACT. The aim of this paper is to provide new results regarding the effective practical computation of t-norm-based arithmetic operations on LR fuzzy numbers.

1 Introduction

The present paper is devoted to the derivation of fast computation formulae for t-norm-based addition and multiplication of LR fuzzy numbers. We denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy sets of the real line and by $\mathcal{FN}(\mathbb{R})$ the set of all fuzzy numbers of LR-type:

$$A(x) = \begin{cases} L\left(\frac{a-x}{\alpha_1}\right) & \text{if } x \in [a-\alpha_1, a], \\ R\left(\frac{x-a}{\alpha_2}\right) & \text{if } x \in [a, a+\alpha_2], \\ 0 & \text{otherwise,} \end{cases}$$

where a is the mean value of A; $L, R\colon [0,1] \to [0,1]$ are the reference (shape) functions of A with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, which are non-increasing, continuous mappings; α_1 and α_2 is the left and the right spread of A correspondingly. We use the notation $A = (a, \alpha_1, \alpha_2)_{LR}$.

Let $A = (a, \alpha_1, \alpha_2)_{LR}$ and $B = (b, \beta_1, \beta_2)_{LR}$ be (for the sake of simplicity positive) fuzzy numbers of the same shape described by reference functions L and R, i.e. $A, B \in \mathcal{FN}(\mathbb{R}^+)$. The addition- and multipliciation rules of such LR fuzzy numbers are well-known in the case of "min"–norm, see, e.g., Dubois and Prade [1]. They verified the following formulae for the sum $A \oplus B$ and for the product $A \odot B$:

$$(a, \alpha_1, \alpha_2)_{LR} \oplus (b, \beta_1, \beta_2)_{LR} = \left(a + b,\ \alpha_1 + \beta_1,\ \alpha_2 + \beta_2\right)_{LR}, \tag{1}$$

$$(a, \alpha_1, \alpha_2)_{LR} \odot (b, \beta_1, \beta_2)_{LR} \simeq \left(a \cdot b,\ a\beta_1 + b\alpha_1,\ a\beta_2 + b\alpha_2\right)_{LR}. \tag{2}$$

Key words: extension principle, t-norms, fuzzy numbers.

The first one is an exact calculation formula for addition and the second is an approximate one for the multiplication assuming that the spreads of A and B are small compared with their mean values, i.e. $\alpha_1, \alpha_2 \ll a$ and $\beta_1, \beta_2 \ll b$. So, the addition is shape preserving and the multiplication is approximately a shape preserving operation if extended with the "min"-norm.

Using the general form of Zadeh's extension principle a real operation $*: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is extended to $\mathcal{F}(\mathbb{R})$ by

$$A * B(z) = \sup_{x*y=z} T(A(x), B(y)); \qquad z \in \mathbb{R}, \tag{3}$$

where $A, B \in \mathcal{F}(\mathbb{R})$ and $T\colon [0,1] \times [0,1] \to [0,1]$ is an arbitrary t-norm. In this case arithmetic operations on fuzzy numbers (even the addition) become much more complicated. We shall derive formulae similar to (1) and (2) for t-norm based arithmetic operations defined by (3) considering a special relationship between the generator function of t-norm and the reference functions of operands.

Recall that the mapping $T : [0,1] \times [0,1] \to [0,1]$ is a t-norm iff it is commutative, associative, non-decreasing in each argument and $T(x,1) = x$, $\forall x \in [0,1]$. A t-norm T is said to be Archimedean iff T is continuous and $T(x,x) < x$, $\forall x \in]0,1[$. Every Archimedean t-norm T is representable by a continuous and decreasing function $g : [0,1] \to [0, \infty[$ with $g(1) = 0$ and

$$T(x,y) = g^{[-1]}(g(x) + g(y)), \tag{4}$$

where $g^{[-1]}$ is the pseudo-inverse of g, defined by

$$g^{[-1]}(y) = g^{-1}(\min\{y, g(0)\}). \tag{5}$$

Function g is called the additive generator of T.

2 g,p-fuzzified addition

Let $A = (a, \alpha_1, \alpha_2)_{RR}$ and $B = (b, \beta_1, \beta_2)_{RR}$ be fuzzy numbers of the same shape described by reference function R both on the left and the right-hand side. Note that A and B are not necessary symmetric as their left and right spreads can differ from each other.

2.1 g-fuzzy numbers.

Now, we propose to define $A \oplus B$ by (3) with the t-norm T_p having the additive generator g^p $(p \geq 1)$ such that $g^{-1} = R$. Obviously T_p is nilpotent, i.e., has zero divisors ($\exists\, x, y > 0\colon T_p(x,y) = 0$) because $g(0) = R^{-1}(0) = 1 < +\infty$.

We will write in the following just g instead of $RR = g^{-1}g^{-1}$ using notations like $A = (a, \alpha_1, \alpha_2)_g$ and call A a g-fuzzy number.

The following theorem holds for the g,p-fuzzified addition:

THEOREM 1. *Let $A = (a, \alpha_1, \alpha_2)_g$ and $B = (b, \beta_1, \beta_2)_g$. If the addition is extended by the t-norm T_p with additive generator g^p $(p \geq 1)$, then $A \oplus B$ is also a g-fuzzy number and*

$$(a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g = \left(a + b,\ \sqrt[q]{\alpha_1^q + \beta_1^q},\ \sqrt[q]{\alpha_2^q + \beta_2^q}\right)_g, \tag{6}$$

where $q \geq 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. A generalized form of Nguyen's formula on level sets [7] was proved in Fullér, Keresztfalvi [2]. It gives an exact expression for level sets of $A \oplus B$. Denoting by $[A]_\lambda$ the λ-level set of A we have in case of an arbitrary continuous two-place function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the following equality

$$[f(A,B)]_\lambda = \bigcup_{T(\xi,\eta) \geq \lambda} f([A]_\xi, [B]_\eta) \qquad \lambda \in]0,1] \tag{7}$$

if the fuzzy sets $A, B \in \mathcal{F}(\mathbb{R})$ have upper semicontinuous, compactly-supported membership functions and the t-norm T is also upper semicontinuous. Applying this formula to our case we obtain for all $\lambda \in]0,1]$

$$\begin{aligned}
[A \oplus B]_\lambda &= \bigcup_{T_p(\xi,\eta) \geq \lambda} [A]_\xi + [B]_\eta = \\
&= \bigcup_{T_p(\xi,\eta) \geq \lambda} [a - \alpha_1 g(\xi)\ ,\ a + \alpha_2 g(\xi)] + [b - \beta_1 g(\eta)\ ,\ b + \beta_2 g(\eta)] = \\
&= \bigcup_{(g(\xi))^p + (g(\eta))^p \leq (g(\lambda))^p} [a + b - \alpha_1 g(\xi) - \beta_1 g(\eta)\ ,\ a + b + \alpha_2 g(\xi) + \beta_2 g(\eta)] = \\
&= a + b + \bigcup_{\|(g(\xi), g(\eta))\|_p \leq g(\lambda)} [-\alpha_1 g(\xi) - \beta_1 g(\eta)\ ,\ \alpha_2 g(\xi) + \beta_2 g(\eta)],
\end{aligned} \tag{8}$$

where $\|(g(\xi), g(\eta))\|_p = \sqrt[q]{(g(\xi))^p + (g(\eta))^p}$.

The next step was firstly proposed by Kovács [5, 6]: there exists a natural bijection between the normed linear space $\mathbb{R}^2$ and the normed linear space of linear functionals on $\mathbb{R}^2$, i.e. $Lin(\mathbb{R}^2, \mathbb{R})$. Moreover, this bijection is an isometry if we normalize $Lin(\mathbb{R}^2, \mathbb{R})$ with the $\|\cdot\|_p$ norm and $\mathbb{R}^2$ with the $\|\cdot\|_q$ norm where $\frac{1}{p} + \frac{1}{q} = 1$. Due to this fact we have for the lower bound

$$\begin{aligned}
\sup_{\|(g(\xi), g(\eta))\|_p \leq g(\lambda)} (\alpha_1 g(\xi) + \beta_1 g(\eta)) &= g(\lambda) \cdot \sup_{\|(h_1, h_2)\|_p \leq 1} (\alpha_1 h_1 + \beta_1 h_2) = \\
&= g(\lambda) \cdot \|(\alpha_1, \beta_1)\|_q = \\
&= g(\lambda) \cdot \sqrt[q]{\alpha_1^q + \beta_1^q}.
\end{aligned} \tag{9}$$

where $(h_1, h_2) = \left(\frac{g(\xi)}{g(\lambda)}, \frac{g(\eta)}{g(\lambda)}\right)$ and in a similar way for the upper bound

$$\sup_{\|(g(\xi),g(\eta))\|_p \le g(\lambda)} (\alpha_2 g(\xi) + \beta_2 g(\eta)) = g(\lambda) \cdot \sqrt[q]{\alpha_2^q + \beta_2^q}. \tag{10}$$

from which we obtain

$$[A \oplus B]_\lambda = a + b + g(\lambda) \cdot \left[-\|(\alpha_1, \beta_1)\|_q, \|(\alpha_2, \beta_2)\|_q \right] \tag{11}$$

This is a λ-level set of a g-fuzzy number, that is

$$\begin{aligned}(a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g &= \left(a + b, \|(\alpha_1, \beta_1)\|_q, \|(\alpha_2, \beta_2)\|_q\right)_g = \\ &= \left(a + b, \sqrt[q]{\alpha_1^q + \beta_1^q}, \sqrt[q]{\alpha_2^q + \beta_2^q}\right)_g \end{aligned} \tag{12}$$

which was to be proved. □

2.2 Extreme cases.

Both extreme cases are worth mentioning:

- It is easy to see that $\lim_{p \to \infty} T_p(x, y) = \min\{x, y\}$. Thus, if $q = 1$ ($p = \infty$), then we arrive at the min-based addition and

$$(a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g = \left(a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2\right)_g \tag{13}$$

  has the greatest spreads:

$$\begin{aligned}\sqrt[1]{\alpha_1^1 + \beta_1^1} &= \alpha_1 + \beta_1, \\ \sqrt[1]{\alpha_2^1 + \beta_2^1} &= \alpha_2 + \beta_2, \end{aligned} \tag{14}$$

  which coincides with (1).
- If $p = 1$ (and $q = \infty$), then

$$(a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g = \left(a + b, \max\{\alpha_1, \beta_1\}, \max\{\alpha_2, \beta_2\}\right)_g \tag{15}$$

  has the smallest spreads:

$$\begin{aligned}\lim_{q \to \infty} \sqrt[q]{\alpha_1^q + \beta_1^q} &= \max\{\alpha_1, \beta_1\} \\ \lim_{q \to \infty} \sqrt[q]{\alpha_2^q + \beta_2^q} &= \max\{\alpha_2, \beta_2\} \end{aligned} \tag{16}$$

Varying the parameter p between 1 and $+\infty$ the spreads of $A \oplus B$ take their values between the above mentioned extremes. We can see from (6) that the spreads decrease while p decreases (q increases), which property provides a good possibility of controlling the growth of spreads, i.e., the growth of uncertainty during the calculations.

2.3 Triangular fuzzy numbers.

Obviously, if $g(x) = 1 - x$, then $A = (a, \alpha_1, \alpha_2)_g$ and $B = (b, \beta_1, \beta_2)_g$ are triangular fuzzy numbers. The t-norm generated by $g^p(x) = (1-x)^p$ is

$$T_{Y,p}(x,y) = \max\left\{0, 1 - \sqrt[p]{(1-x)^p + (1-y)^p}\right\}, \qquad x, y \in [0,1], \quad p \geq 1,$$

which is just Yager's parameterized family. Of course, formulae (13) and (15) remain valid for triangular numbers too:

- $\lim_{p\to\infty} T_{Y,p}(x,y) = \min\{x,y\}$ and so $q = 1$ ($p = \infty$) leads to the min-based addition of triangular fuzzy numbers. The sum $(a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g$ is also a triangular fuzzy number and has the greatest spreads (14).
- If $p = 1$ (and $q = \infty$), then $T_{Y,1} = T_L$ is the Łukasiewicz t-norm: $T_L(x,y) = \max\{0, x + y - 1\}$. The sum $(a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g$ is triangular in this case too and has the smallest spreads (16).

3 t-norm based multiplication

For the sake of simplicity, let $A = (a, \alpha_1, \alpha_2)_g$ and $B = (b, \beta_1, \beta_2)_g$ be now positive g-fuzzy numbers. The following theorem corresponding to Theorem 1 holds for the g, p-fuzzified multiplication:

THEOREM 2. *If the multiplication is extended by the t-norm T_p having an additive generator g^p ($p \geq 1$), then the product of two positive g-fuzzy numbers $(a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g$ can be calculated by the following approximation formula:*

$$(a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq \left(a \cdot b,\ \sqrt[q]{(b\alpha_1)^q + (a\beta_1)^q},\ \sqrt[q]{(b\alpha_2)^q + (a\beta_2)^q}\right)_g \qquad (18)$$

with $\frac{1}{p} + \frac{1}{q} = 1$, provided that the spreads of A and B are small compared with their mean values, i.e., $\alpha_1, \alpha_2 \ll a$ and $\beta_1, \beta_2 \ll b$.

P r o o f. This theorem can be proved in a very similar way as it was done in the case of addition. We just point out some differences. Equation (8) turns into

$$[A \odot B]_\lambda = a \cdot b + \bigcup_{\|(g(\xi),g(\eta))\|_p \le g(\lambda)} \Big([-b\alpha_1 g(\xi) - a\beta_1 g(\eta),\, b\alpha_2 g(\xi) + a\beta_2 g(\eta)] + \\ + [\alpha_1 g(\xi)\beta_1 g(\eta),\, \alpha_2 g(\xi)\beta_2 g(\eta)] \Big). \quad (19)$$

We can neglect the last term according to the hypotheses: $\alpha_1, \alpha_2 \ll a$ and $\beta_1, \beta_2 \ll b$, thus

$$[A \odot B]_\lambda \simeq a \cdot b + \bigcup_{\|(g(\xi),g(\eta))\|_p \le g(\lambda)} [-b\alpha_1 g(\xi) - a\beta_1 g(\eta),\, b\alpha_2 g(\xi) + a\beta_2 g(\eta)] = \\ = a \cdot b + g(\lambda) \cdot \left[-\|(b\alpha_1, a\beta_1)\|_q,\, \|(b\alpha_2, a\beta_2)\|_q \right], \quad (20)$$

from which (18) immediately follows. □

3.1 Extreme cases.

In both extreme cases we obtain formulae similar to those we obtained for addition:

- $\lim_{p\to\infty} T_p(x,y) = \min\{x,y\}$ thus, if $q = 1$ ($p = \infty$), then

$$(a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq (a \cdot b,\, b\alpha_1 + a\beta_1,\, b\alpha_2 + a\beta_2)_g \quad (21)$$

  has the greatest spreads which coincides with (21).
- If $p = 1$ ($q = \infty$), then

$$(a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq (a \cdot b,\, \max\{b\alpha_1, a\beta_1\},\, \max\{b\alpha_2, a\beta_2\})_g$$

  has the smallest spreads.

3.2 Triangular fuzzy numbers.

If $g(x) = 1 - x$, then $A = (a, \alpha_1, \alpha_2)_g$ and $B = (b, \beta_1, \beta_2)_g$ are triangular fuzzy numbers and the t-norm $T_p = T_{Y,p}$ belongs to Yager's parameterized family (see section 2.3). The extreme cases are the same as in the previous section except that instead of g-fuzzy numbers we have triangular ones.

3.3 Remarks.

Finally, we should mention the cases when A and B are not necessarily positive:

(i) If $A \in \mathcal{FN}(\mathbb{R}^-)$ and $B \in \mathcal{FN}(\mathbb{R}^+)$, then (18) turns into

$$(a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq \left(a \cdot b,\ \sqrt[q]{(b\alpha_1)^q + (a\beta_2)^q},\ \sqrt[q]{(b\alpha_2)^q + (a\beta_1)^q}\right)_g \quad (23)$$

(ii) If $A \in \mathcal{FN}(\mathbb{R}^-)$ and $B \in \mathcal{FN}(\mathbb{R}^-)$, then (18) turns into

$$(a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq \left(a \cdot b,\ \sqrt[q]{(b\alpha_2)^q + (a\beta_2)^q},\ \sqrt[q]{(b\alpha_1)^q + (a\beta_1)^q}\right)_g \quad (24)$$

Obviously, all the results of this paper can be generalized to the case of operations on LR fuzzy intervals without any problems.

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