g, p–FUZZIFICATION OF ARITHMETIC OPERATIONS

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ABSTRACT. The aim of this paper is to provide new results regarding the effective practical computation of t-norm-based arithmetic operations on LR fuzzy numbers.

1 Introduction

The present paper is devoted to the derivation of fast computation formulae for t-norm-based addition and multiplication of LR fuzzy numbers. We denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy sets of the real line and by $\mathcal{FN}(\mathbb{R})$ the set of all fuzzy numbers of LR-type:

$$A(x) = \begin{cases} 
L \left( \frac{x-a}{\alpha_1} \right) & \text{if } x \in [a - \alpha_1, a], \\
R \left( \frac{x-a}{\alpha_2} \right) & \text{if } x \in [a, a + \alpha_2], \\
0 & \text{otherwise},
\end{cases}$$

where $a$ is the mean value of $A$; $L, R: [0, 1] \to [0, 1]$ are the reference (shape) functions of $A$ with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, which are non-increasing, continuous mappings; $\alpha_1$ and $\alpha_2$ is the left and the right spread of $A$ correspondingly. We use the notation $A = (a, \alpha_1, \alpha_2)_{LR}$.

Let $A = (a, \alpha_1, \alpha_2)_{LR}$ and $B = (b, \beta_1, \beta_2)_{LR}$ be (for the sake of simplicity positive) fuzzy numbers of the same shape described by reference functions $L$ and $R$, i.e. $A, B \in \mathcal{FN}(\mathbb{R}^+)$. The addition- and multiplication rules of such LR fuzzy numbers are well-known in the case of “min”–norm, see, e.g., Dubois and Prade [1]. They verified the following formulae for the sum $A \oplus B$ and for the product $A \odot B$:

$$\begin{align*}
(a, \alpha_1, \alpha_2)_{LR} \oplus (b, \beta_1, \beta_2)_{LR} &= (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}, \\
(a, \alpha_1, \alpha_2)_{LR} \odot (b, \beta_1, \beta_2)_{LR} &\simeq (a \cdot b, a\beta_1 + b\alpha_1, a\beta_2 + b\alpha_2)_{LR}.
\end{align*}$$

Key words: extension principle, t-norms, fuzzy numbers.
The first one is an exact calculation formula for addition and the second is an approximate one for the multiplication assuming that the spreads of $A$ and $B$ are small compared with their mean values, i.e. $\alpha_1, \alpha_2 \ll a$ and $\beta_1, \beta_2 \ll b$. So, the addition is shape preserving and the multiplication is approximately a shape preserving operation if extended with the "min"-norm.

Using the general form of Zadeh’s extension principle a real operation $*: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is extended to $\mathcal{F}(\mathbb{R})$ by

$$A * B(z) = \sup_{z \times y = z} T(A(x), B(y)); \quad z \in \mathbb{R},$$

where $A, B \in \mathcal{F}(\mathbb{R})$ and $T: [0,1] \times [0,1] \rightarrow [0,1]$ is an arbitrary t-norm. In this case arithmetic operations on fuzzy numbers (even the addition) become much more complicated. We shall derive formulae similar to (1) and (2) for t-norm based arithmetic operations defined by (3) considering a special relationship between the generator function of t-norm and the reference functions of operands.

Recall that the mapping $T: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm iff it is commutative, associative, non-decreasing in each argument and $T(x,1) = x, \forall x \in [0,1]$. A t-norm $T$ is said to be Archimedean iff $T$ is continuous and $T(x,x) < x, \forall x \in [0,1]$. Every Archimedean t-norm $T$ is representable by a continuous and decreasing function $g: [0,1] \rightarrow [0,\infty[$ with $g(1) = 0$ and

$$T(x,y) = g^{-1}[(g(x) + g(y))],$$

where $g^{-1}$ is the pseudo-inverse of $g$, defined by

$$g^{-1}(y) = g^{-1}(\min\{y, g(0)\}).$$

Function $g$ is called the additive generator of $T$.

2 $g,p$-fuzzified addition

Let $A = (a, \alpha_1, \alpha_2)_RR$ and $B = (b, \beta_1, \beta_2)_RR$ be fuzzy numbers of the same shape described by reference function $R$ both on the left and the right-hand side. Note that $A$ and $B$ are not necessary symmetric as their left and right spreads can differ from each other.

2.1 $g$-fuzzy numbers.

Now, we propose to define $A \oplus B$ by (3) with the t-norm $T_p$ having the additive generator $g^p (p \geq 1)$ such that $g^{-1} = R$. Obviously $T_p$ is nilpotent, i.e., has zero divisors ($\exists x, y > 0: T_p(x,y) = 0$) because $g(0) = R^{-1}(0) = 1 < +\infty$.

We will write in the following just $g$ instead of $RR = g^{-1}g^{-1}$ using notations like $A = (a, \alpha_1, \alpha_2)_g$ and call $A$ a $g$-fuzzy number.

The following theorem holds for the $g,p$-fuzzified addition:
**THEOREM 1.** Let $A = (a, \alpha_1, \alpha_2)_q$ and $B = (b, \beta_1, \beta_2)_q$. If the addition is extended by the t-norm $T_p$ with additive generator $g^p$ ($p \geq 1$), then $A \oplus B$ is also a $q$-fuzzy number and

$$(a, \alpha_1, \alpha_2)_q \oplus (b, \beta_1, \beta_2)_q = (a + b, \sqrt[1]{\alpha_1^q + \beta_1^q}, \sqrt[1]{\alpha_2^q + \beta_2^q})_q,$$  \hspace{1cm} (6)

where $q \geq 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** A generalized form of Nguyen’s formula on level sets [7] was proved in Fullér, Keresztfalvi [2]. It gives an exact expression for level sets of $A \oplus B$. Denoting by $[A]_\lambda$ the $\lambda$-level set of $A$ we have in case of an arbitrary continuous two-place function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the following equality

$$[f(A, B)]_\lambda = \bigcup_{T^{(\xi, \eta)} : \lambda} f([A]_{\xi}, [B]_{\eta}) \quad \lambda \in [0, 1]$$  \hspace{1cm} (7)

if the fuzzy sets $A, B \in \mathcal{F}(\mathbb{R})$ have upper semicontinuous, compactly-supported membership functions and the t-norm $T$ is also upper semicontinuous. Applying this formula to our case we obtain for all $\lambda \in [0, 1]$

$$[A \oplus B]_{\lambda} = \bigcup_{T^{(\xi, \eta)} : \lambda} [A]_{\xi} + [B]_{\eta} =$$

$$= \bigcup_{T^{(\xi, \eta)} : \lambda} [a - \alpha_1 g(\xi), a + \alpha_2 g(\xi)] + [b - \beta_1 g(\eta), b + \beta_2 g(\eta)] =$$

$$= \bigcup_{T^{(\xi, \eta)} : \lambda} [a + b - \alpha_1 g(\xi) - \beta_1 g(\eta), a + b + \alpha_2 g(\xi) + \beta_2 g(\eta)] =$$

$$(g(\xi))^p + (g(\eta))^p \leq (g(\lambda))^p$$

$$= a + b + \bigcup_{\|g(\xi)\|_p, \|g(\eta)\|_p \leq g(\lambda)} [-\alpha_1 g(\xi) - \beta_1 g(\eta), \alpha_2 g(\xi) + \beta_2 g(\eta)],$$  \hspace{1cm} (8)

where $\|g(\xi), g(\eta)\|_p = \sqrt{(g(\xi))^p + (g(\eta))^p}$.

The next step was firstly proposed by Kovács [5, 6]: there exists a natural bijection between the normed linear space $\mathbb{R}^2$ and the normed linear space of linear functionals on $\mathbb{R}^2$, i.e. $\text{Lin}(\mathbb{R}^2, \mathbb{R})$. Moreover, this bijection is an isometry if we normalize $\text{Lin}(\mathbb{R}^2, \mathbb{R})$ with the $\| \cdot \|_p$ norm and $\mathbb{R}^2$ with the $\| \cdot \|_q$ norm where $\frac{1}{p} + \frac{1}{q} = 1$. Due to this fact we have for the lower bound

$$\sup_{\|g(\xi), g(\eta)\|_p \leq g(\lambda)} (\alpha_1 g(\xi) + \beta_1 g(\eta)) = g(\lambda) \cdot \sup_{\|h_1, h_2\|_p \leq 1} (\alpha_1 h_1 + \beta_1 h_2) =$$

$$= g(\lambda) \cdot \|\alpha_1, \beta_1\|_q =$$

$$= g(\lambda) \cdot \sqrt{\alpha_1^q + \beta_1^q}.$$  \hspace{1cm} (9)

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where \((h_1, h_2) = (g(\xi) / g(\lambda), g(\eta) / g(\lambda))\) and in a similar way for the upper bound

\[
\sup_{\|g(\xi).g(\eta)\|_p \leq g(\lambda)} \left( \alpha_2 g(\xi) + \beta_2 g(\eta) \right) = g(\lambda) \cdot \sqrt[\frac{p}{q}]{\alpha_2^q + \beta_2^q}.
\]  

(10)

from which we obtain

\[
[A \oplus B]_\lambda = a + b + g(\lambda) \cdot \left[ -\|\alpha_1, \beta_1\|_q, \|\alpha_2, \beta_2\|_q \right] \]  

(11)

This is a \(\lambda\)-level set of a \(g\)-fuzzy number, that is

\[
(a, \alpha_1, \alpha_2) \oplus (b, \beta_1, \beta_2) = (a + b, \|\alpha_1, \beta_1\|_q, \|\alpha_2, \beta_2\|_q)_g = 
= (a + b, \sqrt[q]{\alpha_1^q + \beta_1^q}, \sqrt[q]{\alpha_2^q + \beta_2^q})_g
\]  

(12)

which was to be proved. \(\square\)

2.2 Extreme cases.

Both extreme cases are worth mentioning:

- It is easy to see that \(\lim_{p \to \infty} T_p(x, y) = \min\{x, y\}\). Thus, if \(q = 1\) (\(p = \infty\)), then we arrive at the min-based addition and

\[
(a, \alpha_1, \alpha_2) \oplus (b, \beta_1, \beta_2) = (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_g
\]  

(13)

has the greatest spreads:

\[
\sqrt[q]{\alpha_1^q + \beta_1^q} = \alpha_1 + \beta_1, \quad \sqrt[q]{\alpha_2^q + \beta_2^q} = \alpha_2 + \beta_2
\]  

(14)

which coincides with (1).

- If \(p = 1\) (and \(q = \infty\)), then

\[
(a, \alpha_1, \alpha_2) \oplus (b, \beta_1, \beta_2) = (a + b, \max\{\alpha_1, \beta_1\}, \max\{\alpha_2, \beta_2\})_g
\]  

(15)

has the smallest spreads:

\[
\lim_{q \to \infty} \sqrt[q]{\alpha_1^q + \beta_1^q} = \max\{\alpha_1, \beta_1\}
\]  

\[
\lim_{q \to \infty} \sqrt[q]{\alpha_2^q + \beta_2^q} = \max\{\alpha_2, \beta_2\}
\]  

(16)
Varying the parameter \( p \) between 1 and \( +\infty \) the spreads of \( A \oplus B \) take their values between the above mentioned extremes. We can see from (6) that the spreads decrease while \( p \) decreases (\( q \) increases), which property provides a good possibility of controlling the growth of spreads, i.e., the growth of uncertainty during the calculations.

2.3 Triangular fuzzy numbers.

Obviously, if \( g(x) = 1 - x \), then \( A = (a, \alpha_1, \alpha_2)_g \) and \( B = (b, \beta_1, \beta_2)_g \) are triangular fuzzy numbers. The t-norm generated by \( g^p(x) = (1 - x)^p \) is

\[
T_{Y,p}(x, y) = \max \left\{ 0, 1 - \sqrt[p]{(1 - x)^p + (1 - y)^p} \right\}, \quad x, y \in [0, 1], \quad p \geq 1,
\]

which is just Yager's parameterized family. Of course, formulae (13) and (15) remain valid for triangular numbers too:

- \( \lim_{p \to \infty} T_{Y,p}(x, y) = \min\{x, y\} \) and so \( q = 1 \) \((p = \infty)\) leads to the min-based addition of triangular fuzzy numbers. The sum \((a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g\) is also a triangular fuzzy number and has the greatest spreads (14).

- If \( p = 1 \) \((q = \infty)\), then \( T_{Y,1} = T_L \) is the Lukasiewicz t-norm:
  \( T_L(x, y) = \max\{0, x + y - 1\} \). The sum \((a, \alpha_1, \alpha_2)_g \oplus (b, \beta_1, \beta_2)_g\) is triangular in this case too and has the smallest spreads (16).

3 t-norm based multiplication

For the sake of simplicity, let \( A = (a, \alpha_1, \alpha_2)_g \) and \( B = (b, \beta_1, \beta_2)_g \) be now positive \( g \)-fuzzy numbers. The following theorem corresponding to Theorem 1 holds for the \( g, p \)-fuzzified multiplication:

**Theorem 2.** If the multiplication is extended by the t-norm \( T_p \) having an additive generator \( g^p \) \((p \geq 1)\), then the product of two positive \( g \)-fuzzy numbers \((a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g\) can be calculated by the following approximation formula:

\[
(a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq \left( a \cdot b, \sqrt[p]{(b \alpha_1)^q + (a \beta_1)^q}, \sqrt[p]{(b \alpha_2)^q + (a \beta_2)^q} \right)_g
\]

\((18)\)

with \( \frac{1}{p} + \frac{1}{q} = 1 \), provided that the spreads of \( A \) and \( B \) are small compared with their mean values, i.e., \( \alpha_1, \alpha_2 \ll a \) and \( \beta_1, \beta_2 \ll b \).

**Proof.** This theorem can be proved in a very similar way as it was done in the case of addition. We just point out some differences. Equation (8) turns
into

\[ [A \odot B]_\lambda = a \cdot b + \bigcup_{\|g(\xi), g(\eta)\|_p \leq g(\lambda)} \left( [b \alpha_1 g(\xi) - a \beta_1 g(\eta), b \alpha_2 g(\xi) + a \beta_2 g(\eta)] + [\alpha_1 g(\xi) \beta_1 g(\eta), \alpha_2 g(\xi) \beta_2 g(\eta)] \right). \]  

(19)

We can neglect the last term according to the hypotheses: \( \alpha_1, \alpha_2 \ll a \) and \( \beta_1, \beta_2 \ll b \), thus

\[ [A \odot B]_\lambda \simeq a \cdot b + \bigcup_{\|g(\xi), g(\eta)\|_p \leq g(\lambda)} [-b \alpha_1 g(\xi) - a \beta_1 g(\eta), b \alpha_2 g(\xi) + a \beta_2 g(\eta)] = \]

\[ = a \cdot b + g(\lambda) \cdot \left[ -\|b \alpha_1, a \beta_1\|_q, \|b \alpha_2, a \beta_2\|_q \right], \]  

(20)

from which (18) immediately follows. \( \square \)

3.1 Extreme cases.

In both extreme cases we obtain formulae similar to those we obtained for addition:

- \( \lim_{p \to \infty} T_p(x, y) = \min\{x, y\} \) thus, if \( q = 1 \ (p = \infty) \), then
  \[ (a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq (a \cdot b, b \alpha_1 + a \beta_1, b \alpha_2 + a \beta_2)_g \]  

(21)

has the greatest spreads which coincides with (21).

- If \( p = 1 \ (q = \infty) \), then
  \[ (a, \alpha_1, \alpha_2)_g \odot (b, \beta_1, \beta_2)_g \simeq (a \cdot b, \max\{b \alpha_1, a \beta_1\}, \max\{b \alpha_2, a \beta_2\})_g \]

has the smallest spreads.

3.2 Triangular fuzzy numbers.

If \( g(x) = 1 - x \), then \( A = (a, \alpha_1, \alpha_2)_g \) and \( B = (b, \beta_1, \beta_2)_g \) are triangular fuzzy numbers and the t-norm \( T_p = T_{Y,p} \) belongs to Yager's parameterized family (see section 2.3). The extreme cases are the same as in the previous section except that instead of \( g \)-fuzzy numbers we have triangular ones.

3.3 Remarks.
Finally, we should mention the cases when $A$ and $B$ are not necessarily positive:

(i) If $A \in \mathcal{FN}(\mathbb{R}^-)$ and $B \in \mathcal{FN}(\mathbb{R}^+)$, then (18) turns into

$$(a, \alpha_1, \alpha_2)_g \circ (b, \beta_1, \beta_2)_g \simeq (a \cdot b, \sqrt{(\alpha_1^g + (a \beta_2)^g)} \cdot \sqrt{(b \beta_2^g + (b \alpha_1)^g)}_g \quad (23)$$

(ii) If $A \in \mathcal{FN}(\mathbb{R}^-)$ and $B \in \mathcal{FN}(\mathbb{R}^-)$, then (18) turns into

$$(a, \alpha_1, \alpha_2)_g \circ (b, \beta_1, \beta_2)_g \simeq (a \cdot b, \sqrt{(b \alpha_2^g + (a \beta_2)^g)} \cdot \sqrt{(b \beta_2^g + (b \alpha_1)^g)}_g \quad (24)$$

Obviously, all the results of this paper can be generalized to the case of operations on LR fuzzy intervals without any problems.

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