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TOPOLOGIES GENERATED BY THE FAMILY OF ALL $\mathcal{I}$–SPARSE SETS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. In [1] Filipczak introduced the definition of the $\mathcal{I}$-proximally continuous function. In this note we define the coarsest topology making the family of $\mathcal{I}$-proximally continuous functions continuous.

Let $\mathbb{R}$ denote the real line and let $\mathbb{N}$ denote the family of all positive integers. All topological notations are given with respect to the natural topology except for the case where a topology T is specifically mentioned.

We introduce the following notations:

$\mathcal{I}$ – the σ-ideal of subsets of $\mathbb{R}$ of the first category,

$\mathcal{S}$ – the σ-field of subsets of $\mathbb{R}$ having the Baire property.

We start with the definition of the $\mathcal{I}$-density point which was introduced by W. Wilczyński in [4].

DEFINITION 1 [4]. We shall say that 0 is a *point of $\mathcal{I}$-density of a set* $A \in \mathcal{S}$ if and only if for each unbounded sequence of positive integers $\{n_m\}_{m\in\mathcal{N}}$, there exists a subsequence $\{n_{m_p}\}_{p\in\mathcal{N}}$ such that

$$\{x\colon \chi_{n_{m_p}\cdot A\cap[-1,1]}(x) \not\to 1\} \in \mathcal{I}.$$

A point x_0 is a point of $\mathcal{I}$-density of a set $A \in \mathcal{S}$ if and only if 0 is a point of $\mathcal{I}$-density of the set $A - x_0$. A point x_0 is a *point of $\mathcal{I}$-dispersion of a set* $A \in \mathcal{S}$ if and only if x_0 is a point of $\mathcal{I}$-density of the set $\mathbb{R}\backslash A$.

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LEMMA 2 [4]. *A point* 0 *is a point of* $\mathcal{I}$*-density of a set* $A \in \mathcal{S}$ *if and only if, for each sequence of positive real numbers* $\{t_n\}_{n \in \mathrm{N}}$ *such that* $\lim_{n\to\infty} t_n = \infty$, *there exists a subsequence* $\{t_{n_m}\}_{m \in \mathrm{N}}$ *such that*

$$\{x\colon \chi_{t_{n_m}\cdot A\cap[-1,1]}(x) \nrightarrow 1\} \in \mathcal{I}\,.$$

Let $\{E_n\}_{n\in\mathrm{N}}$ *be an arbitrary sequence of sets and let* $x \in \mathbb{R}$. *We observe that* $\chi_{E_n}(x) \nrightarrow 1$ *if and only if* $\liminf_{n\to\infty} \chi_{E_n}(x) = 0$, *which is equivalent to* $x \in \limsup_{n\to\infty}(\mathbb{R}\setminus E_n)$.

DEFINITION 3 [1]. A point $x \in \mathbb{R}$ is an *upper* $\mathcal{I}$*-density point of a set* $E \in \mathcal{S}$ if and only if there exists a sequence $\{t_n\}_{n\in\mathcal{N}}$ of real numbers, such that $\lim_{n\to\infty} t_n = \infty$ and $\mathbb{R}\setminus \liminf_{n\to\infty} t_n \cdot (E - x) \cap [-1,1] \in \mathcal{I}$. Then we denote $d^*_{\mathcal{I}}(E,x) = 1$.

A point x is a lower $\mathcal{I}$-dispersion point of a set $E \in \mathcal{S}$ if and only if x is the upper $\mathcal{I}$-density point of $\mathbb{R}\setminus E$. Then we write $d_{\mathcal{I}^*}(E,x) = 0$.

We observe that, for each $E \in \mathcal{S}$, $d_{\mathcal{I}^*}(E,x) = 0$ if and only if there exists a sequence $\{t_n\}_{n\in\mathrm{N}}$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\limsup_{n\to\infty} t_n \cdot (E - x) \cap [-1,1] \in \mathcal{I}\,.$$

DEFINITION 4 [1]. A set $E \in \mathcal{S}$ is said to be $\mathcal{I}$*-sparse at a point* $x \in \mathbb{R}$ if and only if, for each $F \in \mathcal{S}$, if $d_{\mathcal{I}^*}(F,x) = 0$, then $d_{\mathcal{I}^*}(E \cup F, x) = 0$.

Let, for $A \in \mathcal{S}$,

$$\Phi_{\mathcal{I}}(A) = \{x \in \mathbb{R}\colon \mathbb{R}\setminus A \text{ is } \mathcal{I}\text{-sparse at } x\}\,.$$

THEOREM 5 [1]. *The family* $T = \{\Phi_{\mathcal{I}}(A)\setminus P\colon A \in \mathcal{S},\, P \in \mathcal{I}\}$ *is a topology on* $\mathbb{R}$ *and* $(\mathbb{R}, T)$ *is Hausdorff but is not a regular space.*

DEFINITION 6 [1], [2]. We shall say that a function f is $\mathcal{I}$-proximally continuous if and only if f is a continuous function with respect to the T topology. The family of all $\mathcal{I}$-proximally continuous functions will be denoted by $\mathcal{C}_T$.

Since the T topology is not regular, we see that it is not the coarsest topology for the family $\mathcal{C}_T$. To find the coarsest topology for the family $\mathcal{C}_T$, we need the following lemmas and theorems.

LEMMA 7. *Let $\{t_n\}_{n\in\mathrm{N}}$ be an arbitrary increasing sequence of real numbers such that $\lim\limits_{n\to\infty} t_n = \infty$ and let $A, B \in \mathcal{S}$ be such that $A\triangle B \in \mathcal{I}$. If $\limsup\limits_{n\to\infty} t_n \cdot A \in \mathcal{I}$, then $\limsup\limits_{n\to\infty} t_n \cdot B \in \mathcal{I}$.*

P r o o f. We assume that $\limsup\limits_{n\to\infty} t_n \cdot A \in \mathcal{I}$. Then

$$\begin{aligned}\limsup_{n\to\infty} t_n \cdot B &= \limsup_{n\to\infty} t_n \cdot (B\backslash A) \cup \limsup_{n\to\infty} t_n \cdot (A \cap B)\\ &\subset \limsup_{n\to\infty} t_n \cdot (B\backslash A) \cup \limsup_{n\to\infty} t_n \cdot A\,.\end{aligned}$$

Thus, by $A\triangle B \in \mathcal{I}$, we have $\limsup\limits_{n\to\infty} t_n \cdot B \in \mathcal{I}$. □

LEMMA 8. *Let $A = G\triangle P$, where G is an open set and $P \in \mathcal{I}$. Then the set A is $\mathcal{I}$-sparse at a point $x \in \mathbb{R}$ if and only if, for each open set H, if $d^*_{\mathcal{I}}(H, x) = 0$, then $d_{\mathcal{I}^*}(Y \cup G, x) = 0$.*

P r o o f. Let A be an $\mathcal{I}$-sparse set at a point $x \in \mathbb{R}$ and H an open set such that $d_{\mathcal{I}^*}(H, x) = 0$. Then, by Definition 4, $d_{\mathcal{I}^*}(H \cup A, x) = 0$. Since $(H \cup A)\triangle(H \cup G) \in \mathcal{I}$, by Lemma 7, we have that $d_{\mathcal{I}^*}(H \cup G, x) = 0$. Now, we assume that, for each open set H, if $d_{\mathcal{I}^*}(H, x) = 0$, then $d_{\mathcal{I}^*}(H \cup G, x) = 0$. Let $F \in \mathcal{S}$ be an arbitrary set such that $d_{\mathcal{I}^*}(F, x) = 0$. Then there exists an open H such that $F\triangle H \in \mathcal{I}$. Therefore, by Lemma 7, we have that $d_{\mathcal{I}^*}(H, x) = 0$ and, by our assumption, $d_{\mathcal{I}^*}(H \cup G, x) = 0$. Since $(F \cup A)\triangle(H \cup G) \in \mathcal{I}$, we have $d_{\mathcal{I}^*}(A \cup F, x) = 0$. Therefore the set A is $\mathcal{I}$-sparse at x. □

T. Filipczak [1] showed that the operator $\Phi_{\mathcal{I}}$ is a lower density [3] in $\mathcal{S}$, thus we have the following lemma.

LEMMA 9. $T = \{A \in \mathcal{S}\colon A \subset \Phi_{\mathcal{I}}(A)\}$.

Let $\tau = \{A \in T\colon$ for each $x \in A$, there exists an open set U such that $\mathbb{R}\backslash A \subset U$ and $x \in \Phi_{\mathcal{I}}(\mathbb{R}\backslash U)\}$.

THEOREM 10. (*Luzin–Menchoff*). *Let A be an arbitrary set and F a closed set such that $F \subset A$. If, for each $x \in F$, there exists an open set H such that $\mathbb{R}\backslash A \subset \mathrm{cl}(H)$ and $x \in \Phi_{\mathcal{I}}(\mathbb{R}\backslash H)$, then there exists a perfect set P such that $F \subset P \subset A$ and $F \subset \Phi_{\mathcal{I}}(P)$.*

P r o o f. For each $x \in \mathbb{R}\backslash A$, let

$$\delta_x = \begin{cases} \varrho^2(x, F) & \text{if } \varrho(x, F) < \frac{1}{2} \\ \frac{1}{4} & \text{if } \varrho(x, F) \geq \frac{1}{2} \end{cases}$$

and let $G = \bigcup_{x\in\mathbb{R}\setminus A} (x-\delta_x, x+\delta_x)$. Then $\mathbb{R}\setminus A \subset G$. For each $x \in \mathbb{R}\setminus A$, $\varrho(x,F) > \delta_x$ and, therefore, $F \cap G = \emptyset$.

We shall show that $F \subset \Phi_{\mathcal{I}}(\mathbb{R}\setminus G)$. Let $x_0 \in F$. We assume that $x_0 = 0$. Then there exists an open set H such that $\mathbb{R}\setminus A \subset \mathrm{cl}(H)$ and H is $\mathcal{I}$-sparse at 0. Let B be an open set such that $d_{\mathcal{I}^*}(B,0) = 0$. Then $d_{\mathcal{I}^*}(B\cup H, 0) = 0$, which means that there exists a sequence of positive real numbers $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\limsup_{n\to\infty} t_n\cdot(H\cup B)\cap[-1,1] \in \mathcal{I}$. Since H and B are open sets, we have that $\limsup_{n\to\infty} t_n\cdot(H\cup B)\cap[0,1]$ is a nowhere dense set. We suppose that there exists an open interval $(a,b) \subset [-1,1]$ such that $\limsup_{n\to\infty} t_n\cdot(B\cup G)\cap(a,b)$ is a dense subset of the interval (a,b). We may assume that $(a,b)\subset(0,1)$. Since $\limsup_{n\to\infty} t_n\cdot(B\cup H)\cap(a,b)$ is a nowhere dense subset of (a,b), there exist a number $k_0 \in \mathbb{N}$ and an interval $(c,d)\subset(a,b)$ such that $\bigcup_{n=k_0}^{\infty} t_n\cdot(B\cup H)\cap(c,d) = \emptyset$. We know that there exists $y \in \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} t_n\cdot(B\cup G)\cap(c,d)$. Thus, for each $k\in\mathbb{N}$, there exists $n_k > k$ such that $y \in t_{n_k}\cdot(B\cup G)\cap(c,d)$. Since, for each $k > k_0$, $t_{n_k}\cdot B\cap(c,d) = \emptyset$, we have that, for each $k > k_0$, $y \in t_{n_k}\cdot G\cap(c,d)$.

Let $k > k_0$. Since $y \in t_{n_k}\cdot G = \bigcup_{x\in\mathbb{R}\setminus A} t_{n_k}\cdot(x-\delta_x,\, x+\delta_x)$ we have that there exists $x_k \in \mathbb{R}\setminus A$ such that $y \in t_{n_k}\cdot(x_k-\delta_{x_k},\, x_k+\delta_{x_k})$. By $\delta_{x_k} \le \varrho^2(x_k,F) \le x_k^2$, we have

$$0 < t_{n_k}\cdot x_k\cdot(1-x_k) \le t_{n_k}\cdot(x_k-\delta_{x_k}) < y\,.$$

We observe that, for each $k\in\mathbb{N}$, if $x_k \ge \frac{1}{2}$, then, by $\delta_{x_k} \le \frac{1}{4}$, we have $x_k - \delta_{x_k} \ge \frac{1}{4}$, and if $x_k < \frac{1}{2}$, then $x_k - \delta_{x_k} \ge x_k(1-x_k) \ge \frac{1}{2}\cdot x_k$. Therefore, by

$$0 \le \lim_{k\to\infty}(x_k-\delta_{x_k}) \le \lim_{k\to\infty}\frac{1}{t_{n_k}}\cdot y\,,$$

we have that $\lim_{k\to\infty} x_k = 0$ and $\{t_{n_k}\cdot x_k\}_{k\in\mathbb{N}}$ is a bounded sequence.

Thus, by

$$\lim_{k\to\infty}\frac{t_{n_k}\cdot\delta_{x_k}}{t_{n_k}\cdot x_k} \le \lim_{k\to\infty} x_k = 0\,,$$

we have that $\lim_{k\to\infty} t_{n_k}\cdot\delta_{x_k} = 0$.

Therefore, $\limsup_{k\to\infty} t_{n_k}\cdot(x_k-\delta_{x_k}) \le y \le \liminf_{k\to\infty} t_{n_k}\cdot(x_k+\delta_{x_k})$ and $\lim_{k\to\infty} t_{n_k}\cdot x_k = y$.

Since $y \in (c,d)$, there exists $k > k_0$ such that

$$t_{n_k} \cdot x_k \in (c,d) \cap t_{n_k} \cdot (\mathbb{R} \backslash A) \subset (c,d) \cap t_{n_k} \cdot \mathrm{cl}(H) .$$

This is impossible since $(c,d) \cap t_{n_k} \cdot H = \emptyset$ and, therefore,

$$\limsup_{n \to \infty} t_n \cdot (B \cup G) \cap [0,1] \in \mathcal{I} .$$

So, by Lemma 8, the set G is an $\mathcal{I}$-sparse set at 0.

Let P be the set of all limit points of the set $\mathbb{R} \backslash G$. Since each point of the set F is a $\mathcal{I}$-sparse point of the set G, therefore $F \subset P$. By $P \triangle (\mathbb{R} \backslash G) \in \mathcal{I}$, we have that $F \subset \Phi_{\mathcal{I}}(P)$. Thus the set P is perfect and the proof of the theorem is completed. □

LEMMA 11. *Let $A \in \tau$. Then, for each $x \in A$, there exists an F_σ set B such that $x \in B \subset A$ and $B \in \tau$.*

Proof. Let $A \in \tau$ and $x \in A$. Then there exists an open set G such that $\mathbb{R} \backslash A \subset G$ and $x \in \Phi_{\mathcal{I}}(\mathbb{R} \backslash G)$. We put $B = \big(\mathbb{R} \backslash \mathrm{cl}(G)\big) \cup \{x\}$. Then B is an F_σ set and $B \subset A$. We have that $\mathbb{R} \backslash B = \mathrm{cl}(G) \cap (\mathbb{R} \backslash \{x\}) \subset \mathrm{cl}(G)$, $x \in \Phi_{\mathcal{I}}(\mathbb{R} \backslash G)$ and $\{x\}$ is a closed set. Therefore, by Theorem 10, there exists a perfect set P such that $\{x\} \subset P \subset B$ and $x \in \Phi_{\mathcal{I}}(P)$. Let $H = \mathbb{R} \backslash P$. Then H is an open set, $\mathbb{R} \backslash B \subset H$ and $x \in \Phi_{\mathcal{I}}(\mathbb{R} \backslash H)$. Therefore $B \in \tau$ and proof of the lemma is complete. □

THEOREM 12 [2]. *If a function f is $\mathcal{I}$-proximally continuous then f is of the first class of Baire.*

LEMMA 13. *Let f be an $\mathcal{I}$-proximally continuous function. Then, for each $a \in \mathbb{R}$, $\{x \in \mathbb{R} \colon f(x) > a\} \in \tau$.*

Proof. Let $a \in \mathbb{R}$. We assume that $a = 0$. We put $A = \{x \colon f(x) > 0\}$. Let $x_0 \in A$. Then there exists $b > 0$ such that $f(x_0) > b$. Let $B = \{x \colon f(x) < b\}$. Then $A, B \in T$. If the set B is empty, then it is obvious that there exists and open set G such that $\mathbb{R} \backslash A \subset G$ and $x_0 \in \Phi_{\mathcal{I}}(\mathbb{R} \backslash G)$. We assume that $B \neq \emptyset$. Then, by Theorem 12, $\mathrm{int}(B) \neq \emptyset$ and

$$\{x \colon f(x) < b\} \subset \mathrm{cl}\big(\mathrm{int}(\{x \colon f(x) < b\})\big) .$$

We put $G = \mathrm{int}(B)$. Then $\mathbb{R} \backslash A \subset \mathrm{cl}(G)$ and, by $\{x \colon f(x) > b\} \in T$, we have that $x_0 \in \Phi_{\mathcal{I}}(\mathbb{R} \backslash G)$. Now, in a similar way as in Lemma 11 we can show that $A \in \tau$. □

THEOREM 14. *Let E be an F_σ set such that $E \in \tau$. There exists an $\mathcal{I}$-proximally continuous function f such that $0 < f(x) \leq 1$ for all $x \in E$ and $f(x) = 0$ for all $x \in \mathbb{R}\backslash E$.*

By using Theorem 10 the proof of this theorem can be carried out exactly as the proof of the theorem of Z. Zahorski [5].

THEOREM 15. *The family τ is a completely regular topology on the real line and*

$$C_T = \{f\colon \mathbb{R} \to \mathbb{R}\colon f \text{ is a continuous function with respect to the } \tau \text{ topology}\}.$$

Proof. It is obvious that $\emptyset, \mathbb{R} \in \tau$. We assume that $A, B \in \tau$. Let $x \in A \cap B$. Then there exist open sets H and G such that $\mathbb{R}\backslash A \subset H$, $\mathbb{R}\backslash B \subset G$ and $x \in \Phi_{\mathcal{I}}(\mathbb{R}\backslash H) \cap \Phi_{\mathcal{I}}(\mathbb{R}\backslash G)$. Then $\mathbb{R}\backslash(A \cap B) \subset H \cup G$, $H \cup G$ is an open set and $x \in \Phi_{\mathcal{I}}\big(\mathbb{R}\backslash(H \cup G)\big) = \Phi_{\mathcal{I}}(\mathbb{R}\backslash H) \cap \Phi_{\mathcal{I}}(\mathbb{R}\backslash G)$. Thus $A \cap B \in \tau$.

Let $\{A_I\}_{I \in L} \subset \tau$ and let $x \in \bigcup_{I \in L} A_I$. Then there exist an $I \in L$ and an open set G such that $\mathbb{R}\backslash A_I \subset G$ and $x \in \Phi_{\mathcal{I}}(\mathbb{R}\backslash G)$. Thus $\mathbb{R}\backslash \bigcup_{I \in L} A_I \subset G$ and $\bigcup_{I \in L} A_I \in \tau$. Therefore τ is a topology on the real line.

Let

$$\mathcal{B}_0 = \{A \in \tau\colon A \text{ is an } F_\sigma \text{ set}\}$$

and let

$$\mathcal{B}_1 = \big\{A \subset \mathbb{R}\colon \text{ there exists a function } f \in C_T \text{ such that } \{x\colon f(x) > 0\} = A\big\}.$$

Then, by Lemma 13 and Theorem 12, $\mathcal{B}_1 \subset \mathcal{B}_0$ and, by Theorem 13, $\mathcal{B}_0 \subset \mathcal{B}_1$. Since, by Lemma 11, the family $\mathcal{B}_0$ is a base of the τ topology, the proof of the theorem is complete. □

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