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ON ALMOST QUASICONTINUITY OF FUNCTIONS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. The almost quasicontinuity is a property of functions which is weaker than quasicontinuity and almost continuity in the sense of Husain. We characterize the almost quasicontinuity by functions of the oscillation type, describe the set B_f of all points at which any function f is almost quasicontinuous and we show the invariance of this property under almost uniform convergence.

Let X, Y be topological spaces and let $f\colon X \to Y$ be a function. If in the definition of continuity at a point $x \in X$ the condition

$$x \in \operatorname{Int} f^{-1}(V) \qquad \text{for each neighbourhood } V \text{ of } f(x)$$

we replace by one of the following: $x \in \operatorname{Cl}\left(\operatorname{Int} f^{-1}(V)\right)$, $x \in \operatorname{Int}\left(\operatorname{Cl} f^{-1}(V)\right)$ or $x \in \operatorname{Cl}\Big(\operatorname{Int}\left(\operatorname{Cl} f^{-1}(V)\right)\Big)$, then this gives the definition of quasicontinuity [8, 9], almost continuity [7] or almost quasicontinuity [1, 10] of f at x, respectively. A function is called *quasicontinuous* (*almost continuous, almost quasicontinuous*) if it has this property at each point. Then we have:

$$\begin{array}{ccccc} & & \text{quasicontinuity} & & \\ & \nearrow & & \searrow & \\ \text{continuity} & & & & \text{almost quasicontinuity} \\ & \searrow & & \nearrow & \\ & & \text{almost continuity} & & \end{array}$$

and none of these implications is invertable [10].

The properties mentioned above are natural since for functions with values in metric spaces almost continuity (almost quasicontinuity) together with cliquishness is equivalent to continuity [6], (resp. quasicontinuity [1]); for the definition of cliquishness see [11].

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In this note we consider functions with values in uniform spaces. We give a characterization of almost quasicontinuity by functions of the oscillation type. Then it is used to description of the set B_f of all almost quasicontinuity points and for the investigation of the invariance of almost quasicontinuity under almost uniform convergence.

A subset A of a topological space X is called:

– *regular closed* if $A = \mathrm{Cl}\,(\mathrm{Int}\, A)$; [2]

– *semi-open* if $A \subset \mathrm{Cl}\,(\mathrm{Int}\, A)$; [9].

As it was shown in [9], the union of any family of semi-open sets is semi-open; the intersection of an open set and a semi-open one is semi-open. Using also these notions we can give the following simple characterization of almost quasicontinuity.

THEOREM 1. *Let X, Y be topological spaces. For a function $f\colon X \to Y$ the following conditions are equivalent:*

(a) *f is almost quasicontinuous;*

(b) $f^{-1}(V) \subset \mathrm{Cl}\left(\mathrm{Int}\,(\mathrm{Cl}\, f^{-1}(V))\right)$ *for each open set $V \subset Y$;*

(c) $\mathrm{Int}\left(\mathrm{Cl}\,(\mathrm{Int}\, f^{-1}(A))\right) \subset f^{-1}(A)$ *for each closed set $A \subset Y$;*

(d) $\mathrm{Cl}\, f^{-1}(V)$ *is semi-open for each open set $V \subset Y$;*

(e) $\mathrm{Cl}\, f^{-1}(V)$ *is regular closed for each open set $V \subset Y$.*

We omit the standard proof of this theorem. □

In the sequel we will consider functions with values in uniform spaces. For a uniform space $(Y, \mathscr{V})$ by $P_{\mathscr{V}}$ we denote a saturated family of pseudometrics on Y inducing the uniformity $\mathscr{V}$. If $y \in Y$, $\varrho \in P_{\mathscr{V}}$ and $r > 0$, then we denote $B(y, \varrho, r) = \{z \in Y\colon \varrho(y, z) < r\}$.

Now let f be a function from a topological space X into a uniform space $(Y, \mathscr{V})$, $x \in X$ and let $\varrho \in P_{\mathscr{V}}$. We define

$$w_{\varrho,f}(x) = \inf_A \inf_M \sup_{x',x'' \in M} \varrho\big(f(x'), f(x'')\big),$$

where infima are taken under all semi-open sets A containing x and all sets M satisfying $x \in M \subset A \subset \mathrm{Cl}\, M$ respectively, and

$$w_f(x) = \sup_{\varrho \in P_{\mathscr{V}}} w_{\varrho,f}(x).$$

A real function $f\colon X \to \mathbb{R} \cup \{\infty\}$ is said to be upper almost quasicontinuous at a point $x \in X$ if for each $\varepsilon > 0$ it holds: $x \in \mathrm{Cl}\left(\mathrm{Int}\,(\mathrm{Cl}\, f^{-1}(-\infty, f(x) + \varepsilon])\right)$.

Thus f is upper almost quasicontinuous at each point $x \in X$ for which $f(x) = \infty$. It is easy to verify that f is upper almost quasicontinuous at $x \in X$ with $f(x) < \infty$ if and only if for each $\varepsilon > 0$ we have $x \in \operatorname{Cl}\Big(\operatorname{Int}\big(\operatorname{Cl} f^{-1}(-\infty, f(x)+\varepsilon)\big)\Big)$. A function is called *upper almost quasicontinuous* if it has this property at each point.

THEOREM 2. *Let f be a function from a topological space X into a uniform space $(Y, \mathscr{V})$. Then:*

(a) *f is almost quasicontinuous at a point $x \in X$ if and only if $w_f(x) = 0$.*

(b) *For each $\varrho \in P_{\mathscr{V}}$ the function $w_{\varrho,f} \colon X \to \mathbb{R} \cup \{\infty\}$ is upper almost quasicontinuous.*

(c) *The set B_f of all points at which f is almost quasicontinuous is of the form*

$$B_f = \bigcap_{\varrho \in P_{\mathscr{V}}} \bigcap_{n=1}^{\infty} D_{\varrho,n},$$

where $D_{\varrho,n+1} \subset D_{\varrho,n} \subset \operatorname{Cl}\big(\operatorname{Int}(\operatorname{Cl} D_{\varrho,n})\big)$ for each $\varrho \in P_{\mathscr{V}}$ and $n \geq 1$.

Proof. (a) Assume that f is almost quasicontinuous at a point $x_0 \in X$ and let us take $\varrho \in P_{\mathscr{V}}$, $\varepsilon > 0$ and $V = B\big(f(x_0), \varrho, \frac{1}{4}\varepsilon\big)$; then $x_0 \in \operatorname{Cl}\Big(\operatorname{Int}\big(\operatorname{Cl} f^{-1}(V)\big)\Big)$. Let us put

$$A = \operatorname{Cl}\Big(\operatorname{Int}\big(\operatorname{Cl} f^{-1}(V)\big)\Big),$$

$$M = \{x_0\} \cup \big[f^{-1}(V) \cap \operatorname{Int}\big(\operatorname{Cl} f^{-1}(V)\big)\big]$$

The set A is semi-open, $x_0 \in M \subset A = \operatorname{Cl} M$ and for any $x', x'' \in M$ we have $\varrho\big(f(x'), f(x'')\big) < \frac{1}{2}\varepsilon$. Thus $\sup_{x',x'' \in M} \varrho\big(f(x'), f(x'')\big) < \varepsilon$, which gives $w_{\varrho,f}(x_0) < \varepsilon$ for any $\varrho \in P_{\mathscr{V}}$, $\varepsilon > 0$. So we have shown $w_f(x_0) = 0$.

Conversely, we suppose $w_f(x_0) = 0$. If W is a neighbourhood of $f(x_0)$, then $V = B\big(f(x_0), \varrho, \varepsilon\big) \subset W$ for some $\varrho \in P_{\mathscr{V}}$, $\varepsilon > 0$. Since $w_{\varrho,f}(x_0) < \varepsilon$ there exist a semi-open set A and a set M with $x_0 \in M \subset A \subset \operatorname{Cl} M$ and $\varrho\big(f(x'), f(x'')\big) < \varepsilon$ for $x', x'' \in M$. This leads to the condition $\varrho\big(f(x), f(x_0)\big) < \varepsilon$ for $x \in M$, i.e. $M \subset f^{-1}(V)$. Hence we obtain $A \subset \operatorname{Cl} M \subset \operatorname{Cl} f^{-1}(V) \subset \operatorname{Cl} f^{-1}(W)$. But A is semi-open, so we have $x_0 \in A \subset \operatorname{Cl}(\operatorname{Int} A) \subset \operatorname{Cl}\Big(\operatorname{Int}\big(\operatorname{Cl} f^{-1}(W)\big)\Big)$, which finishes the proof of (a).

(b) Let $x_0 \in X$, $\varrho \in P_{\mathcal{Y}}$ and $\varepsilon > 0$ be given and let $w_{\varrho,f}(x_0) < \infty$. We can choose a semi-open set A_0 and a set M_0 with $x_0 \in M_0 \subset A_0 \subset \mathrm{Cl}\ M_0$ and $\varrho\big(f(x'), f(x'')\big) < w_{\varrho,f}(x_0) + \frac{1}{2}\varepsilon$ for $x', x'' \in M_0$. Hence for each $x \in M_0$ we have

$$\inf_{M} \sup_{x',x'' \in M} \varrho\big(f(x'), f(x'')\big) < w_{\varrho,f}(x_0) + \varepsilon\,,$$

where the infimum is taken under all sets M with $x \in M \subset A_0 \subset \mathrm{Cl}\,M$. Thus $w_{\varrho,f}(x) < w_{\varrho,f}(x_0) + \varepsilon$ for each $x \in M_0$. Let us put $V = \big(-\infty, w_{\varrho,f}(x_0) + \varepsilon\big)$; then we have $M_0 \subset w_{\varrho,f}^{-1}(V)$. This gives $A_0 \subset \mathrm{Cl}\,M_0 \subset \mathrm{Cl}\,\big(w_{\varrho,f}^{-1}(V)\big)$. Since A_0 is a semi-open set the last inclusions imply $x_0 \in A_0 \subset \mathrm{Cl}\,(\mathrm{Int}\,A_0) \subset \mathrm{Cl}\,\Big(\mathrm{Int}\,\big(\mathrm{Cl}\,w_{\varrho,f}^{-1}(V)\big)\Big)$, which means the upper almost quasicontinuity of $w_{\varrho,f}$ at x_0.

(c) According to the part (a) we have

$$B_f = \{x \in X\colon w_f(x) = 0\} = \bigcap_{\varrho \in P_{\mathcal{Y}}} \{x \in X\colon w_{\varrho,f}(x) = 0\} =$$

$$= \bigcap_{\varrho \in P_{\mathcal{Y}}} \bigcap_{n=1}^{\infty} \{x \in X\colon w_{\varrho,f}(x) < \tfrac{1}{n}\}.$$

Now, applying the part (b) it suffices to take $D_{\varrho,n} = \{x \in X\colon w_{\varrho,f}(x) < \frac{1}{n}\}$. □

Let us observe that analogous results can be formulated for quasicontinuity. A function $f\colon X \to \mathbb{R} \cup \{\infty\}$ is called *upper quasicontinuous at a point* $x \in X$ if for each $\varepsilon > 0$ it holds: $x \in \mathrm{Cl}\,\big(\mathrm{Int}\,f^{-1}(-\infty, f(x) + \varepsilon]\big)$. Then f is upper quasicontinuous at each point $x \in X$ for which $f(x) = \infty$. Furthermore, f is upper quasicontinuous at $x \in X$ with $f(x) < \infty$ if and only if for each $\varepsilon > 0$ we have $x \in \mathrm{Cl}\,\big(\mathrm{Int}\,f^{-1}(-\infty, f(x) + \varepsilon)\big)$. A function is called *upper quasicontinuous* if it has this property at each point [5].

For any function $f\colon X \to Y$ we denote

$$q_{\varrho,f}(x) = \inf_{A} \sup_{x',x'' \in A} \varrho\big(f(x'), f(x'')\big)\,,$$

where the infimum is taken under all semi-open sets A containing x, and

$$q_f(x) = \sup_{q \in P_{\mathcal{Y}}} q_{\varrho,f}(x)\,.$$

PROPOSITION 1. *Let f be a function from a topological space X into a uniform space $(Y, \mathcal{V})$. Then:*

(a) f is quasicontinuous at a point $x \in X$ if and only if $q_f(x) = 0$.

(b) For each $\varrho \in P_{\mathcal{V}}$ the function $q_{\varrho,f} \colon X \to \mathbb{R} \cup \{\infty\}$ is upper quasicontinuous.

The proof is similar to that in Theorem 2, so it is omitted.

A function f from a topological space X into a uniform space $(Y, \mathcal{V})$ is said to be cliquish if for each $x \in X$, $\varrho \in P_{\mathcal{V}}$, $\varepsilon > 0$ and each neighbourhood U of x there exists an open nonempty set $V \subset U$ such that $\varrho\big(f(x'), f(x'')\big) < \varepsilon$ for $x', x'' \in V$, [3].

THEOREM 3. *If f is a cliquish function from a topological space X into a uniform space $(Y, \mathcal{V})$, then $w_{\varrho,f} \le q_{\varrho,f} \le 2w_{\varrho,f}$ for each $\varrho \in P_{\mathcal{V}}$.*

Proof. The inequality $w_{\varrho,f} \le q_{\varrho,f}$ is an immediate consequence of definitions. Now let $x_0 \in X$, $\varrho \in P_{\mathcal{V}}$ and $w_{\varrho,f}(x_0) < \infty$. For $\varepsilon > 0$ we choose a semi-open set A_0 and a set M such that $x_0 \in M \subset A_0 \subset \operatorname{Cl} M$ and

$$\sup_{x',x'' \in M} \varrho\big(f(x'), f(x'')\big) < w_{\varrho,f}(x_0) + \tfrac{1}{4}\varepsilon .$$

Since f is cliquish for each nonempty open set $U \subset A_0$ there exists a nonempty open set $V_U \subset U$ with

$$\sup_{x',x'' \in V_U} \varrho\big(f(x'), f(x'')\big) < \tfrac{1}{4}\varepsilon .$$

Thus for any $x \in V_U$ and $x_1 \in V_U \cap M$ we have

$$\varrho\big(f(x), f(x_0)\big) \le \varrho\big(f(x), f(x_1)\big) + \varrho\big(f(x_1), f(x_0)\big) < \tfrac{1}{2}\varepsilon + w_{\varrho,f}(x_0) .$$

Let us put

$$V = \bigcup \{V_U \colon U \subset A_0,\ U \text{ is open}\}$$

and

$$A_1 = V \cup \{x_0\} .$$

Then the set V is dense in A_0 and A_1 is semi-open. Furthermore $\varrho\big(f(x), f(x_0)\big) < w_{\varrho,f}(x_0) + \frac{1}{2}\varepsilon$ for each $x \in V$, which implies

$$\sup_{x',x'' \in A_1} \varrho\big(f(x'), f(x'')\big) < 2w_{\varrho,f}(x_0) + \varepsilon .$$

Hence we have $q_{\varrho,f}(x) \le 2w_{\varrho,f}(x_0)$ and this completes the proof. □

The above theorem together with Theorem 2 and Proposition 1 gives the following corollaries:

COROLLARY 1. *If f is a cliquish function from a topological space X into a uniform space $(Y, \mathcal{V})$ then the set B_f coincides with the set of all points at which f is quasicontinuous.*

COROLLARY 2. *A function f with values in a uniform space is quasicontinuous if and only if it is cliquish and almost quasicontinuous.*

Let us remark that in the case of a metric space Y the last corollary makes the result presented in [1].

For a function $f\colon X \to Y$, a point $x \in X$, any neighbourhood U of x and for $\varrho \in P_{\mathcal{V}}$ let us put:

$$\omega_{\varrho,f}(x,U) = \inf_G \inf_M \sup_{Z \in M} \varrho\big(f(x), f(z)\big),$$

where infimums are taken under all nonempty open sets $G \subset U$ and all sets M satisfying $M \subset G \subset \mathrm{Cl}\ M$ respectively, and

$$\omega_{\varrho,f}(x) = \sup\{\omega_{\varrho,f}(x,U)\colon U \quad \text{is a neighbourhood of } x\},$$

$$\omega_f(x) = \sup_{\varrho \in P_{\mathcal{V}}} \omega_{\varrho,f}(x).$$

The function ω_f is introduced in [10] for functions f with values in a metric space Y. It is also shown that $B_f = \{x \in X\colon \omega_f(x) = 0\}$, [10, Th. 3.1].

THEOREM 4. *Let X be a topological space and let $(Y, \mathcal{V})$ be a uniform space. Then for any function $f\colon X \to Y$ we have $\omega_{\varrho,f} \le w_{\varrho,f} \le 2\,\omega_{\varrho,f}$ for each $\varrho \in P_{\mathcal{V}}$.*

Proof. Let $x_0 \in X$, $\varrho \in P_{\mathcal{V}}$ and $w_{\varrho,f}(x_0) < \infty$. Then for each $\varepsilon > 0$ we can choose a semi-open set A and a set M_1 such that $x_0 \in M_1 \subset A \subset \mathrm{Cl}\ \ M_1$ and

$$\sup_{x',x'' \in M_1} \varrho\big(f(x'), f(x'')\big) < w_{\varrho,f}(x_0) + \varepsilon.$$

For any neighbourhood U of x_0 we put $G = U \cap \mathrm{Int}\, A$ and $M_2 = G \cap M_1$. Then G is a nonempty open set, $M_2 \subset G \subset \mathrm{Cl}\ \ M_2$ and

$$\sup_{x \in M_2} \varrho\big(f(x_0), f(x)\big) < w_{\varrho,f}(x_0) + \varepsilon.$$

From this we obtain $\omega_{\varrho,f}(x_0, U) < w_{\varrho,f}(x_0) + \varepsilon$ for any neighbourhood U of x_0 and in the consequence

$$\text{if} \quad w_{\varrho,f}(x_0) < \infty, \quad \text{then} \quad \omega_{\varrho,f}(x_0) \le w_{\varrho,f}(x_0). \tag{1}$$

Now, let $\omega_{\varrho,f}(x_0) < \infty$ and let U be an established neighbourhood of x_0. It follows from the definition of $\omega_{\varrho,f}$ that for each neighbourhood V of x_0 there exists a nonempty open set $G_V \subset V$ and a set M_V with $M_V \subset G_V \subset \mathrm{Cl}\ M_V$ and

$$\sup_{z \in M_V} \varrho(f(x_0), f(z)) < \omega_{\varrho,f}(x_0) + \tfrac{1}{2}\varepsilon .$$

We put

$$A = \{x_0\} \cup \bigcup \{G_V \colon V \text{ is a neighbourhood of } x_0 \text{ and } V \subset U\};$$

$$M = \{x_0\} \cup \bigcup \{M_V \colon V \text{ is a neighbourhood of } x_0 \text{ and } V \subset U\}.$$

Then the set A is a semi-open, $x_0 \in M \subset A \subset \mathrm{Cl}\ M$ and

$$\varrho(f(x'), f(x'')) < 2\,\omega_{\varrho,f}(x_0) + \varepsilon \quad \text{for} \quad x', x'' \in M .$$

The last inequality leads to

$$\text{if} \quad \omega_{\varrho,f}(x_0) < \infty, \quad \text{then} \quad w_{\varrho,f}(x_0) \leq 2\,\omega_{\varrho,f}(x_0) . \tag{2}$$

Futhermore, it follows from (1) and (2) that $w_{\varrho,f}(x_0) = \infty$ if and only if $\omega_{\varrho,f}(x_0) = \infty$, so the proof is completed. □

In the above theorem none of the inequalities can be replaced by the equality as the following shows.

EXAMPLE 1. Let $\mathbb{R}$ be the space of real numbers with the natural topology and $\{r_{0,n} \colon n \geq 1\}$ a sequence of irrational numbers which is dense in $\mathbb{R}$ and $r_{0,n} - r_{0,m}$ is not rational for $n, m \geq 1$, $n \neq m$. Then we assume

$$r_{j,n} = r_{0,n} + \tfrac{1}{j} \qquad \text{for} \quad n, j \geq 1;$$

$$X = \{r_{j,n} \colon\ j \geq 0,\ n \geq 1\} \cup \{0\};$$

$$A_j = \{r_{j,n} \colon\ n \geq 1\} \qquad \text{for} \quad j \geq 0 .$$

We will consider X as a subspace of $\mathbb{R}$, thus each of the sets A_j is dense in X. Now let us take the space l^2 with the usual norm and the standard base $\{e_n \colon n \geq 1\}$; and $\Theta = \{0, 0, \dots\} \in l^2$. We define the function $f \colon X \to l^2$ by

$$f(x) = \begin{cases} \Theta, & \text{if } x = 0 , \\ e_k, & \text{if } x = r_{j,n} , \quad \text{for } j \geq 0,\ n \geq 1,\ n + j = k . \end{cases}$$

For any point $x \in X$, $x \neq 0$ it holds $\|f(x) - f(0)\| = 1$, so $\omega_f(0) = 1$. On the other hand, if A is a semi-open set in X and M satisfies $0 \in M \subset A \subset \mathrm{Cl}\, M$, then $f(M)$ is an infinite set. Hence we have $\sup\limits_{x',x'' \in M} \|f(x') - f(x'')\| = \sqrt{2}$ which gives $w_f(0) = \sqrt{2}$.

Let X be a topological space and $(Y, \mathscr{V})$ a uniform one. A net $\{f_s : s \in S\}$ of functions $f_s : X \to Y$ is called almost uniformly convergent to a function $f : X \to Y$ if for each $x \in X$, $\varepsilon > 0$, $\varrho \in P_{\mathscr{V}}$ there exists a neighbourhood U of x and $s_0 \in S$ such that $\varrho\big(f_s(z), f(z)\big) < \varepsilon$ for any $z \in U$, $s \in S$, $s \geq s_0$, [4].

In the sequel we will consider $\mathbb{R} \cup \{\infty\}$ with the generalized metric d given by $d(x, y) = |x - y|$, however we assume

$$-\infty + \infty = \infty - \infty = 0 \quad \text{and} \quad |\pm \infty| = \infty .$$

Theorem 5. *Let X be a topological space and let $(Y, \mathscr{V})$ be a uniform space. If a net $\{f_s : s \in S\}$ of functions $f_s : X \to Y$ almost uniformly converges to a function $f : X \to Y$, then for each $\varrho \in P_{\mathscr{V}}$ the net $\{w_{\varrho, f_s} : s \in S\}$ is almost uniformly convergent to $w_{\varrho, f}$.*

Proof. Let $x_0 \in X$, $\varepsilon > 0$ and $\varrho \in P_{\mathscr{V}}$ be established. The almost uniform convergence implies the existence of a neighbourhood U of x_0 and $s_0 \in S$ such that

$$\varrho\big(f_s(x), f(x)\big) < \tfrac{1}{4}\varepsilon \quad \text{for} \quad s \geq s_0,\ x \in U . \tag{3}$$

We establish a point $x \in U$, then if $w_{\varrho,f}(x) < \infty$ we have

$$\inf_A \inf_M \sup_{x',x'' \in M} \varrho\big(f(x'), f(x'')\big) < w_{\varrho,f}(x) + \tfrac{1}{4}\varepsilon .$$

So we can choose a semi-open set $A_1 \subset U$ and a set M_1 with $x \in M_1 \subset A_1 \subset \mathrm{Cl}\, M_1$ and

$$\varrho\big(f(x')\, f(x'')\big) < w_{\varrho,f}(x) + \tfrac{1}{4}\varepsilon \quad \text{for} \quad x', x'' \in M_1 . \tag{4}$$

Thus from (3) and (4), for any $x', x'' \in M_1$ and $s \geq s_0$ we obtain

$$\varrho\big(f_s(x'), f_s(x'')\big) \leq \varrho\big(f_s(x'), f(x')\big) + \varrho\big(f(x'), f(x'')\big) + \varrho\big(f(x''), f_s(x'')\big) <$$
$$< w_{\varrho,f}(x) + \tfrac{3}{4}\varepsilon ,$$

hence

$$\inf_M \sup_{x',x'' \in M} \varrho\big(f_s(x'), f_s(x'')\big) \leq w_{\varrho,f}(x) + \tfrac{3}{4}\varepsilon \quad \text{for} \quad s \leq s_0 ,$$

where the infimum is taken under all sets M satisfying $x \in M \subset A_1 \subset \mathrm{Cl}\, M$. The last implies $w_{\varrho,f_s}(x) \le w_{\varrho,f}(x) + \frac{3}{4}\varepsilon$ for $s \ge s_0$, so we have shown

$$w_{\varrho,f_s}(x) - w_{\varrho,f}(x) < \varepsilon \qquad \text{for any} \quad x \in U,\ s \ge s_0 . \tag{5}$$

Similarly, for any $x \in U$ and $s \ge s_0$ there exist a semi-open set $A_s \subset U$ and a set M_s such that $x \in M_s \subset A_s \subset \mathrm{Cl}\ \ M_s$ and

$$\varrho\big(f_s(x'), f_s(x'')\big) < w_{\varrho,f_s}(x) + \tfrac{1}{4}\varepsilon \qquad \text{for} \quad x', x'' \in M_s .$$

This inequality and (3) give

$$\varrho\big(f(x'), f(x'')\big) < w_{\varrho,f_s}(x) + \tfrac{3}{4}\varepsilon \qquad \text{for} \quad x', x'' \in M_s;\ s \ge s_0 .$$

From this it follows $w_{\varrho,f}(x) < w_{\varrho,f_s}(x) + \varepsilon$ for $x \in U$ and $s \ge s_0$. Thus, in virtue of (5) we have $|w_{\varrho,f}(x) - w_{\varrho,f_s}(x)| < \varepsilon$ for any $x \in U$, $s \ge s_0$. Now, if $w_{\varrho,f}(x) = \infty$, we have

$$\sup_{x',x'' \in M} \varrho\big(f(x'), f(x'')\big) > n + \varepsilon$$

for each integer n, any semi-open set A containing x and each M satisfying $x \in M \subset A \subset \mathrm{Cl}\, M$. Then for $s \ge s_0$ we have

$$\sup_{x',x'' \in M} \varrho\big(f_s(x'), f_s(x'')\big) > n$$

for any n, A, m as above. From this we obtain $w_{\varrho,f_s}(x) = \infty$ for every $s \ge s_0$, which completes the proof. □

Theorem 6. *Let X be a topological space and let $(Y, \mathscr{V})$ be a uniform space. If a net $\{f_s \colon s \in S\}$ of functions $f_s \colon X \to Y$ uniformly converges to a function $f \colon X \to Y$, then for each $\varrho \in P_{\mathscr{V}}$ the net $\{w_{\varrho,f_s} \colon s \in S\}$ is uniformly convergent to $w_{\varrho,f}$.*

The proof is exactly as in Theorem 5 because in this case we have (3) satisfied for each $x \in X$.

Corollary 3. *Let X be a locally compact space and let $(Y, \mathscr{V})$ be a uniform space. If a net $\{f_s \colon s \in S\}$ of functions $f_s \colon X \to Y$ converges uniformly on compact sets to a function $f \colon X \to Y$, then for each $\varrho \in P_{\mathscr{V}}$ the net $\{w_{\varrho,f_s} \colon s \in S\}$ converges uniformly on compact sets to the functions $w_{\varrho,f}$.*

Proof. If X is a locally compact space, then according to [4, Th. 2.5] the almost uniform convergence coincides with the uniform on compact sets convergence. Thus the conclusion is an immediate consequence of Theorem 5. □

Theorems 5, 6 and 2(a) imply

COROLLARY 4. *Let X be a topological space, $(Y, \mathscr{V})$ a uniform one and let $\{f_s\colon s \in S\}$ be a net of almost quasicontinuous functions $f_s\colon X \to Y$.*

(a) *If the net $\{f_s\colon s \in S\}$ almost uniformly (uniformly) converges to a function $f\colon X \to Y$, then f is almost quasicontinuous.*

(b) *If X is a locally compact space and the net $\{f_s\colon s \in S\}$ converges to f uniformly on compact sets, then f is almost quasicontinuous.*

Finally, we will use the symbol $F(X,Y)$ to denote the space of all functions from X into Y equipped with the topology of the uniform convergence on compact sets and $AQ^+(X,\mathbb{R})$ the space of all upper almost quasicontinuous real functions with the same topology.

COROLLARY 5. *Let X be a locally compact space and (Y,d) a metric one. Then the function $\Psi\colon F(X,Y) \to AQ^+(X,\mathbb{R})$ given by $\Psi(f) = w_f$ is continuous.*

In virtue of Theorem 6 the result analogous to Corollary 5 can be formulated for $F(X,Y)$ and $AQ^+(X,\mathbb{R})$ equipped with the topology of the uniform convergence (without the assumption of the local compactness of X).

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