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ON SOME EXTENSIONS OF ALMOST CONTINUOUS FUNCTIONS AND OF CONNECTIVITY FUNCTIONS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. Let $\hat{f}$ be a real function defined on $\mathbb{R}^2$ by the formula $\hat{f}(x,y) = f(x)$ where f is a real function defined on $\mathbb{R}$. We prove the following theorems: The function $\hat{f}$ is almost continuous if and only if f is continuous. The function $\hat{f}$ is a connectivity function if and only if f is continuous.

Consider the following topological properties that a function f from one topological space, X, to another, Y, may have:

f is a *connectivity function* if for every connected subset E of X the restriction of f to E has the connected graph.

f is *Darboux* if for every connected subset E of X, $f(E)$ is a connected subset of Y.

f is *almost continuous* (in the sense of Stallings [5]) if for every open set $G \subset X \times Y$ containing the graph of f, there exists a continuous function $g\colon X \to Y$ with the graph lying entirely in G.

Let $C(X)$ denote the set of all continuous real functions defined on X, $Cn(X)$ the set of all real connectivity functions defined on X, $D(X)$ the set of all real Darboux functions defined on X, $Ac(X)$ the set of all real almost continuous functions defined on X and $B_1(X)$ the set of all real functions of Baire's first class. Let $\Gamma(f,X)$ denote the graph of f.

It is known that $B_1(\mathbb{R}) \cap D(\mathbb{R}) = B_1(\mathbb{R}) \cap Cn(\mathbb{R})$ (see [3]) and $B_1(\mathbb{R}) \cap D(\mathbb{R}) = B_1(\mathbb{R}) \cap Ac(\mathbb{R})$ [1], but $B_1(\mathbb{R}^2) \cap D(\mathbb{R}^2) \neq B_1(\mathbb{R}^2) \cap Cn(\mathbb{R}^2)$ [2], and $B_1(\mathbb{R}^2) \cap D(\mathbb{R}^2) \neq B_1(\mathbb{R}^2) \cap Ac(\mathbb{R}^2)$ [4].

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The authors of papers [2] and [4] made use of the function $\hat{s}$ of two variables defined by $\hat{s}(x,y) = s(x)$ where $s(0) = 0$ and $s(x) = \sin x^{-1}$ for $x \neq 0$. They proved that $\hat{s} \notin Cn(\mathbb{R}^2) \cup Ac(\mathbb{R}^2)$. Obviously $\hat{s} \notin B_1(\mathbb{R}^2) \cap D(\mathbb{R}^2)$.

The function $\hat{s}$ is an extension of the function s defined on $\mathbb{R}$ to the function defined on $\mathbb{R}^2$. Generally: Let $f\colon \mathbb{R} \to \mathbb{R}$ and $\hat{f}(x,y) = f(x)$. Let F be a family of real functions of one real variable. Denote $\hat{F} = \{\hat{f}\colon f \in F\}$. The operation $\Phi\colon F \to \hat{F}$ defined by $\Phi(f) = \hat{f}$ preserves many properties of functions belonging to F. The continuity, quasi-continuity, approximative continuity, Darboux property, Baire class of f implies continuity, quasi-continuity and so on of $\hat{f}$. The results of [2] and [4] show that almost continuity and connectivity of functions are exceptions.

The aim of this note is to describe the family of all function $f\colon \mathbb{R} \to \mathbb{R}$, such that $\hat{f}$ is almost continuous (Theorem 1) or a connectivity function (Theorem 2).

THEOREM 1. *The function $\hat{f}$ is almost continuous if and only if the function f is continuous.*

P r o o f. We shall prove that $\{f\colon \hat{f} \in Ac(\mathbb{R}^2)\} = C(\mathbb{R})$. Let $f \in C(\mathbb{R})$. Then $\hat{f} \in C(\mathbb{R}^2) \subset Ac(\mathbb{R}^2)$. Thus

$$\{f\colon \hat{f} \in Ac(\mathbb{R}^2)\} \supset C(\mathbb{R}). \tag{1}$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the condition $\hat{f} \in Ac(\mathbb{R}^2)$. We shall prove that $f \in C(\mathbb{R})$. Suppose this is not true and let x_0 be a point of discontinuity of f. Then there exists a sequence $\{x_n\}$ of points such that $f(x_n)$ converges to $z_0 \neq f(x_0)$. z_0 may be equal to $\pm\infty$. Write $\mathrm{d} = \min(1, |f(x_0) - z_0|)$. There exists a positive number δ_1 such that

$$|\hat{f}(x_0,y) - \hat{f}(x_k,y)| \geq d - 10^{-1}d \tag{2}$$

for all $|x_k - x_0| < \delta_1$ and $y \in \mathbb{R}$.

Define open sets A and B by

$$A = \{(x,y,z)\colon |x - x_0| < d \cdot 10^{-1}e^{-y^2},\ |z - f(x_0)| < d \cdot 10^{-1}e^{-y^2}\},$$

$$F = \{(x,y,z)\colon x = x_0\} \cup \bigcup_{n=1}^{\infty}\{(x,y,z)\colon x = x_n,\ |z - f(x_n)| \geq 10^{-1}d\},$$

$$B = \mathbb{R}^3 \setminus F.$$

The set F is closed. Thus $A \cup B$ is open. It is clear that $\Gamma(\hat{f},\mathbb{R}^2) \subset A \cup B$. Because of almost continuity of f there exists a continuous function $g\colon \mathbb{R}^2 \to \mathbb{R}$

such that $\Gamma(g,\mathbb{R}^2) \subset A \cup B$. No point (x_0, y, z) belongs to B. Thus $\big(x_0, 0, g(x_0, 0)\big)$ belongs to $\Gamma(g,\mathbb{R}^2) \cap A$. Hence by the continuity of g there exists a positive number δ_2 such that $\big(x, 0, g(x, 0)\big) \in \Gamma(g, \mathbb{R}^2) \cap A$ for all $x \in (x_0-\delta_2, x_0+\delta_2)$. Let $|x_0 - x_k| < \delta = \min(\delta_1, \delta_2)$. Then $\big(x_k, 0, g(x_k, 0)\big) \in A$. Denote $L_k = \{(x_k, y) \colon y \in \mathbb{R}\}$, $P_k = \{(x_k, y, z) \colon y, z \in \mathbb{R}\}$, $A_k = A \cap P_k$ and $B_k = B \cap P_k$. The sets A_k and B_k are disjoint, open subsets of P_k. Indeed. Because of (2) and the definition of A the distance between A_k and B_k is greater or equal than $d - 3\cdot 10^{-1}d$. The function $g \,|\, L_k$ is a continuous function defined on L_k. The graph of this function $\Gamma(g \,|\, L_k, L_k)$ is contained in $P_k \cap (A \cup B) = A_k \cup B_k$, and is a connected subset of P_k. Take a number y_k which satisfies the inequality $|x_k - x_0| \geq d\cdot 10^{-1} e^{-y^2} k$. Then $(x_k, y_k, z) \notin A$ for all $z \in \mathbb{R}$. Thus $\big(x_k, y_k, g(x_k, y_k)\big) \notin A$. Hence $\big(x_k, y_k, g(x_k, y_k)\big) \in B$. This implies that $\big(x_k, y_k, g(x_k, y_k)\big)$ is a point of $\Gamma(g \,|\, L_k, L_k) \cap B_k$. The point $\big(x_k, 0, g(x_k, 0)\big)$ belongs to $\Gamma(g \,|\, L_k, L_k) \cap A_k$. Also $\Gamma(g \,|\, L_k, L_k) = \big(\Gamma(g \,|\, L_k, L_k) \cap A_k\big) \cup \big(\Gamma(g \,|\, L_k, L_k) \cap B_k\big)$ where $\Gamma(g \,|\, L_k, L_k) \cap A_k \neq \emptyset$, $\Gamma(g \,|\, L_k, L_k) \cap B_k \neq \emptyset$ and A_k, B_k are disjoint open sets of P_k. This contradicts the fact that the graph of a continuous function defined on the line L_k is a connected set.

Therefore if f is discontinuous then f cannot be almost continuous. Thus $\{f \colon \hat{f} \in Ac\mathbb{R}^2\} \subset C(\mathbb{R})$. This and [1] complete the proof. □

THEOREM 2. *The function $\hat{f}$ is a connectivity function if and only if the function f is continuous.*

P r o o f. We shall prove that $\{f \colon \hat{f} \in Cn(\mathbb{R}^2)\} = C\{\mathbb{R}\}$. Let $f \in C(\mathbb{R})$. Then $\hat{f} \in C(\mathbb{R}^2) \subset Cn(\mathbb{R}^2)$. Thus

$$\{f \colon \hat{f} \in Cn(\mathbb{R}^2)\} \supset C(\mathbb{R}). \tag{3}$$

Let $\hat{f} \in Cn(\mathbb{R}^2)$. We shall prove that $f \in C(\mathbb{R})$. Suppose this is not true and let x_0 be a point of discontinuity of f. Then there exists a sequence $\{x_n\}$ such that $x_n \neq x_0$, $\lim\limits_{n\to\infty} f(x_n) \neq f(x_0)$ and $\lim\limits_{n\to\infty} x_n = x_0$.

Put $E = \{(x,y) \colon y = 0\} \cup \{(x_0, 1)\} \cup \bigcup\limits_{n=1} \{(x,y) \colon x = x_n\}$. The set E is a connected subset of $\mathbb{R}^2$. The point $\big(x_0, 1, \hat{f}(x_0, 1)\big)$ is an isolated point of $\Gamma(\hat{f} \,|\, E, E)$. So $\Gamma(\hat{f} \,|\, E, E)$ is not connected. Thus $\hat{f}$ is not a connectivity function. Therefore if f is discontinuous then $\hat{f} \notin Cn(\mathbb{R}^2)$. This and (3) complete the proof. □

It is easy to prove that for any $f \colon \mathbb{R} \to \mathbb{R}$ belonging to $Ac(\mathbb{R})$ the function $\tilde{f} \colon \mathbb{R} \times [0,1] \to \mathbb{R}$, defined by $\tilde{f}(x,y) = f(x)$, belongs to $Ac(\mathbb{R} \times [0,1])$. But

$\hat{f} \subset Cn(\mathbb{R} \times [0,1])$ if and only if $f \in C(\mathbb{R})$. To prove this statement we replace $\hat{f}$ and $\mathbb{R}^2$ by $\tilde{f}$ and $\mathbb{R} \times [0,1]$ in the proof of Theorem 2.

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