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ON INTEGRATION WITH RESPECT TO OPERATOR VALUED MEASURES IN RIESZ SPACES

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. An operator valued measure is considered assigning to every Borel set (in a compact space T) a linear, positive, order continuous operator from a Riesz space X to another Riesz space Y. A Kurzweil type construction is used for integrating functions from T to X.

Introduction

If X, Y are linear spaces and $L(X,Y)$ is a set of linear operators and $(T,\mathcal{S})$ is a measurable space, then an operator valued measure is a mapping $\mu:\mathcal{S}\to L(X,Y)$ satisfying some conditions. If $f:T\to X$ is a simple function

$$f=\sum_{i=1}^{n}\chi_{E_i}x_i\,,$$

then

$$\int f\,d\mu=\sum_{i=1}^{n}\mu(E_i)(x_i)\,,$$

where $\mu(E_i)\in L(X,Y)$, hence $\mu(E_i)(x_i)$ is the value of $\mu(E_i)$ in the element $x_i\in X$. Therefore $\int f\,d\mu\in Y$. The problem is how to extend this integral to a larger family of functions $f:T\to X$.

The problem of operator valued measure has been studied in a series of papers by I. Dobrakov (see [1]) and for locally convex spaces by J. Haluška ([3]). In [4] J. Haluška considered the case of Banach lattices. The basic property is the weak σ-distributivity, which is a necessary and sufficient condition for extendability of Y-valued measures and integrals ([15]).

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To solve the problem stated above we shall use the Henson-Kurzweil construction of an integral. Of course, since we work with a Borel regular measure on the σ-algebra $\mathcal{S}$ of subsets of a compact space T, the obtained integral is of the Lebesgue type. Another solution of the problem is contained in [11].

Special cases of the studied theory are the case of vector measure P with scalar functions $f : T \to R$ (where $X = R$, $\mu(E)(x) = xP(E) \in Y$) and scalar measure P with vector functions $f : T \to X$ (where $X = Y$, $\mu(E) = P(E)x$). As a special case some results of [9] and [13] can be received.

Assumptions

A. Let T be a (Hausdorff) compact topological space and let $\mathcal{S}$ be the σ-algebra generated by the family of all compact (all open) subsets of T.

Let $U(T)$ be the set of all functions $\delta : T \to 2^T$ such that $\delta(t)$ is a neighbourhood of t for every $t \in T$ and let $A(\delta)$ be the set of all partitions D of T such that $D = \{(E_1, t_1), (E_2, t_2), \dots, (E_n, t_n)\}$, where $E_i \in \mathcal{S}$, $t_i \in \overline{E}_i$, $E_i \subset \delta(t_i)$ $(i = 1, 2, \dots, n)$, and $\bigcup_{i=1}^{n} E_i = T$.

Lemma 1. *The set $A(\delta)$ is non-empty for every $\delta \in U(T)$.*

Proof. Let $\delta \in U(T)$. Then for every $t \in T$ there exists an open set $U(t)$ ($U(t) \in \mathcal{S}$) such that $t \in U(t) \subset \delta(t)$. The set $\{U(t); t \in T\}$ is an open covering of T, hence there is a finite open covering $U(t_1), U(t_2), \dots, U(t_n)$ chosen from $\{U(t), t \in T\}$.

Now, choose pairwise disjoint neighbourhoods $F(t_i) \in \mathcal{S}$ $(i = 1, 2, \dots, n)$ and put

$$F = \bigcup_{i=1}^{n} F(t_i),$$

$$E_1 = (U(t_1) \setminus F) \cup F(t_1),$$

$$E_i = \left(U(t_i) \setminus \bigcup_{j=1}^{i-1} E_j \setminus F \right) \cup F(t_i) \qquad \text{for } i = 2, 3, \dots, n.$$

Then $E_i \cap E_j = \emptyset$ $(i \neq j)$, $\bigcup_{i=1}^{n} E_i = T$, $E_i \in \mathcal{S}$, $E_i \subset U(t_i) \subset \delta(t_i)$ and $t_i \in E_i$ for $i = 1, 2, \dots, n$. □

Remark 2. If $E \subset T$, $E \in \mathcal{S}$, then $\overline{E}$ is compact. Let $U'(t_1), \dots, U'(t_n)$ be a finite covering of $\overline{E}$ chosen from $\{U'(t);\, U'(t) \subset \delta(t),\, t \in \overline{E}\}$ and let $F'(t_i) \in \mathcal{S}$ $(i = 1, 2, \dots, n)$ be pairwise disjoint neighbourhoods of t_i.

Put $U(t_i) = U'(t_i) \cap E$, $F(t_i) = F'(t_i) \cap E$ (i = 1,2, ... ,n). The partition $D \in A(\delta/E)$ $(\delta \in U(T)$ or $\delta \in U(\overline{E}))$ can be constructed by the same way as in the proof of the preceding lemma (now, $t_i \in \overline{E}_i$ $(i = 1, 2, \dots, n)$).

B. We shall assume that X and Y are linear lattices, the linear lattice X is boundedly σ-complete, i.e., every bounded sequence $(a_i)_i \subset X$ has the supremum $\bigvee_i a_i$, the linear lattice Y is weakly σ-distributive, i.e., Y is boundedly σ-complete and for every bounded double sequence $(a_{ij})_{i,j} \subset Y$ such that $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \dots)$ there is

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_i a_{i\varphi(i)} = 0 .$$

Lemma 3. *Let Y be a boundedly σ-complete linear lattice, $(a_{nij})_{n,i,j}$ be a triple bounded sequence of elements of Y such that $a_{nij} \searrow 0$ $(j \to \infty,\ n, i = 1, 2, \dots)$. Then to every $a \in X$, $a > 0$ there is a bounded double sequence $(a_{ij})_{i,j} \subset Y$ such that $a_{ij} \searrow 0$, $(j \to \infty,\ i = 1, 2, \dots)$ and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$*

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{ni\varphi(i+n)} \right) \le \bigvee_{i=1}^{\infty} a_{i\varphi(i)} .$$

Proof. See [2] and [12]. □

If $x_n, x \in X$ then $x_n \to x$ (x_n converges to x with respect to the ordering) iff there exists $(a_n)_n \subset X$, $a_n \searrow 0$ and $|x_n - x| \le a_n$ for all n.

It is possible to prove that a sequence $(x_n)_n \subset X$ converges to $x \in X$ iff $(x_n)_n$ is bounded and

$$x = \bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} x_i = \bigvee_{n=1}^{\infty} \bigwedge_{i=n}^{\infty} x_i .$$

We say that $f_n \to f$ *uniformly* $(f_n, f : T \to X)$ iff there exists $(a_n)_n \subset X$, $a_n \searrow 0$ such that

$$|f_n(t) - f(t)| \leqq a_n$$

for every $t \in T$ and all n.

C. By $L(X,Y)$ we shall denote the set of all σ-homomorphisms from X to Y i.e. such mappings $h: X \to Y$ that

(i) $h(x_1 + x_2) = h(x_1) + h(x_2)$ for every $x_1, x_2 \in X$;

(ii) $h(cx) = c\,h(x)$ for every $x \in X$ and $c \in R$;

(iii) If $(x_n)_n \subset X$, $x_n \searrow 0$ then $h(x_n) \searrow 0$.

The properties (i) and (iii) imply $h(0) = 0$ and $h(x_1) \leqq h(x_2)$ for $x_1, x_2 \in X$, $x_1 \leqq x_2$ (that is h is a positive operator).

D. Let $\mu: \mathcal{S} \to L(X,Y)$ be an operator valued measure having the following properties:

(i) If $E \in \mathcal{S}$, $x \in X$, $x \geqq 0$ then $\mu(E)x \geqq 0$;

(ii) If $E_n \in \mathcal{S}$ $(n = 1, 2, \dots, k)$, $E_i \cap E_j = 0$ $(i \neq j)$, then $\mu\left(\bigcup_{n=1}^{k} E_n\right)x = \sum_{n=1}^{k} \mu(E_n)x$ for every $x \in X$.

(iii) μ is regular in the following sense:
For every set $E \in \mathcal{S}$ and every $x \in X$, $x \geqq 0$ there exists a bounded sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geqq 0$, $a_{nk} \searrow 0$ $(k \to \infty,\ n = 1, 2, \dots)$ and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a compact set F and open set U, $F \subset E \subset U$ such that

$$\mu(U \setminus F)x < \bigvee_i a_{i\varphi(i)}.$$

Example. Let $\mu_1 : \mathcal{S} \to R$ be a regular Borel measure. Let $X = Y$. Put $\mu(E)x = \mu_1(E) \cdot x$ for $E \in \mathcal{S}$ and $x \in X$. Then $\mu: \mathcal{S} \to L(X,X)$ is the regular operator valued measure. For $E \in \mathcal{S}$ and $x \in X$, $x \geqq 0$ it is sufficient to put $a_{ij} = \frac{1}{j}x$ $(i, j = 1, 2 \dots)$. Then for $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a compact set F and an open set U, $F \subset E \subset U$ such that

$$\mu(U \setminus F)x = \mu_1(U \setminus F) \cdot x < \frac{1}{\min_i \varphi(i)} x = \bigvee_i a_{i\varphi(i)}.$$

The proof of the following lemma is evident.

Lemma 4. *Let $\mu: T \to L(X,Y)$ be a regular operator valued measure. Then*

(i) *μ is monotone i.e. if $E, F \in \mathcal{S}$, $E \subset F$, then $\mu(E)x \leqq \mu(F)x$ for every $x \in X$, $x \geqq 0$;*

(ii) *μ is subtractive, i.e., if $E, F \in \mathcal{S}$, $E \subset F$, then $\mu(F \setminus E)x = \mu(F)x - \mu(E)x$ for every $x \in X$.*

Integral

If $\delta \in U(T)$ and $D \in A(\delta)$, $D = \{(E_1, t_1), \dots, (E_n, t_n)\}$ then for the function $f : T \to X$ we define

$$S(f, D) = \sum_{i=1}^{n} \mu(E_i)\, f(t_i)\,,$$

where $\mu(E_i)\, f(t_i)$ is the value of the operator $\mu(E_i)$ in $f(t_i)$.

DEFINITION 5. The function $f : T \to X$ is *integrable* if there exists $y \in Y$ and a bounded double sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geqq 0$, $a_{nk} \searrow 0$ ($k \to \infty$, $n = 1, 2, \dots$) such that for every $\varphi : \mathbb{N} \to \mathbb{N}$ there exists $\delta \in U(T)$ so that

$$|S(f, D) - y| < \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \qquad \text{for any } D \in A(\delta)\,.$$

LEMMA 6. *The integral of f is defined uniquely.*

Proof. Let $y_1, y_2 \in Y$ be two elements satisfying the conditions of the preceding definition. Then there exist $(a_{nk})_{n,k}$, $(b_{nk})_{n,k} \subset Y$, $a_{nk} \geqq 0$, $b_{nk} \geqq 0$, $a_{nk} \searrow 0$, $b_{nk} \searrow 0$ ($k \to \infty$, $n = 1, 2, \dots$) and for $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist $\delta_1, \delta_2 \in U(T)$ such that

$$|S(f, D_1) - y_1| < \bigvee_i a_{i\varphi(i+1)}\,, \qquad |S(f, D_2) - y_2| < \bigvee_i b_{i\varphi(i+2)}$$

for $D_1 \in A(\delta_1)$, $D_2 \in A(\delta_2)$. Put $\delta = \delta_1 \cap \delta_2$ ($\delta(t) = \delta_1(t) \cap \delta_2(t)$ for $t \in T$) and take $D \in A(\delta)$ (then $D \in A(\delta_1) \cap A(\delta_2)$, too). Then

$$|y_1 - y_2| \leqq |y_1 - S(f, D)| + |S(f, D) - y_2| <$$

$$< \bigvee_i a_{i\varphi(i+1)} + \bigvee b_{i\varphi(i+2)} \leqq \bigvee_i c_{i\varphi(i)}\,,$$

where $(c_{nk})_{n,k} \subset Y$ is bounded, $c_{nk} \geqq 0$, $c_{nk} \searrow 0$ ($k \to \infty$, $n = 1, 2, \dots$). The sequence $(c_{nk})_{n,k}$ exists by Lemma 3. From the weak σ-distributivity of Y we have

$$|y_1 - y_2| \leqq \bigwedge_{\varphi} \bigvee_i c_{i\varphi(i)} = 0$$

and hence $y_1 = y_2$ $(= \int f \, d\mu)$. □

THEOREM 7. *If $f, g : T \to X$ are integrable and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable and*

$$\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu .$$

P r o o f . We shall prove that if f, g are integrable and $c \in \mathbb{R}$, then $f+g$, cf are integrable too and

$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu , \qquad \int cf\, d\mu = c \int f\, d\mu .$$

If f is integrable, then there exist $y_1 \in Y$ and bounded double sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geqq 0$, $a_{n,k} \searrow 0$ ($k \to \infty$, $n = 1,2,\dots$) such that for $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\delta_1 \in U(T)$ such that

$$|y_1 - S(f, D_1)| < \bigvee_i a_{i\varphi(i+1)}$$

for any $D_1 \in A(\delta_1)$. Similarly there are $y_2 \in Y$, $(b_{nk})_{n,k} \subset Y$ and $\delta_2 \in U(T)$ such that

$$|y_2 - S(g, D_2)| < \bigvee_i b_{i\varphi(i+2)}$$

for any $D_2 \in A(\delta_2)$.

Put $\delta = \delta_1 \cap \delta_2$ and take $D \in A(\delta)$. Then $D \in A(\delta_1) \cap A(\delta_2)$ and

$$\begin{aligned} |S(f+g, D) - y_1 - y_2| &= |S(f,D) + S(g,D) - y_1 - y_2| \leqq \\ &\leqq |S(f,D) - y_1| + |S(g,D) - y_2| < \\ &< \bigvee_i a_{i\varphi(i+1)} + \bigvee_i b_{i\varphi(i+2)} \leqq \bigvee_i c_{i\varphi(i)}, \end{aligned}$$

where $(c_{nk})_{n,k}$ exists by Lemma 3. Hence $f+g$ is integrable and

$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu .$$

For $c \in \mathbb{R}$ we have $|c| a_{nk} \geqq 0$, $|c| a_{nk} \searrow 0$ ($k \to \infty$, $n = 1,2,\dots$) and

$$\begin{aligned} |S(cf, D) - cy_1| &= |c \cdot S(f,D) - cy_1| \leqq |c|\, |S(f,D) - y_1| < \\ &< |c| \bigvee_i a_{i\varphi(i)} = \bigvee_i |c|\, a_{i\varphi(i)} \end{aligned}$$

for $D \in A(\delta_1)$. This implies that cf is integrable and $\int cf\, d\mu = c \int f\, d\mu$. □

THEOREM 8. *If $f : T \to X$ is integrable and $f(t) \geqq 0$ for every $t \in T$, then $\int f\,d\mu \geqq 0$.*

Proof. By the positivity of operators $\mu(E_i)$ we have

$$\sum_{i=1}^{n} \mu(E_i)f(t_i) = S(f,D) \geqq 0$$

for any $D \in A(\delta)$, any $\delta \in U(T)$ and every positive function f.
Let $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $\delta \in U(T)$ such that

$$\left|\int f\,d\mu - S(f,D)\right| < \bigvee_i a_{i\varphi(i)}$$

for any $D \in A(\delta)$. Hence

$$-\bigvee_i a_{i\varphi(i)} \leqq S(f,D) - \bigvee_i a_{i\varphi(i)} < \int f\,d\mu$$

and

$$\bigvee_i a_{i\varphi(i)} > -\int f\,d\mu$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. From the weak σ-distributivity of Y we have

$$-\int f\,d\mu \leqq \bigwedge_{\varphi} \bigvee_i a_{i\varphi(i)} = 0$$

and then

$$\int f\,d\mu \geqq 0 .$$

□

DEFINITION 9. *A function $f : T \to X$ is integrable on a set $E \in \mathcal{S}$, if there exist $y \in Y$ and a bounded double sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geqq 0$, $a_{nk} \searrow 0$ ($k \to \infty$, $n = 1,2,\dots$) and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\delta \in U(\overline{E})$ such that*

$$|S_E(f,D) - y| < \bigvee_i a_{i\varphi(i)}$$

for any $D \in A(\delta/E)$, where $S_E(f,D) = \sum\limits_{i=1}^{n} \mu(E_i)f(t_i)$ and $\bigcup\limits_{i=1}^{n} E_i = E$, $E_i \cap E_j = \emptyset$ ($i \neq j$), $E_i \in \mathcal{S}$, $t_i \in \overline{E}_i$, $E_i \subset \delta(t_i)$ ($i = 1,2,\dots,n$).
The element y will be denoted by $\int\limits_E f\,d\mu$.

Remark 10. The definition 9 is correct. By Remark 2 $A(\delta/E) \neq \emptyset$ for $\delta \in U(\overline{E})$ and $\int f\,d\mu$ is defined uniquely (see Lemma 6).

Lemma 11. *Let Y be boundedly complete (i.e. every bounded subset of Y has a supremum). Then a function $f : T \to X$ is integrable on $E \in \mathcal{S}$ if and only if the following condition is satisfied:*

There exists a bounded sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geqq 0$, $a_{nk} \searrow 0$ $(k \to \infty$, $n = 1, 2, \dots)$ and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\delta \in U(\overline{E})$ such that

$$|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in A(\delta/E)$.

Proof. The necessity of the condition is evident. We shall prove that this condition is sufficient. Let $(a_{nk})_{n,k} \subset Y$ be such a sequence that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\delta(\varphi) \in U(\overline{E})$ such that

$$|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in A(\delta(\varphi)/E)$. Denote

$$I = \{\delta \in U(\overline{E})\,;\ \exists \varphi \in \mathbb{N}^{\mathbb{N}},\ \delta = \delta(\varphi)\}.$$

Then for $\delta \in I$ the set

$$\{S_E(f, D)\,;\ D \in A(\delta/E)\}$$

is bounded. Since Y is boundedly complete, there exists

$$a_\delta = \bigwedge_{D \in A(\delta/E)} S_E(f, D)\,; \qquad b_\delta = \bigvee_{D \in A(\delta/E)} S_E(f, D).$$

For $\delta_1, \delta_2 \in I$ put $\delta = \delta_1 \cap \delta_2$. Then $A(\delta/E) \subset A(\delta_1/E) \cap A(\delta_2/E)$ and hence $\{S_E(f, D); D \in A(\delta/E)\}$ is bounded, too, and

$$a_{\delta_1} = \bigwedge_{D \in A(\delta_1/E)} S_E(f, D) \leqq \bigwedge_{D \in A(\delta/E)} S_E(f, D) \leqq S_E(f, D) \leqq$$

$$\leqq \bigvee_{D \in A(\delta/E)} S_E(f, D) \leqq \bigvee_{D \in A(\delta_2/E)} S_E(f, D) = b_{\delta_2}.$$

Therefore $\bigvee_{\delta \in I} a_\delta \leqq \bigwedge_{\delta \in I} b_\delta$, hence there exists $y \in Y$ such that

$$a_\delta \leqq y \leqq b_\delta$$

for all $\delta \in I$. Now let $\varphi : \mathbb{N} \to \mathbb{N}$. Then there is $\delta(\varphi) \in U(\overline{E})$ such that

$$S_E(f, D) \leqq S_E(f, D_2) + \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in A(\delta(\varphi)/E)$. Fix D_2. Then

$$b_{\delta(\varphi)} \leqq S_E(f, D_2) + \bigvee_i a_{i\varphi(i)} .$$

Since the inequality holds for every $D_2 \in A(\delta(\varphi)/E)$, we have

$$b_{\delta(\varphi)} \leqq a_{\delta(\varphi)} + \bigvee_i a_{i\varphi(i)} .$$

By the weak σ-distributivity of Y we obtain $\bigwedge_\varphi \bigvee_i a_{i\varphi(i)} = 0$ and so

$$\bigwedge_\varphi b_{\delta(\varphi)} - \bigvee_\varphi a_{\delta(\varphi)} = \bigwedge_\varphi \left(b_{\delta(\varphi)} - a_{\delta(\varphi)}\right) = 0$$

hence

$$y = \bigwedge_\varphi b_{\delta(\varphi)} = \bigvee_\varphi a_{\delta(\varphi)}$$

Then for every $D \in A(\delta(\varphi)/E)$

$$S_E(f, D) - y \leqq b_{\delta(\varphi)} - a_{\delta(\varphi)} \leqq \bigvee_i a_{i\varphi(i)}$$

and similarly

$$y - S_E(f, D) \leqq b_{\delta(\varphi)} - a_{\delta(\varphi)} \leqq \bigvee_i a_{i\varphi(i)} ,$$

so that

$$|S_E(f, D) - y| \leqq \bigvee_i a_{i\varphi(i)}$$

and the proof is complete. □

THEOREM 12. *If $E, F, G \in \mathcal{S}$, $E = F \cup G$, $F \cap G = \emptyset$ and $f : T \to X$ is integrable on E, then f is integrable on F and G, too and*

$$\int_E f\,d\mu = \int_F f\,d\mu + \int_G f\,d\mu .$$

P r o o f. By Lemma 11 there is $(a_{nk})_{n,k} \subset Y$ such that for every $\varphi : \mathbb{N} \to \mathbb{N}$ there is $\delta \in U(\overline{E})$ such that

$$|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for every $D_1, D_2 \in A(\delta/E)$. Take $D, D' \in A(\delta/F)$ and $D_0 \in A(\delta/E \setminus F)$. Put $D_1 = D \cup D_0$, $D_2 = D' \cup D_0$. Then $D_1, D_2 \in A(\delta/E)$ and so

$$|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)} .$$

But

$$|S_F(f, D) - S_F(f, D') = |S_F(f, D) + S_{E\setminus F}(f, D_0) - S_{E\setminus F}(f, D_0) -$$

$$- S_F(f, D')| = |S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for all $D, D' \in A(\delta/F)$. Hence f is integrable on F by Lemma 11. Similarly, f is integrable on G, too.

Then for $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist $\delta_1 \in U(\overline{E})$, $\delta_2 \in U(\overline{F})$, $\delta_2 \subset \delta_1/\overline{F}$ and $\delta_3 \in U(\overline{G})$, $\delta_3 \subset \delta_1/\overline{G}$ such that

$$|S_E(f, D_1) - \int_E f\,d\mu| < \bigvee_i a_{i\varphi(i+1)}$$

for all $D_1 \in A(\delta_1/E)$,

$$|S_F(f, D_2) - \int_F f\,d\mu| < \bigvee_i b_{i\varphi(i+2)}$$

for all $D_2 \in A(\delta_2/F)$,

$$|S_G(f, D_3) - \int_G f\,d\mu| < \bigvee_i c_{i\varphi(i+3)}$$

for all $D_3 \in A(\delta_3/G)$. We have $D_2 \cup D_3 \in A(\delta_1/E)$ and so

$$|S_E(f, D_2 \cup D_3) - \int_E f\,d\mu| < \bigvee_i a_{i\varphi(i+1)} .$$

Since $S_E(f, D_2 \cup D_3) = S_F(f, D_2) + S_G(f, D_3)$, we obtain

$$\left|\int_E f\,d\mu - \int_F f\,d\mu - \int_G f\,d\mu\right| \leqq \left|\int_E f\,d\mu - S_E(f, D_2 \cup D_3)\right| +$$

$$+\left|S_F(f, D_2) - \int_F f\,d\mu\right| + \left|S_G(f, D_3) - \int_G f\,d\mu\right| <$$

$$< \bigvee_i a_{i\varphi(i+1)} + \bigvee_i b_{i\varphi(i+2)} + \bigvee_i c_{i\varphi(i+3)} < \bigvee_i d_{i\varphi(i)} .$$

The sequence $(d_{ij})_{i,j}$ exists by Lemma 3. Using the weak σ-distributivity of Y we get

$$\int_E f\,d\mu = \int_F f\,d\mu + \int_G f\,d\mu .$$

□

THEOREM 13. *If $f\colon T \to X$ is a simple measurable function, $f = \sum_{i=1}^{n} x_i\, \chi_{E_i}$, where $x_i \in X$, $E_i \in \mathcal{S}$ $(i = 1, 2, \dots, n)$, $E_i \cap E_j = \emptyset$ $(i \neq j)$, $\bigcup_{i=1}^{n} E_i = T$, and $\mu : \mathcal{S} \to L(X, Y)$ is a regular operator valued measure, then f is integrable and $\int f\,d\mu = \sum_{i=1}^{n} \mu(E_i)x_i$.*

Proof. By Theorem 7 it is sufficient to prove that $x\chi_E$ $(E \in \mathcal{S}, x \in X)$ is integrable and

$$\int x\chi_E\,d\mu = \mu(E)x .$$

First we suppose that $x \geqq 0$. The regularity of the measure μ implies that for $E \in \mathcal{S}$ and $x \in X$, $x \geqq 0$ there exists a bounded sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geqq 0$, $a_{nk} \searrow 0$ $(k \to \infty,\ n = 1, 2, \dots)$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist an open set $U \in \mathcal{S}$ and a compact set $C \in \mathcal{S}$, $C \subset E \subset U$ so that

$$\mu(U \setminus C)x < \bigvee_i a_{i\varphi(i)} .$$

Since C is compact and U is open there exists $\delta \in U(T)$ such that

$$\begin{aligned} \delta(t) &\subset U \quad \text{for } t \in C\,, \\ \delta(t) &\subset U \setminus C \quad \text{for } t \in U \setminus C\,, \\ \delta(t) \cap C &= \emptyset \quad \text{for } t \notin U\,. \end{aligned}$$

Take $D \in A(\delta)$, $D = \{(E_i, t_i),\ i = 1, 2, \dots, n\}$. By Lemma 4 we have

$$\mu(C)x \leqq \mu(E)x \leqq \mu(U)x$$

and

$$\mu(U \setminus C)x = \mu(U)x - \mu(C)x\,.$$

Now

$$\begin{aligned} \mu(E)x - \bigvee_i a_{i\varphi(i)} &\leqq \mu(U)x - \bigvee_i a_{i\varphi(i)} < \mu(C)x \leqq \\ &\leqq \mu\Big(\bigcup_{t_i \in C} E_i\Big)x = \sum_{t_i \in C} \mu(E_i)x = \sum_{i=1}^{n} \chi_C(t_i)\mu(E_i)x \leqq \\ &\leqq \sum_{i=1}^{n} \chi_E(t_i)\mu(E_i)x = \sum_{i=1}^{n} \mu(E_i)\,(x\chi_E(t_i)) = \\ &= S(x\chi_E, D) \leqq \sum_{i=1}^{n} \mu(E_i)\,(x\chi_U(t_i)) = \sum_{t_i \in U} \mu(E_i)x = \\ &= \mu\Big(\bigcup_{t_i \in U} E_i\Big)x \leqq \mu(U)x \leqq \mu(C)x + \bigvee_i a_{i\varphi(i)} \leqq \\ &\leqq \mu(E)x + \bigvee_i a_{i\varphi(i)}. \end{aligned}$$

Then

$$-\bigvee_i a_{i\varphi(i)} \leqq S(x\chi_E, D) - \mu(E)x \leqq \bigvee_i a_{i\varphi(i)}$$

and hence

$$|S(x\chi_E, D) - \mu(E)x| \leqq \bigvee_i a_{i\varphi(i)}$$

for any $D \in A(\delta)$. In the general case for $x \in X$ we get

$$\int x\chi_E \, d\mu = \int (x^+ - x^-)\chi_E \, d\mu = \int x^+\chi_E \, d\mu - \int x^-\chi_E \, d\mu =$$

$$= \mu(E)x^+ - \mu(E)x^- = \mu(E)x \, .$$

□

Limit Theorem

Lemma 14. *If $f_n : T \to X$ is integrable for $n = 1, 2, \dots$, $f_n \to f$ uniformly and f is bounded, then $\lim_{n\to\infty} \int f_n \, d\mu$ exists.*

Proof. It is sufficient to show that the sequence $(\int f_n \, d\mu)_n$ is bounded and

$$\bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \int f_i \, d\mu \leqq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} \int f_j \, d\mu \, .$$

Since the function f is bounded, there exists $h \in X$, $h \geqq 0$, such that $|f(t)| \leqq h$ for all $t \in T$.

From the uniform convergence of f_n there exists a sequence $(a_n)_n \subset X$, $a_n \searrow 0$ $(n \to \infty)$ and for any $t \in T$

$$|f_n(t) - f(t)| \leqq a_n$$

for all n. Hence

$$-h - a_1 \leqq f(t) - a_1 \leqq f(t) - a_n \leqq f_n(t) \leqq f(t) + a_n \leqq h + a_1$$

and

$$|f_i(t) - f_j(t)| \leqq |f_i(t) - f(t)| + |f_j(t) - f(t)| \leqq a_i + a_j \leqq 2a_n$$

for any $t \in T$ and $i, j \geqq n$. It is evident that if for $f : T \to X$, $f(t) = a$ for all $t \in T$, then

$$\sum_{j=1}^{n} \mu(E_j) \, f(t_j) = \sum_{j=1}^{n} \mu(E_j)a = \mu(T)a$$

for any $D \in A(\delta)$ and any δ. By Theorems 7 and 8 for any n we have

$$\mu(T)(-h - a_1) \leqq \int f_n \, d\mu \leqq \mu(T) \, (h + a_1)$$

and

$$\mu(T)\,(-2a_n) \leqq \int (f_i - f_j)\, d\mu = \int f_i\, d\mu - \int f_j\, d\mu \leqq \mu(T)\,(2a_n) \qquad \text{for} \quad i,j \geqq n.$$

Then the sequence $(\int f_n\, d\mu)_n$ is bounded and

$$\mu(T)\,(-2a_n) + \int f_j\, d\mu \leqq \int f_i\, d\mu \leqq \int f_j\, d\mu + \mu(T)\,(2a_n)$$

for $i, j \geqq n$, which implies

$$\bigvee_{i=n}^{\infty} \int f_i\, d\mu \leqq \bigwedge_{j=n}^{\infty} \int f_j\, d\mu + \mu(T)\,(2a_n)$$

for all n, and hence from continuity of $\mu(T)$ we get

$$\bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \int f_i\, d\mu \leqq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} \int f_j\, d\mu .$$

□

THEOREM 15. *Let $f_n : T \to X$ be integrable for $n = 1, 2 \dots, f_n \to f$ uniformly and f is bounded. Then f is integrable and $\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$.*

Proof. By Lemma 14 $\lim_{n\to\infty} \int f_n\, d\mu = c$ exists and hence there exists a sequence $(c_n)_n \subset Y$, $c_n \searrow 0$ $(n \to \infty)$ and

$$\left| \int f_n\, d\mu - c \right| \leqq c_n$$

for any n. The function f_n is integrable and then there exists a bounded double sequence $(a_{nij})_{i,j} \subset Y$ such that $a_{nij} \searrow 0$ $(j \to \infty,\ i, n = 1, 2 \dots)$ and for every $\varphi : \mathbb{N} \to \mathbb{N}$ there exists $\delta_n \in U(T)$ such that for every $D \in A(\delta_n)$

$$\left| \int f_n\, d\mu - S(f_n, D) \right| < \bigvee_i a_{ni\varphi(i+n+1)} .$$

Since $f_n \to f$ uniformly, there exists a sequence $(b_n)_n \subset X$, $b_n \searrow 0$ and $|f_n(t) - f(t)| \leqq b_n$ for any $t \in T$ and all n.

Let $\varphi \in \mathbb{N}^{\mathbb{N}}$. Put $k = \min_j \varphi(j+1)$ and take $D \in A(\delta_k)$

$$D = \{(E_1, t_1),\ (E_2, t_2), \dots, (E_r, t_r)\}.$$

Then

$$\big|S(f,D) - c\big| \leqq \big|S(f,D) - S(f_k,D)\big| + \left|S(f_k,D) - \int f_k\, d\mu\right| + \left|\int f_k\, d\mu - c\right| <$$

$$< \sum_{i=1}^{r} \mu(E_i)\big(|f(t_i) - f_k(t_i)|\big) + \bigvee_i a_{ki\varphi(i+k+1)} + c_k \leqq$$

$$\leqq \sum_{i=1}^{r} \mu(E_i) b_k + \bigvee_i a_{ki\varphi(i+k+1)} + c_k \leqq$$

$$\leqq \mu(T) b_k + c_k + \bigvee_i a_{ki\varphi(i+k+1)} = d_k + \bigvee_i a_{ki\varphi(i+k+1)},$$

where

$d_j = \mu(T) b_j + c_j$ for $j = 1, 2, \dots,$ $(d_j)_j \subset Y$,

$d_j \searrow 0$ $(j \to \infty)$, since $\mu(T) b_j \searrow 0$, $(j \to \infty)$,

$d_k = d_{\min_j \varphi(j+1)} = \bigvee_i d_{\varphi(i+1)}$.

Put $b_{1ij} = d_j$ for $i, j = 1, 2, \dots$ and $b_{m+1ij} = a_{nij}$ for $n, i, j, m = 1, 2 \dots$. Now

$$|S(f,D) - c| < \bigvee_i d_{\varphi(i+1)} + \bigvee_i a_{ki\varphi(i+k+1)} =$$

$$= \bigvee_i b_{1i\varphi(i+1)} + \bigvee_i b_{k+1i\varphi(i+k+1)} \leqq$$

$$\leqq \sum_{n=1}^{\infty} \bigvee_i b_{ni\varphi(i+n)}.$$

There exists $h \in X$, $h \geqq 0$ such that $|f(t)| \leqq h$ for any $t \in T$, since f is bounded. Then

$$|S(f,D) - c| \leqq |S(f,D)| + |c| = \Big|\sum_{i=1}^{r} \mu(E_i) f(t_i)\Big| + |c| \leqq$$

$$\leqq \sum_{i=1}^{r} \mu(E_i)|f(t_i)| + |c| \leqq \mu(T) h + |c| \leqq a,$$

where $a \in X, a > 0$ and

$$|S(f,D) - c| \leqq a \wedge \left(\sum_{n=1}^{\infty} \bigvee_i b_{ni\varphi(i+n)} \right).$$

By Lemma 3 there exists a bounded double sequence $(a_{ij})_{i,j} \subset Y$, $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1,2,\dots)$ and

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} b_{ni\varphi(i+1)} \right) \leqq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}.$$

Therefore there exists $c \in Y$, $c = \lim\limits_{n\to\infty} \int f_n \, d\mu$ and the bounded double sequence $(a_{ij})_{i,j} \subset Y$, $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1,2,\dots)$ and for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\delta \in U(T)$ $(\delta = \delta_{\min_j \varphi(j+1)})$ such that

$$|S(f,D) - c| \leqq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for any $D \in A(\delta)$. Hence f is integrable and

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

□

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