ON INTEGRATION WITH RESPECT TO OPERATOR
VALUED MEASURES IN RIESZ SPACES

B E L O S L A V  R I E Č A N  —  M A R T A  V R Á B E LOVÁ

Dedicated to the memory of Tibor Neubrunn

ABSTRACT. An operator valued measure is considered assigning to every Borel
set (in a compact space $T$) a linear, positive, order continuous operator from a
Riesz space $X$ to another Riesz space $Y$. A Kurzweil type construction is used
for integrating functions from $T$ to $X$.

Introduction

If $X$, $Y$ are linear spaces and $L(X,Y)$ is a set of linear operators and
$(T,S)$ is a measurable space, then an operator valued measure is a mapping
$\mu : S \rightarrow L(X,Y)$ satisfying some conditions. If $f : T \rightarrow X$ is a simple function

$$f = \sum_{i=1}^{n} \chi_{E_i} x_i,$$

then

$$\int f \, d\mu = \sum_{i=1}^{n} \mu(E_i)(x_i),$$

where $\mu(E_i) \in L(X,Y)$, hence $\mu(E_i)(x_i)$ is the value of $\mu(E_i)$ in the element
$x_i \in X$. Therefore $\int f \, d\mu \in Y$. The problem is how to extend this integral to a
larger family of functions $f : T \rightarrow X$.

The problem of operator valued measure has been studied in a series of papers
by I. Dobrakov (see [1]) and for locally convex spaces by J. Haluška ([3]). In [4] J. Haluška considered the case of Banach lattices. The basic property
is the weak $\sigma$-distributivity, which is a necessary and sufficient condition for
extendability of $Y$-valued measures and integrals ([15]).

AMS Subject Classification (1991): 28B15.
Key words: Kurzweil integral, linear ordered spaces, operator valued measures.
To solve the problem stated above we shall use the Henson-Kurzweil construction of an integral. Of course, since we work with a Borel regular measure on the σ-algebra $\mathcal{S}$ of subsets of a compact space $T$, the obtained integral is of the Lebesgue type. Another solution of the problem is contained in [11].

Special cases of the studied theory are the case of vector measure $P$ with scalar functions $f : T \to R$ (where $X = R$, $\mu(E)(x) = xP(E) \in Y$) and scalar measure $P$ with vector functions $f : T \to X$ (where $X = Y$, $\mu(E) = P(E)x$). As a special case some results of [9] and [13] can be received.

Assumptions

A. Let $T$ be a (Hausdorff) compact topological space and let $\mathcal{S}$ be the σ-algebra generated by the family of all compact (all open) subsets of $T$.

Let $U(T)$ be the set of all functions $\delta : T \to 2^T$ such that $\delta(t)$ is a neighbourhood of $t$ for every $t \in T$ and let $A(\delta)$ be the set of all partitions $D$ of $T$ such that $D = \{(E_1, t_1), (E_2, t_2), \ldots, (E_n, t_n)\}$, where $E_i \in \mathcal{S}$, $t_i \in \overline{E_i}$, $E_i \subset \delta(t_i)$ ($i = 1, 2, \ldots, n$), and $\bigcup_{i=1}^{n} E_i = T$.

**Lemma 1.** The set $A(\delta)$ is non-empty for every $\delta \in U(T)$.

**Proof.** Let $\delta \in U(T)$. Then for every $t \in T$ there exists an open set $U(t)$ ($U(t) \in \mathcal{S}$) such that $t \in U(t) \subset \delta(t)$. The set $\{U(t); t \in T\}$ is an open covering of $T$, hence there is a finite open covering $U(t_1), U(t_2), \ldots, U(t_n)$ chosen from $\{U(t), t \in T\}$.

Now, choose pairwise disjoint neighbourhoods $F(t_i) \in \mathcal{S}$ ($i = 1, 2, \ldots, n$) and put

$$F = \bigcup_{i=1}^{n} F(t_i),$$

$$E_1 = (U(t_1) \setminus F) \cup F(t_1),$$

$$E_i = \left( U(t_i) \setminus \bigcup_{j=1}^{i-1} E_j \setminus F \right) \cup F(t_i) \quad \text{for} \quad i = 2, 3, \ldots, n.$$  

Then $E_i \cap E_j = \emptyset$ ($i \neq j$), $\bigcup_{i=1}^{n} E_i = T$, $E_i \in \mathcal{S}$, $E_i \subset U(t_i) \subset \delta(t_i)$ and $t_i \in E_i$ for $i = 1, 2, \ldots, n$. $\square$

150
Remark 2. If $E \subset T$, $E \in \mathcal{S}$, then $\overline{E}$ is compact. Let $U'(t_1), \ldots, U'(t_n)$ be a finite covering of $\overline{E}$ chosen from $\{U'(t); U'(t) \subset \delta(t), t \in \overline{E}\}$ and let $F'(t_i) \in \mathcal{S}$ ($i = 1, 2, \ldots, n$) be pairwise disjoint neighbourhoods of $t_i$.

Put $U(t_i) = U'(t_i) \cap E$, $F(t_i) = F'(t_i) \cap E$ ($i = 1, 2, \ldots, n$). The partition $D \in A(\delta/E)$ ($\delta \in U(T)$ or $\delta \in U(\overline{E})$) can be constructed by the same way as in the proof of the preceding lemma (now, $t_i \in \overline{E}_i$ ($i = 1, 2, \ldots, n$)).

B. We shall assume that $X$ and $Y$ are linear lattices, the linear lattice $X$ is boundedly $\sigma$-complete, i.e., every bounded sequence $(a_i)_i \subset X$ has the supremum $\bigvee_i a_i$, the linear lattice $Y$ is weakly $\sigma$-distributive, i.e., $Y$ is boundedly $\sigma$-complete and for every bounded double sequence $(a_{ij})_{i,j} \subset Y$ such that $a_{ij} \searrow 0$ ($j \to \infty, i = 1, 2, \ldots$) there is

$$\bigwedge_{\varphi \in \mathbb{N}^\mathbb{N}} \bigvee_{i \in \mathbb{N}^\mathbb{N}} a_{i \varphi(i)} = 0.$$ 

**Lemma 3.** Let $Y$ be a boundedly $\sigma$-complete linear lattice, $(a_{nij})_{n,i,j}$ be a triple bounded sequence of elements of $Y$ such that $a_{nij} \searrow 0$ ($j \to \infty, n, i = 1, 2, \ldots$). Then to every $a \in X$, $a > 0$ there is a bounded double sequence $(a_{ij})_{i,j} \subset Y$ such that $a_{ij} \searrow 0$, ($j \to \infty, i = 1, 2, \ldots$) and for every $\varphi \in \mathbb{N}^\mathbb{N}$

$$a \wedge \left( \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n \varphi(i) + n} \right) \leq \bigvee_{i=1}^{\infty} a_{i \varphi(i)}.$$ 

**Proof.** See [2] and [12].

If $x_n, x \in X$ then $x_n \to x$ ($x_n$ converges to $x$ with respect to the ordering) iff there exists $(a_n)_n \subset X$, $a_n \searrow 0$ and $|x_n - x| \leq a_n$ for all $n$.

It is possible to prove that a sequence $(x_n)_n \subset X$ converges to $x \in X$ iff $(x_n)_n$ is bounded and

$$x = \bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} x_i = \bigvee_{n=1}^{\infty} \bigwedge_{i=n}^{\infty} x_i.$$ 

We say that $f_n \to f$ uniformly ($f_n, f : T \to X$) iff there exists $(a_n)_n \subset X$, $a_n \searrow 0$ such that

$$|f_n(t) - f(t)| \leq a_n$$

for every $t \in T$ and all $n$.  

151
C. By $L(X, Y)$ we shall denote the set of all $\sigma$-homomorphisms from $X$ to $Y$ i.e. such mappings $h : X \to Y$ that

(i) $h(x_1 + x_2) = h(x_1) + h(x_2)$ for every $x_1, x_2 \in X$;

(ii) $h(cx) = ch(x)$ for every $x \in X$ and $c \in R$;

(iii) If $(x_n)_n \subseteq X, x_n \searrow 0$ then $h(x_n) \searrow 0$.

The properties (i) and (iii) imply $h(0) = 0$ and $h(x_1) \leq h(x_2)$ for $x_1, x_2 \in X, x_1 \leq x_2$ (that is $h$ is a positive operator).

D. Let $\mu : \mathcal{S} \to L(X, Y)$ be an operator valued measure having the following properties:

(i) If $E \in \mathcal{S}, x \in X, x \geq 0$ then $\mu(E)x \geq 0$;

(ii) If $E_n \in \mathcal{S}$ $(n = 1, 2, \ldots, k), E_i \cap E_j = 0 \ (i \neq j)$, then

\[
\mu\left( \bigcup_{n=1}^{k} E_n \right) x = \sum_{n=1}^{k} \mu(E_n)x \text{ for every } x \in X.
\]

(iii) $\mu$ is regular in the following sense:

For every set $E \in \mathcal{S}$ and every $x \in X, x \geq 0$ there exists a bounded sequence $(a_{nk})_{n,k} \subseteq Y, a_{nk} \geq 0, a_{nk} \searrow 0 \ (k \to \infty, n = 1, 2, \ldots)$

and for every $\varphi \in \mathbb{N}^n$ there exist a compact set $F$ and open set $U, F \subset E \subset U$ such that

\[
\mu(U \setminus F)x < \bigvee_{i} a_{i\varphi(i)}.
\]

**Example.** Let $\mu_1 : \mathcal{S} \to R$ be a regular Borel measure. Let $X = Y$. Put

$\mu(E)x = \mu_1(E) \cdot x$ for $E \in \mathcal{S}$ and $x \in X$. Then $\mu : \mathcal{S} \to L(X, X)$ is the regular operator valued measure. For $E \in \mathcal{S}$ and $x \in X, x \geq 0$ it is sufficient to put $a_{ij} = \frac{1}{2}x \ (i, j = 1, 2, \ldots)$. Then for $\varphi \in \mathbb{N}^n$ there exists a compact set $F$ and an open set $U, F \subset E \subset U$ such that

\[
\mu(U \setminus F)x = \mu_1(U \setminus F) \cdot x < \frac{1}{\min_{i} \varphi(i)} x = \bigvee_{i} a_{i\varphi(i)}.
\]

The proof of the following lemma is evident.

**Lemma 4.** Let $\mu : T \to L(X, Y)$ be a regular operator valued measure. Then

(i) $\mu$ is monotone i.e. if $E, F \in \mathcal{S}, E \subset F$, then

$\mu(E)x \leq \mu(F)x$ for every $x \in X, x \geq 0$;

(ii) $\mu$ is subtractive, i.e., if $E, F \in \mathcal{S}, E \subset F$, then

$\mu(F \setminus E)x = \mu(F)x - \mu(E)x$ for every $x \in X$.
INTEGRATION WITH RESPECT TO OPERATOR VALUED MEASURES IN RIESZ SPACES

Integral

If \( \delta \in U(T) \) and \( D \in A(\delta) \), \( D = \{(E_1, t_1), \ldots, (E_n, t_n)\} \) then for the function \( f : T \to X \) we define

\[
S(f, D) = \sum_{i=1}^{n} \mu(E_i) f(t_i),
\]

where \( \mu(E_i) f(t_i) \) is the value of the operator \( \mu(E_i) \) in \( f(t_i) \).

**Definition 5.** The function \( f : T \to X \) is integrable if there exists \( y \in Y \) and a bounded double sequence \( (a_{nk})_{n,k} \subset Y, \ a_{nk} \geq 0, \ a_{nk} \searrow 0 \ (k \to \infty, n = 1, 2, \ldots) \) such that for every \( \varphi : \mathbb{N} \to \mathbb{N} \) there exists \( \delta \in U(T) \) so that

\[
|S(f, D) - y| < \sum_{i=1}^{\infty} a_{i\varphi(i)} \quad \text{for any} \quad D \in A(\delta).
\]

**Lemma 6.** The integral of \( f \) is defined uniquely.

**Proof.** Let \( y_1, y_2 \in Y \) be two elements satisfying the conditions of the preceding definition. Then there exist \( (a_{nk})_{n,k} \), \( (b_{nk})_{n,k} \subset Y, \ a_{nk} \geq 0, \ b_{nk} \geq 0 \ (k \to \infty, n = 1, 2, \ldots) \) and for \( \varphi \in \mathbb{N}^2 \) there exist \( \delta_1, \delta_2 \in U(T) \) such that

\[
|S(f, D_1) - y_1| < \sum_{i} a_{i\varphi(i+1)}, \quad |S(f, D_2) - y_2| < \sum_{i} b_{i\varphi(i+2)}
\]

for \( D_1 \in A(\delta_1), \ D_2 \in A(\delta_2) \). Put \( \delta = \delta_1 \cap \delta_2 \ (\delta(t) = \delta_1(t) \cap \delta_2(t) \text{ for } t \in T) \) and take \( D \in A(\delta) \) (then \( D \in A(\delta_1) \cap A(\delta_2) \), too). Then

\[
|y_1 - y_2| \leq |y_1 - S(f, D)| + |S(f, D) - y_2| <
\]

\[
< \sum_{i} a_{i\varphi(i+1)} + \sum_{i} b_{i\varphi(i+2)} \leq \sum_{i} c_{i\varphi(i)},
\]

where \( (c_{nk})_{n,k} \subset Y \) is bounded, \( c_{nk} \geq 0, \ c_{nk} \searrow 0 \ (k \to \infty, n = 1, 2, \ldots) \). The sequence \( (c_{nk})_{n,k} \) exists by Lemma 3. From the weak \( \sigma \)-distributivity of \( Y \) we have

\[
|y_1 - y_2| \leq \bigwedge_{i, \varphi} \bigvee_{i} c_{i\varphi(i)} = 0
\]

and hence \( y_1 = y_2 \ (= \int f \, d\mu) \). \( \square \)
THEOREM 7. If \( f, g : T \to X \) are integrable and \( \alpha, \beta \in \mathbb{R} \), then \( \alpha f + \beta g \) is integrable and
\[
\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu.
\]

**Proof.** We shall prove that if \( f, g \) are integrable and \( c \in \mathbb{R} \), then \( f + g, cf \) are integrable too and
\[
\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu, \quad \int cf \, d\mu = c \int f \, d\mu.
\]
If \( f \) is integrable, then there exist \( y_1 \in Y \) and bounded double sequence \( (a_{nk})_{n,k} \subseteq Y, a_{nk} \geq 0, a_{n,k} \searrow 0 \ (k \to \infty, n = 1, 2, \ldots) \) such that for \( \varphi \in \mathbb{N}^* \) there exists \( \delta_1 \in U(T) \) such that
\[
|y_1 - S(f, D_1)| < \sqrt[\alpha]{a_{i\varphi(i+1)}}
\]
for any \( D_1 \in A(\delta_1) \). Similarly there are \( y_2 \in Y, (b_{nk})_{n,k} \subseteq Y \) and \( \delta_2 \in U(T) \) such that
\[
|y_2 - S(g, D_2)| < \sqrt[\alpha]{b_{i\varphi(i+2)}}
\]
for any \( D_2 \in A(\delta_2) \).

Put \( \delta = \delta_1 \cap \delta_2 \) and take \( D \in A(\delta) \). Then \( D \in A(\delta_1) \cap A(\delta_2) \) and
\[
|S(f + g, D) - y_1 - y_2| = |S(f, D) + S(g, D) - y_1 - y_2| \leq
\]
\[
\leq |S(f, D) - y_1| + |S(g, D) - y_2| <
\]
\[
< \sqrt[\alpha]{a_{i\varphi(i+1)}} + \sqrt[\alpha]{b_{i\varphi(i+2)}} \leq \sqrt[\alpha]{c_{i\varphi(i)}},
\]
where \( (c_{nk})_{n,k} \) exists by Lemma 3. Hence \( f + g \) is integrable and
\[
\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu.
\]

For \( c \in \mathbb{R} \) we have \( |c|a_{nk} \geq 0, |c|a_{nk} \searrow 0 \ (k \to \infty, n = 1, 2, \ldots) \) and
\[
|S(cf, D) - cy_1| = |c \cdot S(f, D) - cy_1| \leq |c| |S(f, D) - y_1| <
\]
\[
< |c| \sqrt[\alpha]{a_{i\varphi(i)}} = \sqrt[\alpha]{|c| a_{i\varphi(i)}}
\]
for \( D \in A(\delta_1) \). This implies that \( cf \) is integrable and \( \int cf \, d\mu = c \int f \, d\mu \). \( \square \)
THEOREM 8. If \( f : T \to X \) is integrable and \( f(t) \geq 0 \) for every \( t \in T \), then \( \int f \, d\mu \geq 0 \).

Proof. By the positivity of operators \( \mu(E_i) \) we have
\[
\sum_{i=1}^{n} \mu(E_i) f(t_i) = S(f, D) \geq 0
\]
for any \( D \in A(\delta) \), any \( \delta \in U(T) \) and every positive function \( f \).

Let \( \varphi \in \mathbb{N}^n \) and \( \delta \in U(T) \) such that
\[
|\int f \, d\mu - S(f, D)| < \sqrt{a_{i \varphi(i)}^2}
\]
for any \( D \in A(\delta) \). Hence
\[
- \sqrt{a_{i \varphi(i)}} \leq S(f, D) - \sqrt{a_{i \varphi(i)}} < \int f \, d\mu
\]
and
\[
\sqrt{a_{i \varphi(i)}} > - \int f \, d\mu
\]
for all \( \varphi \in \mathbb{N}^n \). From the weak \( \sigma \)-distributivity of \( Y \) we have
\[
- \int f \, d\mu \leq \bigwedge \sqrt{a_{i \varphi(i)}} = 0
\]
and then
\[
\int f \, d\mu \geq 0.
\]

□

DEFINITION 9. A function \( f : T \to X \) is integrable on a set \( E \in \mathcal{S} \), if there exist \( y \in Y \) and a bounded double sequence \( (a_{nk})_{n,k} \subset Y \), \( a_{nk} \geq 0 \), \( a_{nk} \searrow 0 \) \((k \to \infty, n = 1, 2, \ldots)\) and for every \( \varphi \in \mathbb{N}^n \) there exists \( \delta \in U(E) \) such that
\[
|S_E(f, D) - y| < \sqrt{a_{i \varphi(i)}}
\]
for any \( D \in A(\delta/E) \), where \( S_E(f, D) = \sum_{i=1}^{n} \mu(E_i) f(t_i) \) and \( \bigcup_{i=1}^{n} E_i = E \), \( E_i \cap E_j = \emptyset \) \((i \neq j)\), \( E_i \in \mathcal{S} \), \( t_i \in \overline{E_i} \), \( E_i \subset \delta(t_i) \) \((i = 1, 2, \ldots, n)\).

The element \( y \) will be denoted by \( \int f \, d\mu \).

Remark 10. The definition 9 is correct. By Remark 2 \( A(\delta/E) \neq \emptyset \) for \( \delta \in U(E) \) and \( \int f \, d\mu \) is defined uniquely (see Lemma 6).
**Lemma 11.** Let $Y$ be boundedly complete (i.e. every bounded subset of $Y$ has a supremum). Then a function $f : T \to X$ is integrable on $E \in S$ if and only if the following condition is satisfied:

There exists a bounded sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geq 0$, $a_{nk} \searrow 0$ ($k \to \infty$, $n = 1, 2, \ldots$) and for every $\varphi \in \mathbb{N}^N$ there is $\delta \in U(E)$ such that

$$|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in A(\delta/E)$.

**Proof.** The necessity of the condition is evident. We shall prove that this condition is sufficient. Let $(a_{nk})_{n,k} \subset Y$ be such a sequence that for every $\varphi \in \mathbb{N}^N$ there is $\delta(\varphi) \in U(E)$ such that

$$|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in A(\delta(\varphi)/E)$. Denote

$$I = \{\delta \in U(E); \exists \varphi \in \mathbb{N}^N, \delta = \delta(\varphi)\}.$$ Then for $\delta \in I$ the set

$$\{S_E(f, D); D \in A(\delta/E)\}$$

is bounded. Since $Y$ is boundedly complete, there exists

$$a_\delta = \bigwedge_{D \in A(\delta/E)} S_E(f, D); \quad b_\delta = \bigvee_{D \in A(\delta/E)} S_E(f, D).$$

For $\delta_1, \delta_2 \in I$ put $\delta = \delta_1 \cap \delta_2$. Then $A(\delta/E) \subset A(\delta_1/E) \cap A(\delta_2/E)$ and hence $\{S_E(f, D); D \in A(\delta/E)\}$ is bounded, too, and

$$a_{\delta_1} = \bigwedge_{D \in A(\delta_1/E)} S_E(f, D) \leq \bigwedge_{D \in A(\delta/E)} S_E(f, D) \leq S_E(f, D) \leq \bigvee_{D \in A(\delta_2/E)} S_E(f, D) = b_{\delta_2}.\]

Therefore $\bigvee_{\delta \in I} a_\delta \leq \bigwedge_{\delta \in I} b_\delta$, hence there exists $y \in Y$ such that

$$a_\delta \leq y \leq b_\delta$$

156
for all $\delta \in I$. Now let $\varphi : \mathbb{N} \to \mathbb{N}$. Then there is $\delta(\varphi) \in U(E)$ such that

$$S_E(f, D) \leq S_E(f, D_2) + \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in A(\delta(\varphi)/E)$. Fix $D_2$. Then

$$b_{\delta(\varphi)} \leq S_E(f, D_2) + \bigvee_i a_{i\varphi(i)}.$$

Since the inequality holds for every $D_2 \in A(\delta(\varphi)/E)$, we have

$$b_{\delta(\varphi)} \leq a_{\delta(\varphi)} + \bigvee_i a_{i\varphi(i)}.$$

By the weak $\sigma$-distributivity of $Y$ we obtain $\bigwedge \bigvee a_{i\varphi(i)} = 0$ and so

$$\bigwedge_{\varphi} \bigvee_{i} b_{\delta(\varphi)} - \bigvee_{\varphi} a_{\delta(\varphi)} = \bigwedge_{\varphi} (b_{\delta(\varphi)} - a_{\delta(\varphi)}) = 0$$

hence

$$y = \bigwedge_{\varphi} b_{\delta(\varphi)} = \bigvee_{\varphi} a_{\delta(\varphi)}$$

Then for every $D \in A(\delta(\varphi)/E)$

$$S_E(f, D) - y \leq b_{\delta(\varphi)} - a_{\delta(\varphi)} \leq \bigvee_i a_{i\varphi(i)}$$

and similarly

$$y - S_E(f, D) \leq b_{\delta(\varphi)} - a_{\delta(\varphi)} \leq \bigvee_i a_{i\varphi(i)}$$

so that

$$|S_E(f, D) - y| \leq \bigvee_i a_{i\varphi(i)}$$

and the proof is complete.
THEOREM 12. If \( E, F, G \in S, E = F \cup G, F \cap G = \emptyset \) and \( f : T \to X \) is integrable on \( E \), then \( f \) is integrable on \( F \) and \( G \), too and

\[
\int_E f \, d\mu = \int_F f \, d\mu + \int_G f \, d\mu.
\]

Proof. By Lemma 11 there is \( (a_{nk})_{n,k} \subset Y \) such that for every \( \varphi : \mathbb{N} \to \mathbb{N} \) there is \( \delta \in U(E) \) such that

\[
|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}
\]

for every \( D_1, D_2 \in A(\delta/E) \). Take \( D, D' \in A(\delta/F) \) and \( D_0 \in A(\delta/E \setminus F) \). Put \( D_1 = D \cup D_0, D_2 = D' \cup D_0 \). Then \( D_1, D_2 \in A(\delta/E) \) and so

\[
|S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}.
\]

But

\[
|S_F(f, D) - S_F(f, D')| = |S_F(f, D) + S_{E \setminus F}(f, D_0) - S_{E \setminus F}(f, D_0) - S_F(f, D')| = |S_E(f, D_1) - S_E(f, D_2)| < \bigvee_i a_{i\varphi(i)}
\]

for all \( D, D' \in A(\delta/F) \). Hence \( f \) is integrable on \( F \) by Lemma 11. Similarly, \( f \) is integrable on \( G \), too.

Then for \( \varphi \in \mathbb{N}^N \) there exist \( \delta_1 \in U(E), \delta_2 \in U(F), \delta_2 \subset \delta_1/F \) and \( \delta_3 \in U(G), \delta_3 \subset \delta_1/G \) such that

\[
|S_E(f, D_1) - \int_E f \, d\mu| < \bigvee_i a_{i\varphi(i+1)}
\]

for all \( D_1 \in A(\delta_1/E) \),

\[
|S_F(f, D_2) - \int_F f \, d\mu| < \bigvee_i b_{i\varphi(i+2)}
\]

for all \( D_2 \in A(\delta_2/F) \),

\[
|S_G(f, D_3) - \int_G f \, d\mu| < \bigvee_i c_{i\varphi(i+3)}
\]

158
for all $D_3 \in A(\delta_3 / G)$. We have $D_2 \cup D_3 \in A(\delta_1 / E)$ and so

$$|S_E(f, D_2 \cup D_3) - \int_E f \, d\mu| < \sqrt{a_{i\varphi(i+1)}}.$$ 

Since $S_E(f, D_2 \cup D_3) = S_F(f, D_2) + S_G(f, D_3)$, we obtain

$$\left| \int_E f \, d\mu - \int_F f \, d\mu - \int_G f \, d\mu \right| \leq \int_E f \, d\mu - S_E(f, D_2 \cup D_3) + \left| S_F(f, D_2) - \int_F f \, d\mu \right| + \left| S_G(f, D_3) - \int_G f \, d\mu \right| <$$

$$< \sqrt{a_{i\varphi(i+1)}} + \sqrt{b_{i\varphi(i+2)}} + \sqrt{c_{i\varphi(i+3)}} < \sqrt{d_{i\varphi(i)}}.$$ 

The sequence $(d_{ij})_{i,j}$ exists by Lemma 3. Using the weak $\sigma$-distributivity of $Y$ we get

$$\int_E f \, d\mu = \int_F f \, d\mu + \int_G f \, d\mu.$$  

\[\square\]

**Theorem 13.** If $f : T \to X$ is a simple measurable function, $f = \sum_{i=1}^n x_i \chi_{E_i}$, where $x_i \in X$, $E_i \in \mathcal{S}$ ($i = 1, 2, \ldots, n$), $E_i \cap E_j = \emptyset$ ($i \neq j$), $\bigcup_{i=1}^n E_i = T$, and $\mu : \mathcal{S} \to L(X,Y)$ is a regular operator valued measure, then $f$ is integrable and

$$\int f \, d\mu = \sum_{i=1}^n \mu(E_i)x_i.$$ 

**Proof.** By Theorem 7 it is sufficient to prove that $x \chi_E$ ($E \in \mathcal{S}, x \in X$) is integrable and

$$\int x \chi_E \, d\mu = \mu(E)x.$$ 

First we suppose that $x \geq 0$. The regularity of the measure $\mu$ implies that for $E \in \mathcal{S}$ and $x \in X$, $x \geq 0$ there exists a bounded sequence $(a_{nk})_{n,k} \subset Y$, $a_{nk} \geq 0$, $a_{nk} \searrow 0$ ($k \to \infty$, $n = 1, 2, \ldots$) such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there exist an open set $U \in \mathcal{S}$ and a compact set $C \in \mathcal{S}$, $C \subset E \subset U$ so that

$$\mu(U \setminus C)x < \sqrt{a_{i\varphi(i)}}.$$ 

159
Since $C$ is compact and $U$ is open there exists $δ ∈ U(T)$ such that

\[
δ(t) ⊂ U \quad \text{for } t ∈ C, \\
δ(t) ⊂ U \setminus C \quad \text{for } t ∈ U \setminus C, \\
δ(t) ∩ C = ∅ \quad \text{for } t ∉ U.
\]

Take $D ∈ A(δ)$, $D = \{(E_i, t_i), i = 1, 2, \ldots, n\}$. By Lemma 4 we have

\[
μ(C)x ≤ μ(E)x ≤ μ(U)x
\]

and

\[
μ(U \setminus C)x = μ(U)x - μ(C)x.
\]

Now

\[
μ(E)x - \bigvee_i a_{iφ(i)} ≤ μ(U)x - \bigvee_i a_{iφ(i)} < μ(C)x ≤
\]

\[
≤ μ(\bigcup_{t_i ∈ C} E_i)x = \sum_{t_i ∈ C} μ(E_i)x = \sum_{i=1}^n χ_C(t_i)μ(E_i)x ≤
\]

\[
≤ \sum_{i=1}^n χ_E(t_i)μ(E_i)x = \sum_{i=1}^n μ(E_i)(xχ_E(t_i)) =
\]

\[
= S(xχ_E, D) ≤ \sum_{i=1}^n μ(E_i)(xχ_U(t_i)) = \sum_{t_i ∈ U} μ(E_i)x =
\]

\[
= μ(\bigcup_{t_i ∈ U} E_i)x ≤ μ(U)x ≤ μ(C)x + \bigvee_i a_{iφ(i)} ≤
\]

\[
≤ μ(E)x + \bigvee_i a_{iφ(i)}.
\]

Then

\[
- \bigvee_i a_{iφ(i)} ≤ S(xχ_E, D) - μ(E)x ≤ \bigvee_i a_{iφ(i)}
\]

and hence

\[
|S(xχ_E, D) - μ(E)x| ≤ \bigvee_i a_{iφ(i)}
\]

160
INTEGRATION WITH RESPECT TO OPERATOR VALUED MEASURES IN RIESZ SPACES

for any $D \in A(\delta)$. In the general case for $x \in X$ we get

$$\int x \chi_E d\mu = \int (x^+ - x^-) \chi_E d\mu = \int x^+ \chi_E d\mu - \int x^- \chi_E d\mu =$$

$$= \mu(E)x^+ - \mu(E)x^- = \mu(E)x.$$

□

Limit Theorem

**Lemma 14.** If $f_n : T \to X$ is integrable for $n = 1, 2, \ldots, f_n \to f$ uniformly and $f$ is bounded, then $\lim_{n \to \infty} \int f_n d\mu$ exists.

**Proof.** It is sufficient to show that the sequence $(\int f_n d\mu)_n$ is bounded and

$$\bigvee_{n=1}^{\infty} \bigwedge_{i=n}^{\infty} \int f_i d\mu \leq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} \int f_j d\mu.$$

Since the function $f$ is bounded, there exists $h \in X$, $h \geq 0$, such that $|f(t)| \leq h$ for all $t \in T$.

From the uniform convergence of $f_n$ there exists a sequence $(a_n)_n \subset X$, $a_n \to 0 \ (n \to \infty)$ and for any $t \in T$

$$|f_n(t) - f(t)| \leq a_n$$

for all $n$. Hence

$$-h - a_1 \leq f(t) - a_1 \leq f(t) - a_n \leq f_n(t) \leq f(t) + a_n \leq h + a_1$$

and

$$|f_i(t) - f_j(t)| \leq |f_i(t) - f(t)| + |f_j(t) - f(t)| \leq a_i + a_j \leq 2a_n$$

for any $t \in T$ and $i, j \geq n$. It is evident that if for $f : T \to X$, $f(t) = a$ for all $t \in T$, then

$$\sum_{j=1}^{n} \mu(E_j) f(t_j) = \sum_{j=1}^{n} \mu(E_j) a = \mu(T)a$$

for any $D \in A(\delta)$ and any $\delta$. By Theorems 7 and 8 for any $n$ we have

$$\mu(T)(-h - a_1) \leq \int f_n d\mu \leq \mu(T)(h + a_1).$$
and

\[ \mu(T)(-2a_n) \leq \int (f_i - f_j) \, d\mu = \int f_i \, d\mu - \int f_j \, d\mu \leq \mu(T)(2a_n) \quad \text{for} \quad i, j \geq n. \]

Then the sequence \((\int f_n \, d\mu)_n\) is bounded and

\[ \mu(T)(-2a_n) + \int f_j \, d\mu \leq \int f_i \, d\mu \leq \int f_j \, d\mu + \mu(T)(2a_n) \]

for \(i, j \geq n\), which implies

\[ \bigvee_{i=n}^{\infty} \int f_i \, d\mu \leq \bigwedge_{j=n}^{\infty} \int f_j \, d\mu + \mu(T)(2a_n) \]

for all \(n\), and hence from continuity of \(\mu(T)\) we get

\[ \bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} \int f_i \, d\mu \leq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} \int f_j \, d\mu. \]

> **Theorem 15.** Let \(f_n : T \to X\) be integrable for \(n = 1, 2, \ldots, f_n \to f\) uniformly and \(f\) is bounded. Then \(f\) is integrable and \(\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu\).

**Proof.** By Lemma 14 \(\lim_{n \to \infty} \int f_n \, d\mu = c\) exists and hence there exists a sequence \((c_n)_n \subset Y\), \(c_n \downarrow 0\) \((n \to \infty)\) and

\[ \left| \int f_n \, d\mu - c \right| \leq c_n \]

for any \(n\). The function \(f_n\) is integrable and then there exists a bounded double sequence \((a_{nij})_{i,j} \subset Y\) such that \(a_{nij} \downarrow 0\) \((j \to \infty, i, n = 1, 2, \ldots)\) and for every \(\varphi : \mathbb{N} \to \mathbb{N}\) there exists \(\delta_n \in U(T)\) such that for every \(D \in A(\delta_n)\)

\[ \left| \int f_n \, d\mu - S(f_n, D) \right| < \bigvee_{i} a_{n\varphi(i+n+1)} \cdot \]

Since \(f_n \to f\) uniformly, there exists a sequence \((b_n)_n \subset X\), \(b_n \downarrow 0\) and \(|f_n(t) - f(t)| \leq b_n\) for any \(t \in T\) and all \(n\).
INTEGRATION WITH RESPECT TO OPERATOR VALUED MEASURES IN RIESZ SPACES

Let $\varphi \in \mathbb{N}^\mathbb{N}$. Put $k = \min_j \varphi(j + 1)$ and take $D \in A(\delta_k)$.

$$D = \{(E_1, t_1), (E_2, t_2), \ldots, (E_r, t_r)\}.$$ 

Then

$$|S(f, D) - c| \leq |S(f, D) - S(f_k, D)| + \left| \int f_k \, d\mu \right| + \left| \int f_k \, d\mu - c \right| <$$

$$< \sum_{i=1}^r \mu(E_i) \left| \left( f(t_i) - f_k(t_i) \right) \right| + \sqrt{a_{ki} \varphi(i+k+1)} + c_k \leq$$

$$\leq \sum_{i=1}^r \mu(E_i) b_k + \sqrt{a_{ki} \varphi(i+k+1)} + c_k \leq$$

$$\leq \mu(T) b_k + c_k + \sqrt{a_{ki} \varphi(i+k+1)} = d_k + \sqrt{a_{ki} \varphi(i+k+1)};$$

where

$$d_j = \mu(T) b_j + c_j \quad \text{for} \quad j = 1, 2, \ldots, (d_j) \in Y,$$

$$d_j \searrow 0 \quad (j \to \infty), \quad \text{since} \quad \mu(T) b_j \searrow 0 \quad (j \to \infty),$$

$$d_k = d_{\min_j \varphi(j+1)} = \sqrt{d_{\varphi(i+1)}}.$$

Put $b_{ij} = d_j$ for $i, j = 1, 2, \ldots$ and $b_{m+1 ij} = a_{ni j}$ for $n, i, j, m = 1, 2, \ldots$. Now

$$|S(f, D) - c| < \sqrt{d_{\varphi(i+1)}} + \sqrt{a_{ki} \varphi(i+k+1)} =$$

$$= \sqrt{b_{1 i} \varphi(i+1)} + \sqrt{b_{k+1 i} \varphi(i+k+1)} \leq$$

$$\leq \sum_{n=1}^\infty \sqrt{b_{ni \varphi(i+n)}}.$$

There exists $h \in X$, $h \geq 0$ such that $|f(t)| \leq h$ for any $t \in T$, since $f$ is bounded. Then

$$|S(f, D) - c| \leq |S(f, D)| + |c| = \left| \sum_{i=1}^r \mu(E_i) f(t_i) \right| + |c| \leq$$

$$\leq \sum_{i=1}^r \mu(E_i)|f(t_i)| + |c| \leq \mu(T) h + |c| \leq a,$$

163
where $a \in X$, $a > 0$ and

$$|S(f, D) - c| \leq a \wedge \left( \sum_{n=1}^{\infty} \bigvee_{i} b_{n i \varphi(i+n)} \right).$$

By Lemma 3 there exists a bounded double sequence $(a_{ij})_{i,j} \subset Y$, $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) and

$$a \wedge \left( \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} b_{n i \varphi(i+1)} \right) \leq \bigvee_{i=1}^{\infty} a_{i \varphi(i)}.$$

Therefore there exists $c \in Y$, $c = \lim_{n \to \infty} \int f_n \, d\mu$ and the bounded double sequence $(a_{ij})_{i,j} \subset Y$, $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) and for every $\varphi \in \mathbb{N}^N$ there exists $\delta \in U(T)$ ($\delta = \delta_{\min} \varphi(j+1)$) such that

$$|S(f, D) - c| \leq \bigvee_{i=1}^{\infty} a_{i \varphi(i)}$$

for any $D \in A(\delta)$. Hence $f$ is integrable and

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

\[ \square \]

REFERENCES


164
INTEGRATION WITH RESPECT TO OPERATOR VALUED MEASURES IN RIESZ SPACES


Received September 3, 1992

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
814 73 Bratislava
SLOVAKIA

Department of Mathematics
Faculty of Sciences
Pedagogical University
Parská 3
947 74 Nitra
SLOVAKIA

165