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SOME REMARKS ON METRIC PRESERVING FUNCTIONS

IMRICH POKORNÝ

Dedicated to the memory of Tibor Neubrunn

ABSTRACT. We obtained a characterization of the class of metric preserving functions which are the sum of the identity function and a periodic function.

DEFINITION. We call a function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ *metric preserving* iff $f \circ d\colon M \times M \to \mathbb{R}^+$ is a metric for every metric $d\colon M \times M \to \mathbb{R}^+$, where (M, d) is an arbitrary metric space and $\mathbb{R}^+$ denotes the set of nonnegative reals. We denote by $\mathscr{M}$ the set of all metric preserving functions.

Some properties of metric preserving functions were investigated in the papers [1], [2] and [3].

The purpose of the paper is to characterize the class of metric preserving functions which have the following form $f(x) = x + g(x)$, where g is a periodic function.

The following two functions are examples of such functions

$$f_1(x) = x + |\sin(x)|, \quad x \in \mathbb{R}^+,$$

and

$$f_2(x) = [x] + \sqrt{x - [x]}, \quad x \in \mathbb{R}^+,$$

where $[x]$ denotes the integer part of x.

Throughout this paper we denote by id the identity function on $\mathbb{R}^+$ (i.e., $\mathrm{id}(x) = x$ for each $x \in \mathbb{R}^+$) and by $\mathscr{G}$ the class of all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that the function $f - \mathrm{id}$ is periodic and nonconstant.

OBSERVATION 1. *$f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is subadditive iff $f - \mathrm{id}$ is subadditive.*

The proof is left as an exercise.

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LEMMA 1. *Let $f \in \mathscr{M} \cap \mathscr{G}$. Then $f -$ id has the smallest period.*

Proof. By contradiction. Put $g = f - \mathrm{id}$. Suppose there does not exist the smallest period of g. Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ of positive periods of g such that $T_n \to 0$. In this case $f(T_n) = T_n + g(T_n) = T_n$, since $g(0) = g(T_n) = 0$. By [2, Lemma 2] there is a neighbourhood $\mathcal{U}$ of 0 on which $f(x) = x$ and hence $g(x) = 0$ on $\mathcal{U}$. Then from periodicity of g it follows that $g \equiv 0$, a contradiction. □

PROPOSITION. *Let $f \in \mathscr{M} \cap \mathscr{G}$. Then f is nondecreasing.*

Proof. Put $g = f - \mathrm{id}$. Denote by T_g the smallest period of g.

First we show that f is nondecreasing on $(0, T_g)$. We prove it by contradiction. Suppose that there are $x_1, x_2 \in (0, T_g)$ such that $x_1 < x_2$ and $f(x_1) > f(x_2)$. Let $a = T_g + x_1$, $b = T_g$ and $c = x_2$. Then (a, b, c) is a triangle tripled ($a \le b + c$, $b \le a + c$ and $c \le a + b$) and by [3, Proposition 1]

$$\begin{aligned} f(a) = f(x_1 + T_g) \le f(b) + f(c) = f(T_g) + f(x_2) = \\ = T_g + f(x_2) < T_g + f(x_1) = T_g + x_1 + g(x_1) = \\ = x_1 + T_g + g(x_1 + T_g) = f(x_1 + T_g) = f(a). \end{aligned}$$

We have a contradiction.

Since for each $k \in \mathbb{N}$ and $x \in (0, T_g)$ we have

$$\begin{aligned} f(x + k, T_g) = x + k \cdot T_g + g(x + k \cdot T_g) = x + k \cdot T_g + g(x) = \\ = k \cdot T_g + x + g(x) = k \cdot T_g + f(x), \end{aligned}$$

the function f is nondecreasing on $\mathbb{R}^+$. □

According to [1, Lemma 2.3, Lemma 2.5 and Proposition 1.1] we obtain the following result as an immediate corollary.

THEOREM 1. *Let $f \in \mathscr{G}$. Then $f \in \mathscr{M}$ iff the following conditions hold:*

(i) $\forall a \in \mathbb{R}^+ : f(a) = 0 \iff a = 0$,
(ii) *f is subadditive,*
(iii) *f is nondecreasing.*

LEMMA 2. *Let $f \in \mathscr{M} \cap \mathscr{G}$. Put $g = f - \mathrm{id}$. Then $g(x) \ge 0$ for all $x \in \langle 0, T_g)$, where T_g is the smallest period of g.*

Proof. By contradiction.

Suppose that there is $a \in (0, T_g)$ such that $g(a) < 0$ and hence $f(a) < a$. Then $a - f(a) > 0$ and there is $k \in \mathbb{N}$ such that

$$k \cdot \big(a - f(a)\big) > T_g . \tag{1}$$

There is $\ell \in \mathbb{N}$ such that $\ell \cdot T_g \leq k \cdot a < (\ell + 1) \cdot T_g$ and

$$0 \leq k \cdot a - \ell \cdot T_g < T_g . \tag{2}$$

According to subadditivity of g and the inequalities (2), (1) we have:

$$\begin{aligned} f(k \cdot a - T_g \cdot \ell) &= (k \cdot a - T_g \cdot \ell) + g(k \cdot a - T_g \cdot \ell) = \\ &= g(k \cdot a) + (k \cdot a - T_g \cdot \ell) \leq k \cdot g(a) + (k \cdot a - T_g \cdot \ell) = \\ &= k \cdot \big(f(a) - a\big) + (k \cdot a - T_g \cdot \ell\big) < -T_g + T_g , \end{aligned}$$

i.e., $f(k \cdot a - T_g \cdot \ell) < 0$, what is a contradiction. □

OBSERVATION 2. *Let $f \in \mathscr{M} \cap \mathscr{G}$, $g = f - \mathrm{id}$ and $g(a) = 0$ for some $a \in \mathbb{R}^+$. Then $g(k \cdot a) = 0$ for each $k \in \mathbb{N}$.*

LEMMA 3. *Let $f \in \mathscr{M} \cap \mathscr{G}$, $g = f - \mathrm{id}$ and m, n be relatively prime positive integers such that $g\left(\frac{m}{n} \cdot T_g\right) = 0$, where T_g is the smallest period of g. Then for each $i \in \mathbb{N}$ $g\left(\frac{i}{n} \cdot T_g\right) = 0$.*

Proof. Let $k, \ell \in \mathbb{N}$ such that

$$k \cdot m = \ell \cdot n + 1 . \tag{3}$$

Then by Observation 2 $g\left(k \cdot \frac{m}{n} \cdot T_g\right) = 0$ and from (3) we have

$$\begin{aligned} g\left(\frac{k \cdot m}{n} \cdot T_g\right) &= g\left(\frac{\ell \cdot n + 1}{n} \cdot T_g\right) = g\left(\ell \cdot T_g + \frac{1}{n} \cdot T_g\right) = \\ &= g\left(\frac{1}{n} \cdot T_g\right) = 0 . \end{aligned}$$

By Observation 1 $g\left(\frac{i}{n} \cdot T_g\right) = 0$ for every $i \in \mathbb{N}$. □

THEOREM 2. *Let $f \in \mathscr{M} \cap \mathscr{G}$. Put $g = f - \mathrm{id}$. Then $g(x) > 0$ for every $x \in (0 \cdot T_g)$, where T_g is the smallest period of g.*

Proof. By contradiction.
Suppose that there is $a \in (0, T_g)$ such that $g(a) = 0$.

1) Let $\frac{a}{T_g}$ be a rational number. Let $m, n \in \mathbb{N}$ such that $a = \frac{m}{n} \cdot T_g$. By Lemma 3 we obtain that $g\left(\frac{1}{n}, T_g\right) = 0$. Let $x \in \left(0, \frac{1}{n} \cdot T_g\right)$ and let $k \in \mathbb{N} \cap (1, n)$. Then from subadditivity of g

$$g(x) = g(x) + g\left(\frac{k}{n} \cdot T_g\right) \geq g\left(x + \frac{k}{n} \cdot T_g\right) = g\left(x + \frac{k}{n} \cdot T_g\right) +$$
$$+ g\left(\frac{n-k}{n} \cdot T_g\right) \geq g\left(x + \frac{k}{n} \cdot T_g + \frac{n-k}{n} \cdot T_g\right) =$$
$$= g(x + T_g) = g(x).$$

Therefore $g\left(x + \frac{k}{n} \cdot T_g\right) = g(x)$ which shows that $T = \frac{1}{n} \cdot T_g$ is a period of g. This contradicts the definition of T_g.

2) Let $\frac{a}{T_g}$ be an irrational number. It is well-known that the set $\{k \cdot x - [k \cdot x]; k \in \mathbb{N}\}$ is a dense set on $[0, 1]$ for arbitrary irrational x. Put $A = \left\{k \cdot \frac{a}{T_g} - \left[k \cdot \frac{a}{T_g}\right]; k \in \mathbb{N}\right\}$.

The set $B = T_g \cdot A = \{T_g \cdot x\,;\, x \in A\}$ is a dense set on $[0, T_g]$. From Observation 2 it follows that $g(x) = 0$ for every $x \in B$ (since $x = k \cdot a - \ell \cdot T_g$ for suitable $k, \ell \in \mathbb{N}$). Hence there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in B$ and $x_n \to 0$. Therefore $f(x_n) = x_n$ and by [2. Lemma 2] there is a neighbourhood $\mathcal{U}$ of 0 such that $f = \mathrm{id}$ on $\mathcal{U}$ and hence $g(x) = 0$ on $\mathcal{U}$. Then there is $m \in \mathbb{N}$ such that $b = \frac{1}{m} \cdot T_g$ and $b \in \mathcal{U}$. Then $g(b) = 0$ and $\frac{b}{T_g} = \frac{1}{m}$ is a rational number. This case was discussed in the previous part of this proof. □

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Department of Metallforming
Faculty of Metallurgy
Technical University
Vysokoškolská 4
040 01 Košice
SLOVAKIA