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UNCERTAINTY MEASURES OF FUZZY PROPOSITIONS AND THEIR USE IN FUZZY INFERENCE

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0. Introduction

The connection of fuzzy sets theory with the possibility theory has been studied many times, e.g. by L. Zadeh in his famous paper [1] already in 1979. On the other hand, the book [2] by G. Shafer in 1976 has presented a qualified insight into a subject's propositions and his beliefs are true. These papers and other ones were a particular answer to the unpronounced challenge of modal logic about the need to inquire the relation of the possible and the necessary. While the possibility theory belongs to the solution of the uncertainty problem of logical propositions from the possibility point of view, the belief (or credibility) theory is related rather to the necessity problem in epistemic logic.

However, according to the modal duality of the operators for possibility and necessity, there does not suffice only one measure to define completely the uncertainty problem for propositions of a logical system; there are necessary and sufficient two uncertainty measures, which are in a complementary relation. Therefore, a necessity measure was added to the possibility measure for propositions in classical logic, and a measure of plausibility (playing the role of possibility) was added to the measure of credibility (playing the role of necessity) in epistemic logic.

In the paper, we are trying to solve the uncertainty problem by defining dual measures of possibility and necessity in a non-standard environment – for propositions of fuzzy logic.

1. The possibility and necessity measures of fuzzy propositions

Let $U \neq \emptyset$ be a universe of discourse of object u, and $B \neq \emptyset$ a system of subset covering the universe U. We represent subsets $B_i \in B$ by a unique function $\mu_{B_i} = \mu_i$, which is either the characteristic function of the set B_i (if B_i is a non-fuzzy set), or the membership function of the set B_i (if B_i is a fuzzy set), i.e.

$$\forall u \in U : \mu_i(u) \begin{cases} \in \{0,1\}, & \text{if } B \text{ is a crisp set}, \\ \in [0,1], & \text{if } B \text{ is a fuzzy set}. \end{cases}$$

Let X be a variable in the universe U represented by a fuzzy subset $A \subseteq U$ with the membership function μ_A. It is known [3] that if the variable X is represented by a possibility distribution π_X, the fuzzy proposition "X is A" is equivalent to the relation

$$\pi_X(u) = \mu_A(u), \quad u \in U. \tag{1}$$

Assume that the possibility distribution π_X, as well as all the possibility distributions which will appear in the following text in an implicit or explicit form, are supposed to be normalized, i.e.

$$\exists u \in U : \pi_X(u) = 1.$$

It is our intention to define a pair of set functions,

$$\Pi : B \to [0,1], \quad \text{resp.} \quad N : B \to [0,1],$$

expressing a possibility or necessity measure, respectively, such that the proposition "X is B_i" is true at a given proposition "X is A" represented by the relation (1).

The product $B_i \bigcap A$ characterizes a degree of agreement of the set B_i and A in a natural way. Therefore, it is suitable to use it for the definition of the possibility measure Π, for the propositions "X is B_i" at a given proposition "X is A" [4]:

$$\Pi_X(B_i\,;\,A) = \sup_{u \in U} \min\{\mu_i(u), \mu_A(u)\}, \forall B_i \in B. \tag{2}$$

Hence, for the complementary proposition "X is $\overline{B}_i$" = "X is not B_i",

$$\Pi_X(\overline{B}_i\,;\,A) = \sup_{u \in U} \min\{1 - \mu_i(u), \mu_A(u)\}. \tag{3}$$

The uncertainty measures, Π a N, are special cases of the operators <> of possibility and $\Box$ of necessity from modal logic, which satisfy the formal dual relation

$$<> = \neg \Box \neg,$$

meaning that for every particular pair of uncertainty measures, e.g. Π a N, the following dual modal condition must be fulfilled,

$$\Pi(p) = 1 - N(\neg p), \tag{4}$$

where p's are proper propositions of the given logical system. In our case this equality has the form

$$\Pi_X(B_i\,;\,A) = 1 - N_X(\overline{B}_i\,;\,A), \quad \forall B_i \in B. \tag{5}$$

To fulfil the equality (5) it requires for the possibility measure, Π, defined by (2), to define the necessity measure, N, by

$$N_X(B_i\,;\,A) = \inf_{u\in U} \max\{\mu_i(u), 1 - \mu_A(u)\}, \tag{6}$$

which follows from the duality (5) inserting (3),

$$N_X(B_i\,;\,A) = 1 - \Pi_X(\overline{B}_i; A) = 1 - \sup_{u\in U} \min\{1 - \mu_i(u), \mu_A(u)\} = \\ = \inf_{u\in U} \max\{\mu_i(u), 1 - \mu_A(u)\}.$$

So far we have dealt with the uncertainty of propositions of the form "X is B_i" related to the variable X in the universe U. To define the uncertainty measures, Π and N, we have used a set representation. From the construction used it is obvious that if we denote B_X the system B of the subsets B_i, by which we have described appropriate properties of the variable X at a given A, and if we denote P_X the set of appropriate elementary propositions of the form "X is B_i" at a given fundamental fuzzy proposition "X is A", then the Boolean algebra $\mathcal{B}_X = (B_X, ^-, \cap, \cup)$ of sets is isomorphic to the Boolean algebra $\mathcal{P}_X = (P_X, \neg, \wedge, \vee)$ of propositions.

It is obvious that if the fuzzy set A in the proposition "X is A" is reduced to the (non-fuzzy) singleton $\{u_0\}$, then μ_A is reduced to the corresponding characteristic function with the only nonzero value $\mu_A(u_0) = 1$, otherwise $\emptyset$. If it is the case, the uncertainties, Π a N according to (2) and (6), are

$$\Pi_X(B_i\,;\,A) = \Pi_X(B_i\,;\,\{u_0\}) = \begin{cases} 1, & \text{if } u_0 \in B_i, \\ 0, & \text{otherwise}, \end{cases}$$

$$N_X(B_i\,;\,A) = N_X(B_i\,;\,\{u_0\}) = \begin{cases} 1, & \text{if } u_0 \in B_i, \\ 0, & \text{otherwise}. \end{cases}$$

The expressions (2) and (6) define possibility and necessity measures for elementary propositions. To extend those definitions to a general case of sets from the system B_X or propositions from the Boolean algebra $\mathcal{P}_X$, some elementary definitions for composition of membership functions or truth evaluating for compound fuzzy sets or compound propositions are to be used, e.g.

$$\begin{aligned} \mu_{\overline{F}}(x) &= 1 - \mu_F(x)\,, \\ \mu_{F\cap G}(x) &= \min\{\mu_F(x), \mu_G(x)\}\,, \\ \mu_{F\cup G}(x) &= \max\{\mu_F(x), \mu_G(x)\}\,, \end{aligned}$$

or

$$\begin{aligned} v(\neg F) &= 1 - v(F)\,, \\ v(F \wedge G) &= \min\{v(F), v(G)\}\,, \\ v(F \vee G) &= \max\{v(F), v(G)\}\,. \end{aligned}$$

Then, for the sets B_i, $B_j \in B_X$ we define the possibility measure, Π_X, and the necessity measure, N_X, as follows,

$$\Pi_X(B_i \cup B_j\,;\,A) = \max\{\Pi_X(B_i\,;\,A), \Pi_X(B_j\,;\,A)\}\,, \tag{7}$$

and

$$N_X(B_i \cap B_j\,;\,A) = \min\{N_X(B_i\,;\,A), N_X(B_j\,;\,A)\}\,. \tag{8}$$

respectively.

More generally,

$$\Pi_X\Big(\bigcup B_i\,;\,A\Big) = \sup\{\Pi_X(B_i\,;\,A)\}\,, \quad \forall B_i \in \mathcal{B}_X\,, \tag{9}$$

and

$$N_X\Big(\bigcap B_i\,;\,A\Big) = \inf\{N_X(B_i\,;\,A)\}\,, \quad \forall B_i \in \mathcal{B}_X\,. \tag{10}$$

respectively.

If, for simplicity, we express elementary propositions more briefly,

$$p_0 = \text{“}X \text{ is } A\text{”}, \quad p_i = \text{“}X \text{ is } B_i\text{”}, \quad i = 1, 2, \dots\,,$$

then we can speak about the set $P_X = \{p_i\,;\,p_0\}$ of elementary propositions, or about the Boolean algebra $\mathcal{P}_X$ of propositions $p \cdot q, \dots$, as well as about the uncertainty measures $\Pi_X(p_i\,;\,p_0)$, and $N_X(p_i\,;\,p_0)$ defined by (2) and (6) in the set P_X or in the set $\mathcal{P}_X$ of compound propositions.

According to the simplification above the expressions (7), (8), (9), (10) can be expressed in the form

$$\Pi_X(p_i \vee p_j\,;\,p_0) = \max\{\Pi_X(p_i\,;\,p_0), \Pi_X(p_j\,;\,p_0)\}\,,$$

and

$$N_X(p_i \wedge p_j\,;\,p_0) = \min\{N_X(p_i\,;\,p_0), N_X(p_j\,;\,p_0)\}\,,$$

or more generally for propositions $p, q \in \mathcal{P}_X$,

$$\Pi_X(p \vee q) = \max\{\Pi_X(p), \Pi_X(q)\}\,, \quad \forall p, q \in \mathcal{P}_X\,,$$

and

$$N_X(p \wedge q) = \min\{N_X(p), N_X(q)\}\,, \quad \forall p, q \in \mathcal{P}_X\,,$$

or for an infinite number of propositions $p, q, \dots \in \mathcal{P}_X$,

$$\Pi_X(p \vee q \vee \dots) = \sup\{\Pi_X(p), \Pi_X(q), \dots\}\,,$$

and

$$N_X(p \wedge q \wedge \dots) = \inf\{N_X(p), N_X(q), \dots\}\,,$$

respectively.

It can be proved that if X, Y are independent variables in the universe U, with elementary propositions "Y is D_i" for a given variable Y at a given proposition "Y is C", then the following expressions [3] hold,

$$\begin{aligned}
\Pi_{X,Y}(B_i \times D_j\,;\,A \times C) &= \min\{\Pi_X(B_i\,;\,A), \Pi_Y(D_j\,;\,C)\}\,,\\
\Pi_{X,Y}(B_i + D_j\,;\,A \times C) &= \max\{\Pi_X(B_i\,;\,A), \Pi_Y(D_j\,;\,C)\}\,,\\
N_{X,Y}(B_i \times D_j\,;\,A \times C) &= \min\{N_X(B_i\,;\,A), N_Y(D_j\,;\,C)\}\,,\\
N_{X,Y}(B_i + D_j\,;\,A \times C) &= \max\{N_x(B_i\,;\,A), N_Y(D_j\,;\,C)\}\,,
\end{aligned}$$

where the values of possibility and necessity can be determined from proper possibility distributions according to (1).

2. Implication functions for fuzzy propositions

Generally, a natural requirement is accepted that the truth value of compound logical formulas or propositions in any logical system is a function of the truth values of their components. Thus, for fuzzy logic too, if $v(p)$, $v(q)$, $v(p \rightarrow q)$ are the truth values of the fuzzy propositions p, q, $p \rightarrow q$ from the set $\mathcal{P}_X$, then it is possible to define various functions f which determine the truth value $v(p \rightarrow q)$ of implication depending on the values $v(p)$, $v(q)$,

$$v(p \rightarrow q) = f(v(p), v(q)), \forall p, q \in \mathcal{P}_X .$$

Because in definitions of those functions there appear the so-called triangular functions, we introduce briefly definitions of those auxiliary functions which play the role of general logical operators for conjunction, disjunction and negation.

Let $f : [0,1] \times [0,1] \rightarrow [0,1]$ be a function the arguments and values of which are in the interval $[0,1]$. Then f is called a

(i) *triangular norm* denoted *, if for $\forall x, y \in [0,1]$ the following expressions hold,

$$\begin{aligned}
{}^*(0,0) &= 0, \\
{}^*(x,1) &= {}^*(1,x) = x, \\
{}^*(x,y) &= {}^*(y,x), \\
{}^*(x,y) &\leq {}^*(r,s), \quad \text{if} \quad x \leq r \quad \text{and} \quad y \leq s, \\
{}^*(x, {}^*(y,r)) &= {}^*({}^*(x,y), r);
\end{aligned}$$

(ii) *triangular conorm* denoted $\perp$, if for $\forall x, y \in [0,1]$ the following properties hold,

$$\begin{aligned}
\perp(1,1) &= 1, \\
\perp(0,x) &= \perp(x,0) = x, \\
\perp(x,y) &= \perp(y,x), \\
\perp(x,y) &\leq \perp(r,s), \quad \text{if} \quad x \leq r \quad \text{and} \quad y \leq s, \\
\perp(x, \perp(y,r)) &= \perp(\perp(x,y), r).
\end{aligned}$$

Notice that it is practical to write triangular norms and conorms in the form

$$\begin{aligned}
{}^*(x,y) &= x^* y, \\
\perp(x,y) &= x \perp y.
\end{aligned}$$

It can be shown that the maximum norm * is the function min, and the minimum norm is the function

$$T_W(x,y) = \begin{cases} x & \text{if} \quad y = 1, \\ y & \text{if} \quad x = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Thus, $T_W(x,y) \le x^*y \le \min(x,y)$ holds.

Further interesting norms are: the product $x \cdot y$, and the function max, $x^*y = \max(0, x+y-1)$, where the ordering

$$T_W(x,y) \le \max(0, x+y-1) \le x \cdot y \le \min(x,y)$$

holds.

The following dual relation holds between the norm * and conorm $\perp$

$$x \perp y = 1 - (1-x)^*(1-y).$$

From the properties above it follows that the norm * plays the role of a conjunction operator, and the conorm the role of a disjunction operator.

For the sake of completeness, we introduce also the operator n of negation which for all $x \in [0,1]$ fulfills the conditions [6]:

$$n(0) = 1,$$
$$n(n(x)) = x,$$

n is a continuously decreasing function in $[0,1]$.

For various fields of application different types of implication functions are available. Most of them can be derived from two basic schemes representing a general algorithm for calculating values $v(p \to q)$ depending on $v(p)$ a $v(q)$ by means of triangular functions.

The basic schemes for definition of implication functions are as follows,

$$v(p \to q) = n[v(p)] \perp v(q), \tag{11}$$

where n is an operator for negation and $\perp$ is a triangular conorm for disjunction, and

$$v(p \to q) = \sup_{x \in [0,1]} \{x | v(p)^* x \le v(q)\}, \tag{12}$$

where * is a triangular norm for conjunction.

The schema (11) is often used mainly in classical logic, where the formulas $p \to q$ and $\neg p \vee q$ are equivalent, and therefore

$$v(p \to q) = v(\neg q \to \neg p) \tag{13}$$

holds. However, the condition (13) is not fulfilled in many logical systems. Hence, the proper implication functions are defined according to the schema (12).

Among such implication functions are involved, e.g.
(1) *Gödel's implication function*,

$$f(v(p), v(q)) = \begin{cases} 1 & \text{if} \quad v(p) \leq v(q), \\ v(q) & \text{if} \quad v(p) > v(q). \end{cases}$$

As a drawback of this function is its inconditinuity for $v(p) > v(q)$ on the boundary $v(p) = v(q)$.
(2) *Goguen's implication function*,

$$f(v(p), v(q)) = \begin{cases} 1 & \text{if} \quad v(p) = 0, \\ \min(1, v(q)/v(p)) & \text{if} \quad v(p) \neq 0. \end{cases}$$

(3) *Lukasiewicz's implication function*,

$$f(v(p), v(q)) = \min(1, 1 - v(p) + v(q)),$$

which can be derived from both schemes, (11) and (12).

3. The use of possibility distributions in fuzzy inference

As stated in section 1, if p, q, $p \rightarrow q$ are fuzzy propositions, their membership functions can be represented according to (1) by adequate possibility distributions.

Let X be a variable in the universe U, and Y a variable in the universe V. Let

$$\text{IF } X \text{ is } A \text{ THEN } Y \text{ is } B \tag{14}$$

be a particular rule of the type IF–THEN expressing the implication $p \rightarrow q$, when $p =$ "X is A", $q =$ "Y is B" are fuzzy propositions with membership function μ_A, μ_B, where $p \in \mathcal{P}_X$, and $q \in \mathcal{P}_Y$.

If π_X, π_Y are the corresponding possibility distributions of the variables X, Y and $\pi_{Y|X}$ is the possibility distribution corresponding accordingly (1) to the proposition (14), then it can be proved [3] that the possibility distribution $\pi_{X,Y}$ representing the pair (X, Y) of variables X, Y, is

$$\pi_{X,Y}(u, v) = \pi_{Y|X}(v, u)^* \pi_X(u), \qquad \forall u \in U, \quad \forall v \in V,$$

where * is a triangular norm. From this distribution the possibility distribution π_Y of the variable Y can be obtained by the operation of projection,

$$\pi_Y(v) = \sup_{u \in U} \pi_{X,Y}(u, v) = \sup_{u \in U} (\pi_{Y|X}(v, u)^* \pi_X(u)). \tag{15}$$

Since, according to (1),

$$\pi_X(u) = \mu_A(u), \quad \pi Y(v) = \mu_B(v),$$

from (15) there follows

$$\mu_B(v) \geq \pi_{Y|X}(v,u)^* \mu_A(u).$$

From this it is possible, according to [7], to express the unknown distribution $\pi_{Y|X}$ in the form

$$\pi_{Y|X}(v,u) = \mu_A(u) \otimes \mu_B(v), \tag{16}$$

where $\otimes$ is a triangular norm defined by

$$r \otimes s = \sup_{t \in [0,1]} \{t \mid t^* r \leq s\},$$

or in our case

$$\mu_A(u) \otimes \mu_B(v) = \sup_{t \in [0,1]} \{t \mid t^* \mu_A(u) \leq \mu_B(v)\}.$$

If the norm * is min, then using this expression in (16) we obtain the answer,

$$\pi_{Y|X}(v,u) = \begin{cases} 1 & \text{if} \quad \mu_A(u) \leq \mu_B(v), \\ \mu_B(v) & \text{if} \quad \mu_A(u) > \mu_B(v), \end{cases} \tag{17}$$

which corresponds with Gödel's implication function.

When the norm * is realized by product (Goguen), then we obtain for (16)

$$\pi_{Y|X}(v,u) = \begin{cases} 1 & \text{if} \quad \mu_A(u) = 0, \\ \min(1, \mu_B(v)/\mu_A(u)) & \text{if} \quad \mu_A(u) \neq 0. \end{cases} \tag{18}$$

And if the norm is; $^* = \max(0, v(p) + v(q) - 1)$ (Lukasiewicz), then (16) implies,

$$\pi_{Y|X}(v,u) = \min\{1, 1 - \mu_A(u) + \mu_B(v)\}. \tag{19}$$

It may be seen from the expressions (17), (18), (19) that the possibility distribution appropriate to the rule (11) is identical with the corresponding implication function in the sense of the used norm.

This result can be used for computing values of the membership function of a logical consequence gained by the modus ponens inference rule generalized for fuzzy premises,

$$\begin{array}{l}\text{IF } X \text{ is } A \text{ THEN } Y \text{ is } B \\ X \text{ is } A' \\ \hline \qquad\qquad Y \text{ is } B'\end{array}$$

where $\mu_{A'} = \pi_X$ is given. Since $\pi_Y = \mu_{B'}$, we can compute

$$\mu_{B'}(v) = \sup_{u\in U}[(\mu_A(u) \otimes \mu_B(v))^* \mu_A(u)] , \quad \forall v \in V ,$$

from (15) using (16), which is a solution of the task to determine fuzzy consequence from fuzzy premises.

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