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ON ESSENTIAL DERIVED NUMBERS OF TYPICAL CONTINUOUS FUNCTIONS

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Dedicated to the memory of Tibor Neubrunn

ABSTRACT. For a typical continuous function f on $[0,1]$, f has a bilateral essential derived number at each point $x \in (0,1)$.

Let $\mathscr{C}$ denote the set of continuous real valued functions defined on $[0,1]$ furnished with the metric of uniform convergence. When we say a typical $f \in \mathscr{C}$ has a certain property $\mathcal{P}$, we shall mean that the set of $f \in \mathscr{C}$ with this property is residual in $\mathscr{C}$.

Let $x \in (0,1)$, $y \in \overline{\mathbb{R}}$ and $d \geqq 0$. Following [1], we shall say that y is a derived number of $f \in \mathscr{C}$ at x with a *density* d (*upper density* d, *right upper density* d, *symmetrical upper density* $d, \dots$) if there exists a set $E \subset \mathbb{R}$ such that the density (upper density, right upper density, symmetrical upper density,...) of E at x equals to d and $\lim\limits_{t \to x, t \in E} \big(f(t) - f(x)\big)\ (t-x)^{-1} = y$.

If y is a derived number of f at x with a right (left, symmetrical) upper density 1, we say that y is a *right* (*left, symmetrical*) *essential derived number of* f *at* x (cf. [2]).

In the following $\|f\|$ stands for the norm in $\mathscr{C}$, λ for the Lebesgue measure on the real line and $B(f,r)$ for the open ball in $\mathscr{C}$ with center f and radius r.

The following theorem is a slight improvement of a result presented in [4] (Theorem 4 (i)) without a proof. The only difference is that in [4] a weaker notion of a bilateral essential derived number is considered. The author thanks to prof. L. Mišík, since the article was inspired by one his (unpublished) question.

THEOREM. *For a typical* $f \in \mathscr{C}$ *and each* $x \in (0,1)$ *there exists* $y \in \overline{\mathbb{R}}$ *which is a symmetrical essential derived number of* f *at* x.

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Before the proof of Theorem we shall formulate two simple lemmas. The following lemma has a standard proof (cf. Theorem 14.3 from [3]) which will be omitted.

LEMMA 1. *Let $f \in \mathscr{C}$ and $x_0 \in (0,1)$. Then $y \in \overline{\mathbb{R}}$ is a symmetrical derived number of f at x_0 iff*

$$\overline{\lim_{h \to 0+}} \lambda\{x \in (x_0 - h,\, x_0 + h) \colon \big(f(x) - f(x_0)\big)(x - x_0)^{-1} \in U\}(2h)^{-1} = 1 \quad (1)$$

for each neighbourhood U of y.

LEMMA 2. *Let $f \in \mathscr{C}$, $x_0 \in (0,1)$ and let $(h_n)_1^\infty$, $(A_n)_1^\infty$, $(z_n)_1^\infty$ be sequences of real numbers such that $h_n \searrow 0$, $z_n \searrow 0$ and*

$$\lim_{n \to \infty} \lambda\{x \in (x_0 - h_n, x_0 + h_n) \colon |\big(f(x) - f(x_0)\big)(x - x_0)^{-1} - A_n| < z_n\}(2h_n)^{-1} = 1. \quad (2)$$

Then there exists $y \in \overline{\mathbb{R}}$ which is a symmetrical essential derived number of f at x.

Proof. Denote by y a cluster point of the sequence (A_n) and choose an arbitrary neighbourhood U of y. Then we can clearly choose an increasing sequence of natural numbers $(n_k)_1^\infty$ such that $(A_{n_k} - z_{n_k}, A_{n_k} + z_{n_k}) \subset U$ for each k. On account of (2) we immediately obtain that

$$\lim_{k \to \infty} \lambda\{x \in (x_0 - h_{n_k},\, x_0 + h_{n_k}) \colon \big(f(x) - f(x_0)\big)(x - x_0)^{-1} \in U\}\left(2\, h_{n_k}\right)^{-1} = 1,$$

which implies (1). □

Proof of Theorem. Let $(P_k)_1^\infty$ be a sequence of polynomials which is dense in $\mathscr{C}$. For each k put $M_k = \| P_k'' \|$ (the norm of the second derivative of P_k) and choose $0 < \delta_k < (k\, M_k)^{-1}$ such that $\delta_k \searrow 0$. Further put

$$G \colon = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} B\left(P_k,\, \delta_k (4k^2)^{-1}\right).$$

It is easy to see that G is a dense G_δ subset of $\mathscr{C}$. Choose an arbitrary $f \in G$. It is sufficient to prove that for each $x \in (0,1)$ there exists a symmetrical essential derived number of f at x. To this end choose an arbitrary $x_0 \in (0,1)$. Since $f \in G$, we can choose an increasing sequence of natural numbers $(k_n)_1^\infty$ such that $f \in B\left(P_{k_n},\, \delta_{k_n} \cdot \left(4k_n^2\right)^{-1}\right)$ for each n.

We shall show that the assumptions of Lemma 2 are satisfied for $h_n = \delta_{k_n}$, $A_n = P'_{k_n}(x_0)$ and $z_n = (k_n)^{-1}$. To this end consider n so big that $(x_0 - h_n, x_0 + h_n) \subset (0,1)$. To prove (2) it is sufficient to show that for such n and for

$$x \in \left(x_0 - h_n,\, x_0 - h_n(k_n)^{-1}\right) \cup \left(x_0 + h_n(k_n)^{-1},\, x_0 + h_n\right)$$

we have

$$\left|\, \big(f(x) - f(x_0)\big)(x - x_0)^{-1} - A_n \,\right| < z_n. \tag{3}$$

Thus suppose that $x \in \left(x_0 + h_n(k_n)^{-1},\, x_0 + h_n\right)$ is given (the other case is quite analogical). We have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{P_{k_n}(x) - P_{k_n}(x_0)}{x - x_0} \right| \leq$$

$$\leq \frac{|\, f(x) - P_{k_n}(x) \,| + |\, f(x_0) - P_{k_n}(x_0) \,|}{|\, x - x_0 \,|} < \frac{2 \cdot \delta_{k_n} \left(4\, k_n^2\right)^{-1}}{\delta_{k_n}\, (k_n)^{-1}} = (2\, k_n)^{-1}. \tag{4}$$

Further, by the Taylor formula, for some $\xi \in (0,1)$,

$$\left| \frac{P_{k_n}(x) - P_{k_n}(x_0)}{x - x_0} - P'_{k_n}(x_0) \right| = \left|\, (1/2)\, P''_{k_n}(\xi)(x - x_0) \,\right| <$$

$$< (1/2) \cdot M_{k_n} \cdot \delta_{k_n} < \left(2\, k_n\right)^{-1}. \tag{5}$$

Since (4) and (5) obviously imply (3), the proof is over. □

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