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ON SOME PROPERTIES OF DIFFERENCE OPERATOR WITH SOME CHARACTERIZATIONS

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ABSTRACT. In this paper, we introduce the generalized spaces of the form $\mathcal{H}(f, g, \Delta_n^m)$, where $\mathcal{H}$ represents one of the spaces ℓ_∞, c or c_0. The köthe duals corresponding to these spaces will be computed and construction of the Schauder bases for $\mathcal{H} \in \{c, c_0\}$ will be given. Also, some matrix characterizations concerning these spaces will be computed. Moreover, the characterization of some classes of compact operators on the spaces $\ell_\infty(f, g, \Delta_n^m)$ and $c_0(f, g, \Delta_n^m)$ by employing the Hausdorff measure of non-compactness will be determined.

1. Introduction

Let Ω be the space of real sequences. Any vector subspace of Ω is called a sequence space. By ℓ_∞, c, c_0 and $\ell_p(1 < p < \infty)$, we denote the sequence spaces of all bounded, convergent, null sequences and p-absolutely convergent series, respectively. Also, we shall write ϕ for the set of all finite sequences that terminate in zeros, $e = (1, 1, 1, \dots)$ and $e^{(j)}$ for the sequence whose only non-zero entry 1 is at the jth place for each $j \in \mathbf{N}$, where $\mathbf{N} = \{0, 1, 2, \dots\}$.

Let $\mathfrak{G} = (g_{ij})$ be an infinite matrix of real numbers g_{ij} $(i, j \in \mathbf{N})$ and $\mathfrak{G}_i$ denote the sequence in the i-th row of $\mathfrak{G}$, i.e., $\mathfrak{G}_i = (g_{ij})_{j=0}^{\infty}$ for every $i \in \mathbf{N}$. In addition, if $\zeta = (\eta_j) \in \Omega$ then we define the $\mathfrak{G}$-transform of η as the sequence $\mathfrak{G}\eta = \{(\mathfrak{G}\eta)_i\}$, where

$$(\mathfrak{G}\eta)_i = \sum_j g_{ij}\eta_j \qquad (i \in \mathbf{N}), \tag{1.1}$$

provided the series on the right-hand side exists for each $i \in \mathbf{N}$ as can be seen in [3, 4, 5, 8, 9, 39, 41].

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Let Γ and Υ be two sequence spaces. By (Γ, Υ), we represent the class of all matrices $\mathfrak{G}$ such that

$$\mathfrak{G} : \Gamma \to \Upsilon.$$

Thus, $\mathfrak{G} \in (\Gamma, \Upsilon)$ if and only if $\mathfrak{G}\eta = \{(\mathfrak{G}\eta)_i\}_{i \in \mathbf{N}} \in \Upsilon$ for all $\eta \in \Gamma$. Also, the matrix domain $\Gamma_{\mathfrak{G}}$ of an infinite matrix $\mathfrak{G}$ in sequence space Γ is defined by

$$\Gamma_{\mathfrak{G}} = \{\eta = (\eta_i) \in \Omega : \mathfrak{G}\eta \in \Gamma\}. \tag{1.2}$$

The construction of new sequence spaces were analysed by several authors, see, for instance, [2, 11] [14]–[21] [25, 32]–[35]. Also in the written work, many papers concerning these spaces which are derived by the domain of average mean or the difference operator of order m as in [2, 15, 26, 28, 29, 38].

2. The sequence spaces $\mathcal{H}\big(f, g, \Delta_n^{(m)}\big)$ for $\mathcal{H} \in \{\ell_\infty, c, c_0\}$

In this section, we introduce the spaces $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$, $c\big(f, g, \Delta_n^{(m)}\big)$ and $c_0\big(f, g, \Delta_n^{(m)}\big)$. We will also show that these spaces are the BK-spaces and are linearly isomorphic to the spaces ℓ_∞, c, c_0. Also, we give the bases for the spaces $c\big(f, g, \Delta_n^{(m)}\big)$ and $c_0\big(f, g, \Delta_n^{(m)}\big)$.

If a normed space Ψ contains a sequence (v_i) with the property that for every $\eta \in \Psi$, there is one and only one sequence of scalars (β_i) such that

$$\lim_{i\to\infty} \|\eta - (\beta_0 v_0 + \beta_1 v_1 + \cdots + \beta_i v_i)\| = 0,$$

then (v_i) is called a Schauder basis for Ψ. The series $\sum \beta_j v_j$ which has the sum η is then called the expansion ofη with respect to (v_i) and written as $\eta = \sum \beta_i v_i$.

We call a space Γ to be an *FK space* if it is a complete linear metric space with continuous coordinates $p_i : \Gamma \to \mathbf{R}$ $(i \in \mathbf{N})$, where $\mathbf{R}$ denotes the real field and $p_i(\eta) = \eta_i$ for all $\eta = (\eta_j) \in \Gamma$ and every $i \in \mathbf{N}$. Also, a *BK space* is a Banach space with continuous coordinates. The space ℓ_p $(1 \leq p < \infty)$ is BK space with $\|\eta\|_p = \Big(\sum_{j=0}^{\infty} \big|\eta_j\big|^p\Big)^{1/p}$ and c_0, c and ℓ_∞ are BK spaces with $\|\eta\|_\infty = \sup_j |\eta_j|$.

In [18], the author has defined new techniques and the space $W(\triangle)$ as follows

$$W(\triangle) = \{\eta = (\eta_j) \in \Omega : (\triangle \eta_j) \in W\},$$

where $W \in \{l_\infty, c, c_0\}$ and $\triangle \eta_j = \eta_j - \eta_{j+1}$.

It was further generalized in [16] as follows.

For an integer $m \geq 0$, we define

$$W(\Delta_m) = \{\eta = (\eta_i) : \Delta_m \eta \in W\},$$

where $W = l_\infty, c$ and c_0 with $\Delta_m \eta_i = \eta_i - \eta_{i+m}$.

Also, it was studied in [6] and [7]. Moreover, an interesting generalization of it is due to Tripathy and Esi [42] who used the above notions and unified them as follows:

$$\Delta_m^n \eta_k = \{\eta \in \Omega : (\Delta_n^m \eta_k) \in W\},$$

where

$$\Delta_m^n \eta_k = \sum_{\mu=0}^{n} (-1)^\mu \binom{n}{r} \eta_{k+m\mu}$$

and

$$\Delta_0^n \eta_k = \eta_k \forall\ k \in \mathbf{N}.$$

We shall write $\Delta = \Delta^{(1)}$ for short and follow the sign convention that any term with a negative subscript is equal to naught.

By choosing $m = 1$, we have the spaces introduced by Et and Çolak [16] and by taking $m = 1 = n$ we get what was studied Kizmaz in [18].

By $\mathcal{F}$ we denote the set of all sequences $f = (f_n)$ such that $f_i \neq 0$ for all $i \in \mathbf{N}$. For $f \in \mathcal{F}$, let $\frac{1}{f} = \left(\frac{1}{f_n}\right)$. Let $f, g \in \mathcal{F}$ and define the matrix $\mathcal{H}(f, g) = (h_{nk})$ by

$$h_{nk} = \begin{cases} f_n g_k, & (0 \leq k \leq n) \\ 0, & (k > n) \end{cases}$$

for all $k, n \in \mathbf{N}$, where f_n depends only on n and g_k only on k. The matrix $\mathcal{H}(f, h)$, defined above, is called the generalized weighted mean or factorable matrix as has been defined in [33] and many more.

Malkowsky and Savaş [26] have defined the sequence spaces $H(f, g, \Gamma)$ which consist of all sequences such that $\mathcal{H}(f, g)$-transforms of them are in $\Gamma \in \{\ell_\infty, c, c_0, \ell_p\}$. Recently, Polat, Karakaya and Şimsek [35] have studied the sequence spaces $\lambda(f, g, \Delta)$ which consist of all sequences such that $\mathcal{H}(f, g, \Delta)$-forms of them are in $\lambda \in \{\ell_\infty, c, c_0\}$, where $\mathcal{H}(f, g, \Delta) = \mathcal{H}(f, g)_\Delta$.

Following [3, 9, 26, 28, 35], we define the sequence spaces $H\big(f, g, \Delta_n^{(m)}\big)$ for $H \in \{\ell_\infty, c, c_0\}$ by

$$H\big(f, g, \Delta_n^{(m)}\big) = \Big\{\eta = (\eta_k) \in \Omega : \xi = \Big((\mathcal{H}_n^{(m)} \eta)_k\Big) \in H\Big\},$$

where the sequence $\xi = (\xi_k)$ is the $\mathcal{H}_n^{(m)} = \mathcal{H}(f, g).\Delta_n^m$-transform of a sequence $\eta = (\eta_k)$, that is,

$$\xi_k = (\mathcal{H}_n^{(m)} \eta)_k = f_k \sum_{j=0}^{k} \left[\sum_{i=j}^{k} \binom{m}{i-j} (-1)^{i-j} g_i\right] \eta_j; \quad (k \in \mathbf{N}). \tag{2.1}$$

With the notation of (1.2), we can redefine the spaces $H\big(f,g,\Delta_n^{(m)}\big)$ for $H\in\{\ell_\infty, c, c_0\}$ as the matrix domains of the triangle $\mathcal{H}_n^{(m)}$ in the spaces $H\in\{\ell_\infty, c, c_0\}$, that is

$$H\big(f,g,\Delta_n^{(m)}\big) = H_{\mathcal{H}_n^{(m)}}. \tag{2.2}$$

The definition in (2.2) includes the following special cases:

(i) If $m = 1 = n$, then $H\big(f,g,\Delta_n^{(m)}\big) = \lambda(f,g,\Delta)$ (cf. [35, 28]).

(ii) If $g = (\lambda_k - \lambda_{k-1})$, $f = (1/\lambda_n)$, $m = 1 = n$ and $H = c, c_0$, then $H\big(f,g,\Delta_n^{(m)}\big) = c_0^\lambda(\Delta)$, $c^\lambda(\Delta)$ (cf. [32]).

(iii) If $g = (1+r^k)$, $f = (1/(n+1))$, $m = 1 = n$ and $H = c, c_0, \ell_\infty$, then $H\big(f,g,\Delta_n^{(m)}\big) = a_0^r(\Delta)$, $a_c^r(\Delta)$, $a_\infty^r(\Delta)$ ([12, 13]).

Throughout we shall assume that the sequences $\eta = (\eta_k)$ and $\xi = (\xi_k)$ are connected by the relation (2.1), that is, ξ is the $\mathcal{H}_n^{(m)}$-transform of η. Then the sequence η is in any of the spaces $c_0\big(f,g,\Delta_n^{(m)}\big)$, $c\big(f,g,\Delta_n^{(m)}\big)$ or $\ell_\infty\big(f,g,\Delta_n^{(m)}\big)$ if and only if ξ is in the respective one of the spaces c_0, c or ℓ_∞.

Now, we may begin with the following result which is essential in the text.

Theorem 2.1. *The sequence spaces $H\big(f,g,\Delta_n^{(m)}\big)$ for $H \in \{\ell_\infty, c, c_0\}$ are Banach spaces with the norm given by*

$$\|\eta\|_{H(f,g,\Delta_n^{(m)})} = \|\xi\|_\infty = \sup_k \left|(\mathcal{H}_n^{(m)}\eta)_k\right|. \tag{2.3}$$

Proof. Let H be any of the spaces c_0, c or ℓ_∞. It is trivial that it is a linear space and is a normed space under the norm defined by (2.3). Thus, to establish the result, we show that every Cauchy sequence in $H\big(f,g,\Delta_n^{(m)}\big)$ is convergent. Suppose $(x^{(n)})_{n=0}^\infty$ is a Cauchy sequence in $X\big(f,g,\Delta_n^{(m)}\big)$. Thus,

$$(\forall\varepsilon>0)\ (\exists N\in\mathbf{N})\ (\forall r,s\ge N);$$

$$\left(\left\|\mathcal{H}_n^{(m)}x^{(r)} - \mathcal{H}_n^{(m)}x^{(s)}\right\|_X = \left\|x^{(r)} - x^{(s)}\right\|_{X(f,g,\Delta_n^{(m)})} < \varepsilon\right).$$

So the sequence $(\mathcal{H}_n^{(m)}x^{(n)})_{n=0}^\infty$ in X is Cauchy and since X is Banach, there exsits $x\in X$ such that

$$\left\|\mathcal{H}_n^{(m)}x^{(n)} - x\right\|_X \to 0 \quad \text{as} \quad n\to\infty.$$

But $x = \big(\mathcal{H}_n^{(m)}\big)\ \big(\mathcal{H}_n^{(m)}\big)^{-1}x$, so

$$\left\|\mathcal{H}_n^{(m)}x^{(n)} - (\mathcal{H}_n^{(m)})(\mathcal{H}_n^{(m)})^{-1}x\right\|_X = \left\|x^{(n)} - (\mathcal{H}_n^{(m)})^{-1}x\right\|_{X(f,g,\Delta_n^{(m)})} \to 0 \quad \text{as} \quad n\to\infty.$$

Now since $(\mathcal{H}_n^{(m)})^{-1}x \in X\big(f,g,\Delta_n^{(m)}\big)$, this completes the proof. □

Theorem 2.2. *Let H be any of the spaces c_0, c or ℓ_∞. Then the sequence space $H\big(f, g, \Delta_n^{(m)}\big)$ is linearly isomorphic to the space H, that is $H\big(f, g, \Delta_n^{(m)}\big) \cong H$.*

Proof. Let us define

$$\mho : H\big(f, g, \Delta_n^{(m)}\big) \to H \quad \text{by} \quad \mho(\eta) = \mathcal{H}_n^{(m)}\eta.$$

Then it is trivial that $\mho$ is linear, bijective and norm preserving, and the result follows. □

Theorem 2.3. *Define the sequences*

$$\nu^{(k)} = \Big\{\nu_n^{(k)}\Big\}_{n\in\mathbf{N}} \quad \text{and} \quad \nu^{(-1)} = \Big\{\nu_n^{(-1)}\Big\}$$

by

$$\nu_n^{(k)} = \begin{cases} 0, & (n < k), \\ \sum_{j=k}^{k+1} \binom{m+n-j-1}{n-j} \frac{(-1)^{j-k}}{f_k g_j} & (n \geq k), \end{cases} \qquad (n \in \mathbf{N}) \tag{2.4}$$

and

$$\nu_n^{(-1)} = \sum_{j=0}^{n} \sum_{i=j}^{j+1} \binom{m+n-i-1}{n-i} \frac{(-1)^{i-j}}{f_j g_i} \qquad (n \in \mathbf{N}).$$

a) *Then the sequence $\big(\nu^{(k)}\big)_{k=0}^{\infty}$ is a basis for the space $c_0\big(f, g, \Delta_n^{(m)}\big)$ and every $\zeta \in c_0\big(f, g, \Delta_n^{(m)}\big)$ has a unique representation of the form*

$$\zeta = \sum_k (\mathcal{H}_n^{(m)} x)_k \nu^{(k)}.$$

b) *Then $\big(\nu^{(k)}\big)_{k=-1}^{\infty}$ is a Schauder basis for $c\big(f, g, \Delta_n^{(m)}\big)$ and every $\zeta \in c\big(f, g, \Delta_n^{(m)}\big)$ has a unique representation of the form*

$$\zeta = l\nu^{(-1)} + \sum_k \Big[(\mathcal{H}_n^{(m)}\zeta)_k - l\Big] \nu^{(k)},$$

where $l = \lim_{k\to\infty}(\mathcal{H}_n^{(m)} x)_k$.

Proof. This is an immediate consequence of [15, Lemma 2.3]. □

3. Köthe duals for $H(f, g, \Delta_n^{(m)})$ for $H \in \{\ell_\infty, c, c_0\}$

In this section, we give the method for obtaining the $\alpha-$, $\beta-$ and $\gamma-$duals of the spaces $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$, $c\big(f, g, \Delta_n^{(m)}\big)$ and $c_0\big(f, g, \Delta_n^{(m)}\big)$.

For the sequence spaces κ and ν, the set $S(\kappa,\nu)$ defined by

$$S(\kappa,\nu)=\{\gamma=(\gamma_j)\in\Omega:\eta\gamma=(\eta_j\gamma_j)\in\nu \ \text{ for all } \ \eta\in\kappa\} \tag{3.1}$$

is called the multiplier space of κ and ν. With the help of (3.1), we denote $\alpha-,\beta-$ and $\gamma-$ duals for the space κ, respectively, as λ^α, λ^β and λ^γ and are given by

$$\kappa^\alpha=S(\kappa,\ell_1),\quad \kappa^\beta=S(\kappa,cs)\quad\text{and}\quad \kappa^\gamma=S(\kappa,bs),$$

where ℓ_1, cs and bs are the spaces of all absolutely, convergent and bounded series, respectively.

Throughout, let $\mathcal{F}$ denote the collection of all nonempty and finite subsets of $\mathbf{N}$.

Now we give the following lemmas (see [37]) which are necessary to prove Theorems 3.3–3.5.

Lemma 3.1. *$B\in(c_0,\ell_1)=(c,\ell_1)=(\ell_\infty,\ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}b_{nk}\right|<\infty.$$

Lemma 3.2. *$B\in(c_0,\ell_\infty)=(c,\ell_\infty)=(\ell_\infty,\ell_\infty)$ if and only if*

$$\sup_{n\in\mathbf{N}}\sum_k|b_{nk}|<\infty. \tag{3.2}$$

Now we shall state the following results without proof:

Theorem 3.3. *The α-dual of the spaces $H\big(f,g,\Delta_n^{(m)}\big)$ for $H\in\{\ell_\infty,c,c_0\}$ is the set*

$$d_1=\left\{a=(a_n)\in\Omega:\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}c_{nk}\right|<\infty\right\},$$

where the matrix $C=(c_{nk})$ is defined via the sequence $a=(a_n)$ by

$$c_{nk}=\begin{cases}\sum\limits_{j=k}^{k+1}\binom{m+n-j-1}{n-j}\frac{(-1)^{j-k}}{f_kg_j}a_n & (0\le k\le n),\\ 0 & (k>n),\end{cases}\qquad (n,k\in\mathbf{N}).$$

Theorem 3.4. *Define the sets d_2, d_3, d_4 and d_5 as follows:*

$$d_2=\left\{a=(a_k)\in\Omega:\sup_{n\in\mathbf{N}}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\nabla_n^{(m)}(j,k)a_j\right|<\infty\right\},$$

$$d_3=\left\{a=(a_k)\in\Omega:\sum_{j=k}^{\infty}\nabla_n^{(m)}(j,k)a_j\ \text{exists for each}\ k\in\mathbf{N}\right\},$$

$$d_4 = \left\{ a = (a_k) \in \Omega : \lim_{n\to\infty} \sum_{k=0}^{n} \sum_{j=k}^{n} \nabla_n^{(m)}(j,k) a_j \quad \textit{exists} \right\}$$

and

$$d_5 = \left\{ a = (a_k) \in \Omega : \lim_{n\to\infty} \sum_{k} \left| \sum_{j=k}^{n} \nabla_n^{(m)}(j,k) a_j \right| = \sum_{k} \left| \sum_{j=k}^{\infty} \nabla^{(m)}(j,k) a_j \right| \right\}.$$

Then

$$\left\{ c_0\big(f,g,\Delta_n^{(m)}\big) \right\}^{\beta} = d_2 \cap d_3,$$

$$\left\{ c\big(f,g,\Delta_n^{(m)}\big) \right\}^{\beta} = d_2 \cap d_3 \cap d_4$$

and

$$\left\{ \ell_\infty\big(f,g,\Delta_n^{(m)}\big) \right\}^{\beta} = d_3 \cap d_5.$$

Theorem 3.5. *The γ-dual of the spaces $H\big(f,g,\Delta_n^{(m)}\big)$ for $X \in \{\ell_\infty, c, c_0\}$ is the set d_2.*

4. Certain matrix transformation on $H\big(f,g,\Delta_n^{(m)}\big)$ for $H \in \{\ell_\infty, c, c_0\}$

This section deals with the characterization of various matrix classes on the given spaces $c_0\big(f,g,\Delta_n^{(m)}\big)$, $c\big(f,g,\Delta_n^{(m)}\big)$ and $\ell_\infty\big(f,g,\Delta_n^{(m)}\big)$.

For an infinite matrix $B = (b_{nk})$, for simplification, we denote

$$\bar{b}_{nk}^{\ell} = \sum_{j=k}^{\ell} \nabla_n^{(m)}(j,k) b_{nj} \qquad (k < m)$$

and

$$\bar{b}_{nk} = \sum_{j=k}^{\infty} \nabla_n^{(m)}(j,k) b_{nj} \tag{4.1}$$

for all $n, k, \ell \in \mathbf{N}$ provided the series on the right-hand side to be convergent.

Now, let us consider the following conditions:

$$\sup_n \left(\sum_{k=0}^{\infty} \left| \bar{b}_{nk} \right| \right) < \infty, \tag{4.2}$$

$$\lim_{\ell\to\infty} \sum_{k=0}^{\infty} \left| \bar{b}_{nk}^{\ell} \right| = \sum_{k=0}^{\infty} \left| \bar{b}_{nk} \right| \quad (n \in \mathbf{N}), \tag{4.3}$$

$$\bar{b}_{nk} \quad \text{exists for all} \quad k, n \in \mathbf{N}, \tag{4.4}$$

$$\sup_{\ell\in\mathbf{N}}\sum_{k=0}^{\ell}\left|\bar{b}_{nk}^{\ell}\right|<\infty \qquad (n\in\mathbf{N}), \tag{4.5}$$

$$\lim_{n\to\infty}\bar{b}_{nk}=\bar{\alpha}_k \quad (k\in\mathbf{N}), \tag{4.6}$$

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\left|\bar{b}_{nk}-\bar{\alpha}_k\right|=0, \tag{4.7}$$

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\bar{b}_{nk}=\alpha, \tag{4.8}$$

$$\sum_{k=0}^{\infty}\bar{b}_{nk} \quad \text{converges for all} \quad n\in\mathbf{N}, \tag{4.9}$$

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\left|\bar{b}_{nk}\right|=0, \tag{4.10}$$

$$\lim_{n\to\infty}\bar{b}_{nk}=0 \quad \text{for all} \quad k\in\mathbf{N}, \tag{4.11}$$

$$\sup_{K\in\mathcal{F}}\left(\sum_{n=0}^{\infty}\left|\sum_{k\in K}\bar{b}_{nk}\right|^{p}\right)<\infty \qquad (1\le p<\infty), \tag{4.12}$$

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\bar{b}_{nk}=0. \tag{4.13}$$

Then by combining Theorem 3.4 with the results of Stieglitz and Tietz [37], we immediately derive the following results.

Theorem 4.1. *We have:*

(a) $B\in\big(\ell_\infty\big(f,g,\Delta_n^{(m)}\big),\ell_\infty\big)$ *if and only if* (4.2), (4.3) *and* (4.4) *hold.*

(b) $B\in\big(c\big(f,g,\Delta_n^{(m)}\big),\ell_\infty\big)$ *if and only if* (4.2), (4.4) *and* (4.5) *hold.*

(c) $B\in\big(c_0\big(f,g,\Delta_n^{(m)}\big),\ell_\infty\big)$ *if and only if* (4.2), (4.4) *and* (4.5) *hold.*

Theorem 4.2. *We have:*

(a) $B\in\big(\ell_\infty\big(f,g,\Delta_n^{(m)}\big),c\big)$ *if and only if* (4.3), (4.4), (4.6) *and* (4.7) *hold.*

(b) $B\in\big(c\big(f,g,\Delta_n^{(m)}\big),c\big)$ *if and only if* (4.2), (4.4), (4.5), (4.6) *and* (4.8) *hold.*

(c) $B\in\big(c_0\big(f,g,\Delta_n^{(m)}\big),c\big)$ *if and only if* (4.2), (4.4), (4.5) *and* (4.6) *hold.*

Theorem 4.3. *We have:*

(a) $B\in\big(\ell_\infty\big(f,g,\Delta_n^{(m)}\big),c_0\big)$ *if and only if* (4.3), (4.4) *and* (4.10) *hold.*

(b) $B\in\big(c\big(f,g,\Delta_n^{(m)}\big),c_0\big)$ *if and only if* (4.2), (4.4), (4.5), (4.11) *and* (4.13) *hold.*

(c) $B\in\big(c_0\big(f,g,\Delta_n^{(m)}\big),c_0\big)$ *if and only if* (4.2), (4.4), (4.5) *and* (4.11) *hold.*

Theorem 4.4. *Let $1 \leq p < \infty$. Then, we have:*

(a) $B \in \big(\ell_\infty\big(f, g, \Delta_n^{(m)}\big), \ell_p\big)$ *if and only if* (4.3), (4.4) *and* (4.12) *hold.*

(b) $B \in \big(c\big(f, g, \Delta_n^{(m)}\big), \ell_p\big)$ *if and only if* (4.4), (4.5), (4.9) *and* (4.12) *hold.*

(c) $B \in \big(c_0\big(f, g, \Delta_n^{(m)}\big), \ell_p\big)$ *if and only if* (4.4), (4.5) *and* (4.12) *hold.*

Now, we may present the lemma given by Başar and Altay [1, Lemma 5.3] which is useful for obtaining the characterization of some new matrix classes from Theorems 4.1–4.3.

Lemma 4.5. *Let λ, μ be any two sequence spaces, A be an infinite matrix and B a triangle matrix. Then, $B \in (\lambda, \mu_B)$ if and only if $T = BB \in (\lambda, \mu)$.*

We should finally note that if b_{nk} is replaced by

$$t_{nk} = f_n \sum_{j=0}^{n} \left[\sum_{i=j}^{n} \binom{m}{i-j} (-1)^{i-j} v_i \right] b_{jk}$$

for all $k, n \in \mathbf{N}$ in Theorems 4.1–4.3, then one can derive the characterization of the classes

$$\lambda\Big(\big(f, g, \Delta_n^{(m)}\big), \mu\big(f, g, \Delta_n^{(m)}\big)\Big)$$

from Lemma 4.5 with $B = \mathcal{H}_n^{(m)}$, where $\lambda, \mu \in \{c_0, c, \ell_\infty\}$.

5. Measure of noncompactness of matrix operators on the sequence spaces $c_0\big(f, g, \Delta_n^{(m)}\big)$ and $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$

In this section, we characterize some classes of compact operators on the spaces $c_0\big(f, g, \Delta_n^{(m)}\big)$ and $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$ by using the Hausdorff measure of noncompactness.

It is quite natural to find conditions for a matrix map between BK-spaces to define a compact operator since a matrix transformation between BK-spaces is continuous. This can be achieved by applying the Hausdorff measure of noncompactness. In the past, several authors characterized classes of compact operators given by infinite matrices on the some sequence spaces by using this method. For example see [10]–[13], [15],[17],[25], [27]–[29],[31],[36]. Recently, Malkowsky and Rakočević [23], Djolović and Malkowsky [14] and Mursaleen and Noman [30] have established some identities or estimates for the operator norms and Hausdorff measures of noncompactness of linear operators given by infinite matrices that map an arbitrary BK-space or the matrix domains of triangles in arbitrary BK-spaces.

For a normed space H_1, we define S_{H_1} as follows:

$$S_{H_1} = \{\eta \in H_1 : \|\eta\| = 1\},$$

i.e., a unit sphere in H_1. If H_1 and H_2 are Banach spaces then $B(H_1, H_2)$ is the set of all continuous linear operators $\mathcal{L} : H_1 \to H_2$; $B(H_1, H_2)$ is the Banach space with the operator norm defined by $\|\mathcal{L}\| = \sup\{\|\mathcal{L}\eta\| : \|\eta\| \leq 1\}$ for all $\mathcal{L} \in B(H_1, H_2)$.

If $(H, \|.\|)$ is a normed sequence space, then we write

$$\|b\|_H^* = \sup_{\eta \in S_H} \left| \sum_{k=0}^{\infty} b_k \eta_k \right| \tag{5.1}$$

for $b \in \Omega$ provided the expression on the right-hand side exists and is finite that is the case whenever H is a BK space and $b \in H^\beta$ [43, p.107].

We recall that if H_1 and H_2 are Banach spaces and L is a linear operator from H_1 to H_2, then L is said to be compact if its domain is all of H_1 and for every bounded sequence (x_n) in H_1, the sequence $((\mathcal{L}\zeta)_n)$ has a convergent subsequence in H_2. We denote the class of such operators by $K(H_1, H_2)$.

If (H, d) is a metric space, we write M_H for the class of all bounded subsets of H. By $B(\eta, r) = \{\xi \in H : d(\eta, \xi) < r\}$ we denote the open ball of radius $r > 0$ with centre in η. Then the Hausdorff measure of noncompactness of the set $Q \in M_X$ denoted by $\mho(Q)$, is given by

$$\mho(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{i=0}^{n} B(\eta_i, r_i), \eta_i \in H, r_i < \varepsilon (i = 0, 1, \ldots, n), n \in \mathbf{N} \right\}.$$

The function $\mho : M_H \to [0, \infty)$ is called the Hausdorff measure of noncompactness.

The basic properties of the Hausdorff measure of noncompactness can be found in [22], for example if Q, Q_1 and Q_2 are bounded subsets of a metric space (H, d), then

$$\mho(Q) = 0 \quad \text{if and only if} \quad Q \quad \text{is totally bounded},$$

$$Q_1 \subset Q_2 \quad \text{implies} \quad \mho(Q_1) \leq \mho(Q_2).$$

Further if H is a normed space, the function $\mho$ has some additional properties connected with the linear structure, e.g.,

$$\mho(Q_1 + Q_2) \leq \mho(Q_1) + \mho(Q_2),$$

$$\mho(\alpha Q) = |\alpha| \, \mho(Q) \quad \text{for all} \quad \alpha \in \mathbf{C},$$

where $\mathbf{C}$ is the complex field.

We shall need the following known results for our investigation.

Lemma 5.1. [28, Lemma 3.1]. *Let H denote any of the spaces c_0 and ℓ_∞. If $B \in (H, c)$, then we have*

$$\alpha_k = \lim_{n\to\infty} b_{nk} \quad \text{exists for every} \quad k \in \mathbf{N},$$

$$\alpha = (\alpha_k) \in \ell_1,$$

$$\sup_n \left(\sum_{k=0}^{\infty} |b_{nk} - \alpha_k| \right) < \infty,$$

$$\lim_{n\to\infty} (B\eta)_n = \sum_{k=0}^{\infty} \alpha_k \eta_k \quad \text{for all} \quad \eta = (\eta_k) \in H.$$

Lemma 5.2. [28, Lemma 1.1]. *Let H denote any of the spaces c_0, c or ℓ_∞. Then we have $H^\beta = \ell_1$ and $\|b\|_H^* \|b\|_{\ell_1}$ for all $b \in \ell_1$.*

Lemma 5.3. [43, Theorem 4.2.8]. *Let H_1 and H_2 be BK-spaces. Then we have $(H_1, H_2) \subset B(H_1, H_2)$, that is, every $B \in (H_1, H_2)$ defines a linear operator $\mathcal{L}_B \in B(H_1, H_2)$, where $\mathcal{L}_B(\eta) = A\eta$ for all $\eta \in H$.*

Lemma 5.4. [15, Lemma 5.2]. *Let $H_1 \supset \phi$ be BK-space and H_2 be any of the spaces c_0, c or ℓ_∞. If $B \in (H_1, H_2)$, then*

$$\|\mathcal{L}_B\| = \|B\|_{(H_1,\ell_\infty)} = \sup_n \|B_n\|_H^* < \infty.$$

Lemma 5.5. [28, Lemma 1.5]. *Let $Q \in M_{c_0}$ and $P_r : c_0 \to c_0$ $(r \in \mathbf{N})$ be the operator defined by $P_r(\eta) = (\eta_0, \eta_1, \ldots, \eta_r, 0, 0, \ldots)$ for all $\eta = (\eta_k) \in c_0$. Then we have*

$$\mho(Q) = \lim_{r\to\infty} \left(\sup_{\eta\in Q} \|(I - P_r)(\eta)\|_{\ell_\infty} \right),$$

where I is the identity operator on c_0.

Further, we know by [22, Theorem 1.10] that every $\zeta = (\zeta_n) \in c$ has a unique representation $\zeta = \bar{\zeta}e + \sum_{n=0}^{\infty} (\zeta_n - \bar{\zeta})e^{(n)}$, where $\bar{\zeta} = \lim_{n\to\infty} \zeta_n$. Thus, we define the projectors $P_r : c \to c$ $(r \in \mathbf{N})$ by

$$P_r(\zeta) = \bar{\zeta}e + \sum_{n=0}^{r} (\zeta_n - \bar{\zeta})e^{(n)} \qquad (r \in \mathbf{N}) \tag{5.2}$$

for all $\zeta = (\zeta_n) \in c$ with $\bar{\zeta} = \lim_{n\to\infty} \zeta_n$. In this situation, the following result gives an estimate for the Hausdorff measure of noncompactness in the BK space c.

Lemma 5.6. [28, Lemma 1.6]. *Let $Q \in M_c$ and $P_r : c \to c$ $(r \in \mathbf{N})$ be the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, \ldots, e^{(r)}\}$. Then, we have*

$$\frac{1}{2} \lim_{r\to\infty} \left(\sup_{\eta\in Q} \|(I - P_r)(\eta)\|_{\ell_\infty} \right) \le \mho(Q) \le \lim_{r\to\infty} \left(\sup_{\eta\in Q} \|(I - P_r)(\eta)\|_{\ell_\infty} \right),$$

where I is the identity operator on c.

The next lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

Lemma 5.7. [22, Thereom 2.25, Corollary 2.26]. *Let H_1 and H_2 be Banach spaces and $L \in B(H_1, H_2)$. Then we have*

$$\|\mathcal{L}\|_{\mho} = \mho\big(\mathcal{L}(S_{H_1})\big) \tag{5.3}$$

and

$$\mathcal{L} \in K(H_1, H_2) \quad \textit{if and only if} \quad \|\mathcal{L}\|_{\mho} = 0. \tag{5.4}$$

The following results will be needed in establishing our results.

Lemma 5.8. *Let H denote any of the spaces $c_0\big(f, g, \Delta_n^{(m)}\big)$ or $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$. If $a = (a_k) \in H^\beta$ then $\bar{a} = (\bar{a}_k) \in \ell_1$ and the equality*

$$\sum_{k=0}^{\infty} a_k \eta_k = \sum_{k=0}^{\infty} \bar{a}_k \xi_k \tag{5.5}$$

holds for every $\eta = (\eta_k) \in H$, where $\xi = G^{(m)}\eta$ is the associated sequence defined by (2.1) and

$$\bar{a}_k = \sum_{j=k}^{\infty} \nabla^{(m)}(j, k) a_j \qquad (k \in \mathbf{N}).$$

Proof. This follows immediately by [32, Theorem 5.6]. □

Lemma 5.9. *Let H denote any of the spaces $c_0\big(f, g, \Delta_n^{(m)}\big)$ or $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$. Then, we have*

$$\|a\|_H^* = \|\bar{a}\|_{\ell_1} = \sum_{k=0}^{\infty} |\bar{a}_k| < \infty$$

for all $a = (a_k) \in H^\beta$, where $\bar{a} = (\bar{a}_k)$ is as in Lemma 5.8.

Proof. Let H_2 be the respective one of the spaces c_0 or ℓ_∞, and take any $a = (a_k) \in H_1^\beta$. Then we have by Lemma 5.8 that $\bar{a} = (\bar{a}_k) \in \ell_1$ and the equality (5.5) holds for all sequences $\eta = (\eta_k) \in H_1$ and $\xi = (\xi_k) \in H_2$ which are connected by the relation (2.1). Further, it follows by (2.3) that $\eta \in S_{H_1}$ if and only if $\xi \in S_{H_2}$. Therefore, we derive from (5.1) and (5.5) that

$$\|a\|_{H_1}^* = \sup_{\eta \in S_{H_1}} \left| \sum_{k=0}^{\infty} a_k \eta_k \right| = \sup_{\xi \in S_{H_2}} \left| \sum_{k=0}^{\infty} \bar{a}_k \xi_k \right| = \|\bar{a}\|_{H_2}^*$$

and since $\bar{a} \in \ell_1$, we obtain from Lemma 5.2 that

$$\|a\|_{H_1}^* = \|\bar{a}\|_{H_2}^* = \|\bar{a}\|_{\ell_1} < \infty$$

which concludes the proof. □

Lemma 5.10. *Let H_1 be any of the spaces $c_0\big(f, g, \Delta_n^{(m)}\big)$ or $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$, H_2 the respective one of the spaces c_0 or ℓ_∞, Z a sequence space and $\mathcal{A} = (b_{nk})$ an infinite matrix. If $B \in (H_1, Z)$, then $\bar{\mathcal{A}} \in (H_2, Z)$ such that $\mathcal{A}\eta = \bar{\mathcal{A}}\xi$ for all sequences $\eta \in H_1$ and $\xi \in H_2$ which are connected by the relation* (2.1), *where $\bar{\mathcal{A}} = (\bar{a}_{nk})$ is the associated matrix defined as in* (4.1).

P r o o f. This is immediate by [28, Lelmma 2.3]. □

Now let $\mathcal{A} = (b_{nk})$ be an infinite matrix and $\bar{\mathcal{A}} = (\bar{a}_{nk})$ the associated matrix defined by (4.1). Then we have the following result.

Theorem 5.11. *Let X denote any of the spaces $c_0\big(f, g, \Delta_n^{(m)}\big)$ or $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$. Then, we have*

(a) *If $B \in (H_1, c_0)$, then*
$$\|\mathcal{L}_B\|_{\mho} = \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\overline{a}_{nk}| . \tag{5.6}$$
(b) *If $B \in (H_1, c)$, then*
$$\frac{1}{2}\limsup_{n\to\infty} \sum_{k=0}^{\infty} |\overline{a}_{nk} - \overline{a}_k| \le \|\mathcal{L}_B\|_{\mho} \le \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\overline{a}_{nk} - \overline{\alpha}_k| , \tag{5.7}$$
where $\overline{a}_k$ is defined as in (4.6) *for all $k \in \mathbf{N}$.*

(c) *If $B \in (H_1, \ell_\infty)$, then*
$$0 \le \|\mathcal{L}_B\|_{\mho} \le \limsup_{n\to\infty} \sum_{k=0}^{\infty} |\overline{a}_{nk}| . \tag{5.8}$$

P r o o f. Let us remark that the expressions in (5.6) and (5.8) exist by Theorems 4.3 and 4.1. Also, by combining Lemmas 5.1 and 5.10, we deduce that the expression in (5.7) exists.

We write $S = S_{H_1}$, for short. Then, we obtain by (5.3) and Lemma 5.3 that
$$\|\mathcal{L}_A\|_{\mho} = \mho(\mathcal{A}S). \tag{5.9}$$

For (a), we have $\mathcal{A}S \in M_{c_0}$. Thus, it follows by applying Lemma 5.5 that
$$\mho(\mathcal{A}S) = \lim_{r\to\infty} \left(\sup_{x\in S} \|(I - P_j)(\mathcal{A}\eta)\|_{\ell_\infty} \right), \tag{5.10}$$
where $P_j : c_0 \to c_0$ $(j \in \mathbf{N})$ is the operator defined by
$$P_j(\eta) = (\eta_0, \eta_1, \dots, \eta_r, 0, 0, \dots) \quad \text{for all} \quad \eta = (\eta_k) \in c_0.$$
This yields that $\|(I-P_j)(\mathcal{A}\eta)\|_{\ell_\infty} = \sup_{n>j} |(\mathcal{A}\eta)_n|$ for all $\eta \in H_1$ and every $j \in \mathbf{N}$. Therefore, by using (1.1), (5.1) and Lemma 5.9, we have for every $j \in \mathbf{N}$ that
$$\sup_{\eta\in S} \|(I - P_j)(\mathcal{A}\eta)\|_{\ell_\infty} = \sup_{n>j} \|\mathcal{A}_n\|^*_{H_1} = \sup_{n>j} \big\|\bar{\mathcal{A}}_n\big\|_{\ell_1} .$$

Together with (5.10), it implies that

$$\mho(\mathcal{A}S) = \lim_{j\to\infty}\left(\sup_{n>j}\left\|\bar{\mathcal{A}}_n\right\|_{\ell_1}\right) = \limsup_{n\to\infty}\left\|\bar{\mathcal{A}}_n\right\|_{\ell_1}.$$

Hence we obtain that (5.6) from (5.9).

To prove (b), we have $\mathcal{A}S \in M_c$. Thus, we are going to apply Lemma 5.6 to get an estimate for the value of $\mho(\mathcal{A}S)$ in (5.9). For this, let $P_j : c \to c$ $(j \in \mathbf{N})$ be the projectors defined by (5.2). Then, we have for every $j \in \mathbf{N}$ that

$$(I - P_j)(\xi) = \sum_{n=j+1}^{\infty}(\xi_n - \bar{\xi})e^{(n)}$$

and hence,

$$\|(I - P_j)(\xi)\|_{\ell_\infty} = \sup_{n>j}\left|\xi_n - \bar{\xi}\right| \tag{5.11}$$

for all $\xi = (\xi_n) \in c$ and every $\xi \in \mathbf{N}$, where $\bar{\xi} = \lim_{n\to\infty}\xi_n$ and I is identity operator on c.

Now, by using (5.9) we obtain by applying Lemma 5.6 that

$$\frac{1}{2}\lim_{j\to\infty}\left(\sup_{\eta\in S}\|(I - P_j)(\mathcal{A}\eta)\|_{\ell_\infty}\right) \le \|\mathcal{L}_A\|_{\mho} \le \lim_{j\to\infty}\left(\sup_{\eta\in S}\|(I - P_j)(\mathcal{A}\eta)\|_{\ell_\infty}\right). \tag{5.12}$$

On the other hand, it is given that $H_1 = c_0\big(f, g, \Delta_n^{(m)}\big)$ or $\ell_\infty\big(f, g, \Delta_n^{(m)}\big)$, and let H_2 be the respective one of the spaces c_0 or ℓ_∞. Also for every given $\eta \in H_1$, let $\xi \in H_2$ be the associated sequence defined by (2.1). Since $B \in (H_1, c)$, we have by Lemma 5.10 that

$$\bar{\mathcal{A}} \in (H_2, c) \quad \text{and} \quad \mathcal{A}\eta = \bar{\mathcal{A}}\xi.$$

Further, it follows from Lemma 5.1 that the limits $\overline{\alpha}_k = \lim_{n\to\infty}\overline{\mathcal{A}}_{nk}$ exist for all k,

$$(\overline{\alpha}_k) \in \ell_1 = H_2^{\beta} \quad \text{and} \quad \lim_{n\to\infty}(\bar{\mathcal{A}}\xi)_n = \sum_{k=0}^{\infty}\overline{\alpha}_k\xi_k.$$

Consequently, we derive from (5.11) that

$$\begin{aligned}\|(I - P_j)(\mathcal{A}\eta)\|_{\ell_\infty} &= \left\|(I - P_j)(\bar{\mathcal{A}}\xi)\right\|_{\ell_\infty}\\ &= \sup_{n>j}\left|(\bar{\mathcal{A}}\xi)_n - \sum_{k=0}^{\infty}\overline{\alpha}_k\xi_k\right|\\ &= \sup_{n>j}\left|\sum_{k=0}^{\infty}(\overline{a}_{nk} - \overline{\alpha}_k)\,\xi_k\right| \qquad \text{for all} \quad r \in \mathbf{N}.\end{aligned}$$

Moreover, since $\eta \in S = S_{H_1}$ if and only if $\xi \in S_{H_2}$, we obtain by (5.1) and Lemma 5.2 that

$$\begin{aligned}\|(I-P_j)(\mathcal{A}\eta)\|_{\ell_\infty} &= \sup_{n>j}\left(\sup_{\xi\in S_{H_2}}\left|\sum_{k=0}^{\infty}(\overline{a}_{nk}-\overline{\alpha}_k)\xi_k\right|\right)\\ &= \sup_{n>j}\left\|\bar{\mathcal{A}}_n-\bar{\alpha}\right\|_{H_2}^{*}\\ &= \sup_{n>j}\left\|\bar{\mathcal{A}}_n-\bar{\alpha}\right\|_{\ell_1}\end{aligned}$$

for all $j \in \mathbf{N}$. Thus we get (5.7) from (5.12).

To prove (c), we consider the projectors $P_j : \ell_\infty \to \ell_\infty$ ($j \in \mathbf{N}$) as in the proof of part (a) for all $\eta = (\eta_k) \in \ell_\infty$. Then, we have

$$\mathcal{A}S \subset P_j(\mathcal{A}S) + (I-P_j)(\mathcal{A}S) \qquad (j \in \mathbf{N}).$$

Thus, it follows by the elementary properties of the function $\mho$ that

$$\begin{aligned}0 &\le \mho(\mathcal{A}S) \le \mho\big(P_r(\mathcal{A}S)\big) + \mho\big((I-P_j)(\mathcal{A}S)\big)\\ &= \mho\big((I-P_j)(\mathcal{A}S)\big) \le \sup_{\eta\in S}\|(I-P_j)(\mathcal{A}\eta)\|_{\ell_\infty}\\ &= \sup_{n>j}\left\|\bar{\mathcal{A}}_n\right\|_{\ell_1}\end{aligned}$$

for all $j \in \mathbf{N}$ and hence

$$0 \le \mho(\mathcal{A}S) \le \lim_{j\to\infty}\left(\sup_{n>j}\left\|\bar{\mathcal{A}}_n\right\|_{\ell_1}\right) = \limsup_{j\to\infty}\left\|\bar{\mathcal{A}}_n\right\|_{\ell_1}.$$

Together with (5.9), it implies (5.8) and completes the proof. □

Corollary 5.12.
Let H denote any of the spaces $c_0\big(f,g,\Delta_n^{(m)}\big)$ or $\ell_\infty\big(f,g,\Delta_n^{(m)}\big)$. Then we have

(a) *If $B \in (H, c_0)$, then*

$$\mathcal{L}_B \text{ is compact if and only if } \lim_{n\to\infty}\sum_{k=0}^{\infty}|\overline{a}_{nk}| = 0.$$

(b) *If $B \in (H, c)$, then*

$$\mathcal{L}_B \text{ is compact if and only if } \lim_{n\to\infty}\sum_{k=0}^{\infty}|\overline{a}_{nk}-\overline{\alpha}_k| = 0.$$

(c) *If $B \in (H, \ell_\infty)$, then*

$$\mathcal{L}_B \text{ is compact if } \lim_{n\to\infty}\sum_{k=0}^{\infty}|\overline{a}_{nk}| = 0.$$

P r o o f. This result follows from Theorem 5.11 by using (5.4). □

Finally, we have the following observation.

COROLLARY 5.13. *For every matrix*

$$B \in \Big(\ell_\infty\big(f, g, \Delta_n^{(m)}\big), c_0\Big) \quad or \quad B \in \Big(\ell_\infty\big(f, g, \Delta_n^{(m)}\big), c\Big),$$

the operator $\mathcal{L}_B$ is compact.

Proof. Let $B \in \big(\ell_\infty\big(f, g, \Delta_n^{(m)}\big), c_0\big)$. Then we have by Theorem 4.3 (a) that $\lim_{n\to\infty} \big(\sum_{k=0}^{\infty} |\overline{a}_{nk}|\big) = 0$. This leads us with Corollary 5.12 (a) to the consequence that $\mathcal{L}_B$ is compact. Similarly, if $B \in \big(\ell_\infty\big(f, g, \Delta_n^{(m)}\big), c\big)$ then, from Theorem 4.2 (a), we have that $\lim_{n\to\infty} \big(\sum_{k=0}^{\infty} |\overline{a}_{nk} - \overline{\alpha}_k|\big) = 0$, where $\overline{\alpha}_k = \lim_{n\to\infty} \overline{a}_{nk}$ for all k. Hence, we deduce from Corollary 5.12 (b) that $\mathcal{L}_B$ is compact. □

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