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MODULUS OF SMOOTHNESS AND K-FUNCTIONALS CONSTRUCTED BY GENERALIZED LAGUERRE-BESSEL OPERATOR

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ABSTRACT. In this paper, we prove the equivalence between a K-functional and a modulus of smoothness generated by Laguerre-Bessel operator on

$$\mathbb{K} = [0, +\infty[\times[0, +\infty[.$$

1. Introduction and preliminaries

In [2, Theorem 1], using a generalized translation operator, they prove the equivalence theorem for a K-functional and modulus of smoothness for Laguerre type operator $L = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2 t \frac{\partial^2}{\partial t^2}$. Theorem 1 (see [2]) has been studied and generalized by many authors ([1], [3], [4], [5]).

In this paper, we introduce the modulus of smoothness associated with the translation operator, based on the Laguerre-Bessel operator we define Sobolev--type space and K-functionals, and we prove the equivalence theorem for a K--functional and a modulus of smoothness for the Laguerre-Bessel transform $\mathcal{W}_{LB}$.

We resume some facts about harmonic analysis related to the Laguerre-Bessel transform, for $(\lambda, m) \in [0, +\infty[\times\mathbb{N}$, the initial value problem

$$\begin{cases} \mathcal{D}_\alpha u = -\lambda^2 u, \\ \mathcal{L}_\alpha u = -4\lambda\big(m + \frac{\alpha+1}{2}\big)u, \\ u(0,0) = 1, \frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial t}(0,0) = 0, \end{cases}$$

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where $\mathcal{L}_\alpha$ is the Laguerre-Bessel operator given by

$$\mathcal{L}_\alpha = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\mathcal{D}_\alpha \quad \text{and} \quad \mathcal{D}_\alpha = \frac{\partial^2}{\partial t^2} + \frac{2\alpha}{t}\frac{\partial}{\partial t}.$$

For all $(x,t) \in \mathbb{K}$ and $\alpha \geq 0$, it has a unique solution $\varphi_{\lambda,m}$ given by

$$\psi_{\lambda,m}(x,t) = j_{\alpha-\frac{1}{2}}(\lambda t)\mathfrak{L}_m^\alpha\big(\lambda x^2\big), \quad \forall (x,t) \in \mathbb{K}, \tag{1}$$

where $\mathfrak{L}_m^\alpha$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathfrak{L}_m^\alpha(x) = e^{-\frac{x}{2}}\frac{L_m^\alpha(x)}{L_m^\alpha(0)},$$

and L_m^α is the Laguerre polynomial of degree m and order α given by

$$L_m^\alpha(x) = \sum_{k=0}^{m}(-1)^k\frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)}\frac{1}{k!(m-k)!}x^k, \tag{2}$$

and j_α is the normalized Bessel function given by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{x}{2}\right)^{2k}. \tag{3}$$

Lemma 1.1 ([6]). *For all $(\lambda,m) \in [0,+\infty[\times\mathbb{N}$, the functions $\psi_{\lambda,m}$ are infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable, and we have*

$$sup_{(x,t)\in\mathbb{K}}|\psi_{\lambda,m}(x,t)| = 1.$$

Let $\alpha \geq 0$ be a fixed number. The weighted Lebesgue measure dm_α on $\mathbb{K}$ is given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}t^{2\alpha}}{\pi\Gamma(\alpha+1)}dxdt. \tag{4}$$

We denote by (see [8]):

- $\mathcal{S}_*(\mathbb{K})$ the space of C^∞ functions on $\mathbb{R}^2$, even with respect to each variable and rapidly decreasing together with all their derivatives, i.e., for all $k,p,q \in \mathbb{N}$,

$$N_{k,p,q}(f) = \sup_{(x,t)\in\mathbb{K}}\left\{\big(1+x^2+t^2\big)^k\left|\frac{\partial^{p+q}}{\partial x^p\partial t^q}f(x,t)\right|\right\} < +\infty.$$

- $L_\alpha^p(\mathbb{K}), p \in [1,+\infty]$, the spaces of measurable functions on $\mathbb{K}$ such that

$$\|f\|_{p,\alpha} = \left[\int_{\mathbb{K}}|f(x,t)|^p dm_\alpha(x,t)\right]^{\frac{1}{p}} < +\infty, \quad \text{if } p \in [1,+\infty[,$$

$$\|f\|_{\infty,\alpha} = \text{ess sup}_{(x,t)\in\mathbb{K}}|f(x,t)| < +\infty.$$

- $L^p_{\gamma_\alpha}([0,+\infty[\times\mathbb{N}), p\in[1,+\infty]$, the spaces of measurable functions on $[0,+\infty[\times\mathbb{N}$ such that

$$\|g\|_{\gamma_\alpha,p} = \left[\int_{[0,+\infty[\times\mathbb{N}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right]^{\frac{1}{p}} < +\infty, \quad \text{if } p\in[1,+\infty[$$

$$\|g\|_{\gamma_\alpha,\infty} = \text{ess sup}_{(\lambda,m)\in[0,+\infty[\times\mathbb{N}} |g(\lambda,m)| < +\infty,$$

where γ_α is the positive measure defined on $[0,+\infty[\times\mathbb{N}$ by

$$\int_{[0,+\infty[\times\mathbb{N}} g(\lambda,m)d\gamma_\alpha(\lambda,m) = \frac{1}{2^{2\alpha-1}\Gamma\left(\alpha+\frac{1}{2}\right)}\sum_{m=0}^{\infty} L^\alpha_m(0)\int_0^{+\infty} g(\lambda,m)\lambda^{3\alpha+1}d\lambda.$$

- The homogeneous norm on $\mathbb{K}$ defined by

$$|\, x,t \,| = |\,(x,t)\,|_{\mathbb{K}} = \left(x^4+4t^2\right)^{\frac{1}{4}}, \quad \text{for all } (x,t)\in\mathbb{K}.$$

- The quasinorm on $[0,+\infty[\times\mathbb{N}$ defined by

$$|\,\lambda,m\,| = |\,(\lambda,m)\,|_{[0,+\infty[\times\mathbb{N}} = 4\lambda\left(m+\tfrac{\alpha+1}{2}\right), \quad \text{for all } (\lambda,m)\in[0,+\infty[\times\mathbb{N}.$$

- $\mathbb{B}_r$ the ball with center 0 and radius r defined by

$$\mathbb{B}_r = \{(\lambda,m)\in[0,+\infty[\times\mathbb{N};\, |\,\lambda,m\,| < r\} \quad \text{and} \quad \mathbb{B}^c_r = ([0,+\infty[\times\mathbb{N})\backslash\mathbb{B}_r.$$

Let $f\in\mathcal{S}_*(\mathbb{K})$. For all (x,t) and $(y,s)\in\mathbb{K}$, a generalized translation operator is defined by

$$T^{(\alpha)}_{(x,t)}f(y,s) = \begin{cases} \frac{1}{4\pi}\sum\limits_{i,j=0}^{1}\int\limits_0^\pi f\big(\Delta_\theta(x,y),\, Y+(-1)^i t+(-1)^j s\big)d\theta & \text{if } \alpha=0, \\ b_\alpha \int\limits_{[0,\pi]^3} f\big(\Delta_\theta(x,y), \Delta_\theta(x,y)\xi\big)\; d\mu_\alpha(\xi,\psi,\theta) & \text{if } \alpha>0. \end{cases}$$

where

$$\Delta_\theta(x,y) = \sqrt{x^2+y^2+2xy\cos\theta}, \quad b_\alpha = \frac{(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\pi^{\frac{3}{4}}\Gamma(\alpha)}, \quad Y = xy\sin\theta$$

and

$$d\mu_\alpha(\xi,\psi,\theta) = (\sin\xi)^{2\alpha-1}(\sin\psi)^{2\alpha-1}(\sin\theta)^{2\alpha}d\xi d\psi d\theta.$$

The Fourier-Laguerre-Bessel transform of a function in $L^1_\alpha(\mathbb{K})$ is given by

$$\mathcal{W}_{LB}f(\lambda,m) = \int_{\mathbb{K}} f(x,t)\psi_{\lambda,m}(x,t)dm_\alpha(x,t), \quad (\lambda,m)\in[0,+\infty[\times\mathbb{N}.$$

From [6], it is well-known that Fourier-Laguerre-Bessel transform can be inverted to

$$\mathcal{W}^{-1}_{LB}f(x,t) = \int_{[0,+\infty[\times\mathbb{N}} f(\lambda,m)\psi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \quad (x,t)\in\mathbb{K}.$$

It is well-known (see [6], [7], [8]) that the Fourier-Laguerre-Bessel transform $\mathcal{W}_{LB}$ satisfies the following properties:

- We have the following Plancherel formula

$$\|f\|_{2,\alpha} = \|\mathcal{W}_{LB}f\|_{\gamma_\alpha,2}, \quad \text{for } f \in L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K}). \tag{5}$$

- We also have the inverse formula of the generalized Fourier transform

$$f(x,t) = \int\limits_{[0,+\infty[\times\mathbb{N}} \mathcal{W}_{LB}f(\lambda,m)\psi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \quad (x,t) \in \mathbb{K} \tag{6}$$

  provided $\mathcal{W}_{LB}f \in L^1_{\gamma_\alpha}([0,+\infty[\times\mathbb{N})$.

- For $f \in L^1_\alpha(\mathbb{K})$, we have

$$\mathcal{W}_{LB}\left(T^{(\alpha)}_{(x,t)}f\right)(\lambda,m) = \psi_{\lambda,m}(x,t)\mathcal{W}_{LB}(f)(\lambda,m),$$

and

$$\mathcal{W}_{LB}\left(T^{(\alpha)}_{(x,t)}f - f\right)(\lambda,m) = \big(\psi_{\lambda,m}(x,t) - 1\big)\mathcal{W}_{LB}(f)(\lambda,m), \tag{7}$$

where

$$(x,t) \in \mathbb{K}, (\lambda,m) \in [0,+\infty[\times\mathbb{N}.$$

- For $f \in L^p_\alpha(\mathbb{K}), p \in [1,+\infty]$, we have

$$T^{(\alpha)}_{(x,t)}f \in L^p_\alpha(\mathbb{K}) \quad \text{and} \quad \left\|T^{(\alpha)}_{(x,t)}f\right\|_{p,\alpha} \le \|f\|_{p,\alpha}. \tag{8}$$

Let $\mathcal{W}^k_{2,\alpha}(\mathbb{K})$ be the Sobolev space constructed by the $\mathcal{L}_\alpha$ operator that is,

$$\mathcal{W}^k_{2,\alpha}(\mathbb{K}) := \left\{f \in L^2_\alpha(\mathbb{K}) : \mathcal{L}^j_\alpha f \in L^2_\alpha(\mathbb{K}),\ j = 1,2,\dots,k\right\},$$

where

$$\mathcal{L}^0_\alpha f = f, \quad \mathcal{L}^j_\alpha f = \mathcal{L}_\alpha(\mathcal{L}^{j-1}_\alpha f), \quad j = 1,2,\dots,k.$$

Let $f \in L^2_\alpha(\mathbb{K})$ and $\delta > 0$. Then, the generalized modulus of smoothness is defined by

$$w_k(f,\delta)_{2,\alpha} = \sup_{0<|x,t|\le\delta} \left\|\Delta^k_{(x,t)}f\right\|_{2,\alpha},$$

where

$$\Delta^k_{(x,t)}f(y,s) = \left(T^{(\alpha)}_{(x,t)} - I\right)^k f(y,s), \quad k \in \mathbb{N}, \tag{9}$$

and I denotes the unit operator.

The generalized K-functional is defined by

$$K_k(f,\delta)_{2,\alpha} = \inf\left\{\|f-g\|_{2,\alpha} + \delta\left\|\mathcal{L}^k_\alpha g\right\|_{2,\alpha} : g \in \mathcal{W}^k_{2,\alpha}(\mathbb{K})\right\}.$$

Lemma 1.2. *Let $f \in L^2_\alpha(\mathbb{K})$ and $(x,t) \in \mathbb{K}$. We have*

$$\mathcal{W}_{LB}\left(\Delta^k_{(x,t)}f\right)(\lambda,m) = \big(\psi_{\lambda,m}(x,t) - 1\big)^k\mathcal{W}_{LB}(f)(\lambda,m). \tag{10}$$

Proof. The result easily follows by using (7), (9) and induction on k. □

Proposition 1.3. *For $f \in \mathcal{W}^k_{2,\alpha}(\mathbb{K})$, we have*

$$\mathcal{W}_{LB}\big(\mathcal{L}^k_\alpha f\big)(\lambda, m) = (-1)^k \mid \lambda, m \mid^k \mathcal{W}_{LB}(f)(\lambda, m), k \in \mathbb{N}. \tag{11}$$

P r o o f. From [6], we have

$$\mathcal{W}_{LB}(\mathcal{L}_\alpha f)(\lambda, m) = - \mid \lambda, m \mid \mathcal{W}_{LB}(f)(\lambda, m).$$

The result easily follows induction on k. □

Throughout this paper, C denotes a positive constant which can differ from line to line.

2. Main results

In order to give the main results, we begin with auxiliary results interesting in themselves. The behavior of the characters $\psi_{\lambda,m}(x,t)$ in 0 could be deduced from relations (1), (2) and (3) as follows:

$$\psi_{\lambda,m}(x,t) = 1 - \frac{(\lambda t)^2}{4(\alpha + \frac{1}{2})} - \frac{\mid \lambda, m \mid x^2}{4(\alpha+1)} + \kappa_{\alpha,m}\lambda^2 x^4 + o\big(\lambda^2 \mid x, t \mid^4\big), \tag{12}$$

where $\kappa_{\alpha,m} = \frac{m^2}{2(\alpha+1)(\alpha+2)} + \frac{m}{2(\alpha+2)} + \frac{1}{8}$.

Lemma 2.1. *Let $v > 0$.*

(i): *There exists $C > 0$ such that for all $(\lambda, m) \in \overline{\mathbb{B}_v}$ and $(x,t) \in \mathbb{K}$,*

$$\mid \psi_{\lambda,m}(x,t) - 1 \mid \geq C \mid \lambda, m \mid\mid x, t \mid^2. \tag{13}$$

(ii): *There exists $C > 0$ such that for all $(x,t) \in \mathbb{K}$,*

$$\mid \lambda, m \mid > v \Rightarrow \mid \psi_{\lambda,m}(x,t) - 1 \mid \geq C. \tag{14}$$

(iii): *There exists $C > 0$ such that for all $(\lambda, m) \in [0, +\infty[\times\mathbb{N}$ and $(x,t) \in \mathbb{K}$,*

$$\mid \psi_{\lambda,m}(x,t) - 1 \mid \leq C \mid \lambda, m \mid\mid x, t \mid^2. \tag{15}$$

P r o o f.

(i): Denote $v = \frac{\eta}{|x,t|^2}$, for $(\lambda, m) \in \overline{\mathbb{B}_v}$. Using relation (12) yields to

$$\lim_{|\lambda,m||x,t|^2 \to 0} \frac{\mid \psi_{\lambda,m}(x,t) - 1 \mid}{\mid \lambda, m \mid\mid x,t \mid^2} = \frac{1}{4(\alpha+1)} > 0.$$

Consequently, there exists a constant C and $\eta > 0$ such that if

$$\mid \lambda, m \mid\mid x, t \mid^2 < \eta,$$

then

$$\mid \psi_{\lambda,m}(x,t) - 1 \mid \geq C \mid \lambda, m \mid\mid x, t \mid^2.$$

(ii): From [10, Lemma 4.3], we have

$$\lim_{|\lambda,m|\to+\infty}\varphi_{\lambda,m}(x,t)=0,$$

where $\varphi_{\lambda,m}(x,t)=e^{i\lambda t}\mathfrak{L}_m^{\alpha}(\lambda x^2)$ is the Laguerre Kernel, and from [17], we have the asymptotic formula for the normalized Bessel function j_α when $x\longrightarrow+\infty$:

$$j_\alpha(x)=\frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{2}{x}\right)^{\alpha+\frac{1}{2}}\cos\left(x-(2\alpha+1)\frac{\pi}{4}\right)+o\left(\frac{1}{x^{\frac{3}{2}}}\right).$$

Hence as

$$\psi_{\lambda,m}(x,t)=j_{\alpha-\frac{1}{2}}(\lambda t)\frac{1}{e^{i\lambda t}}\varphi_{\lambda,m}(x,t),$$

then

$$\lim_{|\lambda,m|\to+\infty}\psi_{\lambda,m}(x,t)=0. \tag{16}$$

We get

$$\lim_{|\lambda,m|\to+\infty}|\ \psi_{\lambda,m}(x,t)-1\ |=1.$$

Hence, there exist $C'>0$ and $A>0$ such that

$$|\ \lambda,m\ |>A\Rightarrow|\ \psi_{\lambda,m}(x,t)-1\ |\geq C'.$$

If $v<A$. Take

$$m_2=\min_{v\leq|\lambda,m|\leq A}|\ \psi_{\lambda,m}(x,t)-1\ |\ .$$

Therefore

$$|\ \psi_{\lambda,m}(x,t)-1\ |\geq C,\quad\text{for}\quad|\ \lambda,m\ |>v.$$

Where $C=\min(m_2,C')$.

(iii): Denote $r=\frac{\eta}{|x,t|^2}$, for $(\lambda,m)\in\mathbb{B}_r$. Using relation (12), there exist $C'>0$ and $\eta>0$ such that for all $(x,t)\in\mathbb{K}$,

$$|\ \psi_{\lambda,m}(x,t)-1\ |\leq C'\ |\ \lambda,m\ ||\ x,t\ |^2.$$

Using (16), we get

$$\lim_{|\lambda,m|\to+\infty}\frac{|\ \psi_{\lambda,m}(x,t)-1\ |}{|\ \lambda,m\ ||\ x,t\ |^2}=0.$$

Hence, there exist $C'>0$ and $A>0$ such that

$$|\ \lambda,m\ |>A\Rightarrow|\ \psi_{\lambda,m}(x,t)-1\ |\leq C'\ |\ \lambda,m\ ||\ x,t\ |^2\ .$$

If $\frac{\eta}{|x,t|^2}<A$. Take $m_1=\max_{\frac{\eta}{|x,t|^2}\leq|\lambda,m|\leq A}\frac{|\psi_{\lambda,m}(x,t)-1|}{|\lambda,m||x,t|^2}$ Therefore.

$$|\ \psi_{\lambda,m}(x,t)-1\ |\leq C'\ |\ \lambda,m\ ||\ x,t\ |^2,\quad\text{for}\ \ (\lambda,m)\in\mathbb{B}_r^c.$$

Where $C''=min(m_1,C')$. Hence, the result where $C=\max(C',C'')$. □

Lemma 2.2. *Let $f\in L^2_\alpha(\mathbb{K})$. Then*

$$\left\|\Delta^k_{(x,t)}f\right\|^2_{2,\alpha}\leq 2^k\|f\|_{2,\alpha}.$$

P r o o f. We use the proof of recurrence for k and formula (8). □

Lemma 2.3. *If $f \in \mathcal{W}^k_{2,\alpha}(\mathbb{K})$, then*

$$w_k(f,\delta)_{2,\alpha} \leq C\delta^{2k}\|\mathcal{L}^k_\alpha f\|_{2,\alpha}. \tag{17}$$

P r o o f. If $f \in \mathcal{W}^k_{2,\alpha}(\mathbb{K})$, then by (10), (11), (15) and Plancherel formula we have

$$\begin{aligned}
\|\Delta^k_{(x,t)}f\|^2_{2,\alpha} &= \int\limits_{[0,+\infty[\times\mathbb{N}} |\ \psi_{\lambda,m}(x,t)-1\ |^{2k}|\ \mathcal{W}_{LB}(f)(\lambda,m)\ |^2\ d\gamma_\alpha(\lambda,m),\\
&\leq \int\limits_{[0,+\infty[\times\mathbb{N}} C^k\ |\ \lambda,m\ |^{2k}|\ x,t\ |^{4k}|\ \mathcal{W}_{LB}(f)\ |^2\ d\gamma_\alpha(\lambda,m),\\
&\leq C^k\ |\ x,t\ |^{4k}\int\limits_{[0,+\infty[\times\mathbb{N}} |\ \lambda,m\ |^{2k}|\ \mathcal{W}_{LB}(f)\ |^2\ d\gamma_\alpha(\lambda,m),\\
&\leq C^k\ |\ x,t\ |^{4k}\int\limits_{[0,+\infty[\times\mathbb{N}} |\ \mathcal{W}_{LB}(\mathcal{L}^k_\alpha f)\ |^2\ d\gamma_\alpha(\lambda,m).
\end{aligned}$$

Therefore

$$\left\|\Delta^k_{(x,t)}f\right\|_{2,\alpha} \leq C\ |\ x,t\ |^{2k}\ \left\|\mathcal{L}^k_\alpha f\right\|_{2,\alpha}.$$

The lemma is proved. □

For any function $f \in L^2_\alpha(\mathbb{K})$ and any number $v > 0$ we define the function

$$\begin{aligned}
P_v(f)(x,t) &:= \int\limits_{\overline{\mathbb{B}_v}} \mathcal{W}_{LB}f(\lambda,m)\psi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m)\\
&= \mathcal{W}^{-1}_{LB}\big(\mathcal{W}_{LB}f(\lambda,m)\chi_v(\lambda,m)\big),
\end{aligned}$$

where

$$\chi_v(\lambda,m) = \begin{cases} 1, & \text{if} \quad (\lambda,m) \in \overline{\mathbb{B}_v}\\ 0, & \text{if} \quad (\lambda,m) \in \overline{\mathbb{B}_v}^c. \end{cases}$$

$\mathcal{W}^{-1}_{LB}$ is the inverse Fourier-Laguerre transform. One can easily prove that the function $P_v(f)$ is infinitely differentiable and belongs to all classes

$$\mathcal{W}^k_{2,\alpha}(\mathbb{K}), k\in\mathbb{N}.$$

Lemma 2.4. *If $f \in L^2_\alpha(\mathbb{K})$, then*

$$\|f - P_v(f)\|_{2,\alpha} \leq Cw_k(f,\delta)_{2,\alpha}. \tag{18}$$

P r o o f. Using the Plancherel identity, we have

$$\|f-P_v(f)\|_{2,\alpha}^2=\int\limits_{[0,+\infty[\times\mathbb{N}}|\,1-\chi_v(\lambda,m)\,|^2|\,\mathcal{W}_{LB}f(\lambda,m)\,|^2\,d\gamma_\alpha(\lambda,m)$$
$$=\int\limits_{\overline{\mathbb{B}_v}^c}|\,\mathcal{W}_{LB}f(\lambda,m)\,|^2\,d\gamma_\alpha(\lambda,m).$$

By (14), we have $\quad|\,\psi_{\lambda,m}(x,t)-1\,|\geq C\quad$ for $\quad|\,\lambda,m\,|>v.$

Therefore, from (10) and the Plancherel identity we deduce that

$$\|f-P_v(f)\|_{2,\alpha}^2\leq C^{-2k}\int\limits_{\overline{\mathbb{B}_v}^c}|\,\psi_{\lambda,m}(x,t)-1\,|^{2k}|\,\mathcal{W}_{LB}f(\lambda,m)\,|^2\,d\gamma_\alpha(\lambda,m)$$
$$=C^{-2k}\int\limits_{\overline{\mathbb{B}_v}^c}\left|\mathcal{W}_{LB}\left(\left(T^{(\alpha)}_{(x,t)}-I\right)^kf\right)(\lambda,m)\right|^2d\gamma_\alpha(\lambda,m)$$
$$\leq C^{-2k}\int\limits_{[0,+\infty[\times\mathbb{N}}\left|\mathcal{W}_{LB}\left(\left(T^{(\alpha)}_{(x,t)}-I\right)^kf\right)(\lambda,m)\right|^2d\gamma_\alpha(\lambda,m),$$
$$=C^{-2k}\left\|(T^{(\alpha)}_{(x,t)}-I)^kf\right\|_{2,\alpha}^2.$$

Hence

$$\|f-P_v(f)\|_{2,\alpha}\leq C^{-k}\left\|(T^{(\alpha)}_{(x,t)}-I)^kf\right\|_{2,\alpha}\leq C^{-k}w_k(f,\delta)_{2,\alpha},$$

The lemma is proved. □

Lemma 2.5. *For any $f\in L^2_\alpha(\mathbb{K})$ and $v>0$ we have*

$$\left\|\mathcal{L}^k_\alpha\big(P_v(f)\big)\right\|_{2,\alpha}\leq C\,|\,x,t\,|^{-2k}\,\left\|\Delta^k_{(x,t)}f\right\|_{2,\alpha},\quad k\in\mathbb{N}. \tag{19}$$

P r o o f. By (10), (11), (13) and the Plancherel identity we have

$$\left\|\mathcal{L}^k_\alpha\big(P_v(f)\big)\right\|_{2,\alpha}^2=\int\limits_{\overline{\mathbb{B}_v}}|\,\lambda,m\,|^{2k}|\,\mathcal{W}_{LB}f(\lambda,m)\,|^2\,d\gamma_\alpha(\lambda,m),$$
$$\leq C^{-2k}|\,x,t\,|^{-4k}\int\limits_{\overline{\mathbb{B}_v}}|\,\psi_{\lambda,m}(x,t)-1\,|^{2k}|\,\mathcal{W}_{LB}f(\lambda,m)\,|^2\,d\gamma_\alpha(\lambda,m),$$
$$\leq C^{-2k}|\,x,t\,|^{-4k}\int\limits_{[0,+\infty[\times\mathbb{N}}\left\|\mathcal{W}_{LB}\big(\Delta^k_{(x,t)}f\big)(\lambda,m)\right\|^2d\gamma_\alpha(\lambda,m),$$
$$=C^{-2k}|\,x,t\,|^{-4k}\,\left\|\Delta^k_{(x,t)}f\right\|_{2,\alpha}^2.$$

Hence
$$\left\|\mathcal{L}_\alpha^k\big(P_v(f)\big)\right\|_{2,\alpha} \le C^{-k}\,|\,x,t\,|^{-2k}\,\left\|\Delta^k_{(x,t)}f\right\|_{2,\alpha}.$$
This proves (19). □

Corollary 2.6. *The inequality*
$$\left\|\mathcal{L}_\alpha^k\big(P_v(f)\big)\right\|_{2,\alpha} \le C\delta^{-2k}w_k(f,\delta)_{2,\alpha}, \tag{20}$$
holds for any $f \in L^2_\alpha(\mathbb{K}), k \in \mathbb{N}$ *and* $\delta > 0$.

Theorem 2.7. *There are two positive constants* $c_1 = c(k)$ *and* $c_2 = c(k)$ *such that*
$$c_1 w_k(f,\delta)_{2,\alpha} \le K_k\big(f,\delta^{2k}\big)_{2,\alpha} \le c_2 w_k(f,\delta)_{2,\alpha} \tag{21}$$
for all $f \in L^2_\alpha(\mathbb{K})$ *and* $\delta > 0$.

Proof. To prove the left-hand inequality in (21), it is sufficient to show that
$$w_k(f,\delta)_{2,\alpha} \le CK_k\big(f,\delta^{2k}\big)_{2,\alpha}. \tag{22}$$
Let $g \in \mathcal{W}^k_{2,\alpha}(\mathbb{K})$. From Lemma 2.2 and Lemma 2.3 we obtain
$$\begin{aligned} w_k(f,\delta)_{2,\alpha} &\le w_k(f-g,\delta)_{2,\alpha} + w_k(g,\delta)_{2,\alpha} \\ &\le 2^k\|f-g\|_{2,\alpha} + C'\delta^{2k}\|\mathcal{L}^k_\alpha g\|_{2,\alpha} \\ &\le C\big(\|f-g\|_{2,\alpha} + \delta^{2k}\left\|\mathcal{L}^k_\alpha g\right\|_{2,\alpha}\big), \end{aligned}$$
where $C = max(2^k, C')$. Taking the infimum over all $g \in \mathcal{W}^k_{2,\alpha}(\mathbb{K})$, we arrive at inequality (22).

Now, we prove the right-hand inequality in (21). If $g = P_v(f)$ for $v > 0$, then it follows from the definition of $K_k(f,\delta)_{2,\alpha}$ that
$$K_k\big(f,\delta^{2k}\big)_{2,\alpha} \le \|f - P_v(f)\|_{2,\alpha} + \delta^{2k}\left\|\mathcal{L}^k_\alpha(P_v(f))\right\|_{2,\alpha}. \tag{23}$$
It follows from Lemma 2.4 and Corollary 2.6 that
$$K_k(f,\delta^{2k})_{2,\alpha} \le 2Cw_k(f,\delta)_{2,\alpha},$$
which proves the right-hand inequality in (21). □

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