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Tatra Mountains
Mathematical Publications
DOI: 10.2478/tmmp-2022-0010
Tatra Mt. Math. Publ. **81** (2022), 129–144

GLOBAL PHASE PORTRAITS OF QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS HAVING AS SOLUTION SOME CLASSICAL PLANAR ALGEBRAIC CURVES OF DEGREE 6

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ABSTRACT. The main goal of this paper is to classify the global phase portraits of seven quadratic polynomial differential systems, exhibiting as invariant algebraic curves seven well-known algebraic curves of degree six.

We prove that these systems have five topologically different phase portraits in the Poincaré disc.

1. Introduction and statement of the main results

We call quadratic differential systems, simply *quadratic systems* or **(QS)**, the differential systems of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where P and Q are real polynomials in the variables x and y, such that the $\max\{\deg(P), \deg(Q)\} = 2$. Here, the dot denotes as usual differentiation with respect to the time t. To such systems one can always associate the quadratic vector field

$$\mathcal{X} = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y.$$

2020 Mathematics Subject Classification: 34C15, 34C25.
Keywords: quadratic differential system, Poincaré disc, invariant algebraic curve.

If system (1) has an algebraic trajectory curve, which is defined by a zero set of a polynomial $h(x, y) = 0$. Then it is clear that the derivative of h with respect to the time will not change along the curve

$$h(x, y) = 0,$$

and by Hilbert's Nullstellensatz (see for instance [10]) we have

$$\begin{aligned} \frac{dh(x,y)}{dt} &= \frac{\partial h(x,y)}{\partial x} P(x,y) + \frac{\partial h(x,y)}{\partial y} Q(x,y) \\ &= h(x,y)k(x,y), \end{aligned} \tag{2}$$

where $k(x, y)$ is a polynomial in the variables x and y of degree at most 1, called the *cofactor* of he *invariant algebraic curve* $h(x, y) = 0$. For more details on the invariant algebraic curves of polynomial differential systems, see [9, Chapter 8].

Recently, the quadratic differential systems have been intensely studied using algebraic, geometric, analytic, and numerical tools. More than one thousand papers on these systems have been published, see for instance Ye Yanquian et al. [16], Reyn [15] and Artés et al. [2], and the references therein.

In [6], Benterki and Llibre classified the global phase portraits of 14 quadratic polynomial differential systems having classical quartic algebraic curves as invariant algebraic curves, which are formed by orbits of the quadratic polynomial differential system. They proved that these systems have 28 topologically different phase portraits in the Poincaré disc. In [3–5] the authors categorized the dynamics of many classes of polynomial vector fields with different invariant algebraic curves by investigating the geometry solutions within the Poincaré disc.

In this paper, we are interested in characterizing and classifying the global phase portraits in the Poincaré disc of seven of quadratic systems having classical invariant algebraic curves of degree 6.

This paper is structured as follows: In Table 1 we present seven usual algebraic curves of degree six, then we give the seven differential systems that admit the seven invariant curves mentioned in Table 1 as invariant algebraic curves, and afterward the Theorem of the main result that characterizes the global phase portraits of the studied systems. In Section 2, we introduce some fundamental concepts needed to clarify our results. For the differential systems under study, Section 3 is designed to state the Propositions and their proofs of finite and infinite singular points. In the final section, we describe how we were able to draw the global phase portraits of these systems.

Table 1. The seven classical algebraic curves realizable by quadratic systems.

Name	Curve $h_i(x,y)=0,\ \ i=1\ldots 7$
Kiss curve	$h_1(x,y)=a^4y^2-\left(a^2-x^2\right)^3,\quad a\neq 0$
Nephroid curve	$h_2(x,y)=4(x^2+y^2-a^2)^3-27a^4y^2,\quad a\neq 0$
Egg curve	$h_3(x,y)=(x^2+y^2)^3-a^2x^4,\quad a\neq 0$
Dipole curve	$h_4(x,y)=(x^2+y^2)^3-a^4x^2,\quad a\neq 0$
Radial curve	$h_5(x,y)=(a^2x^2+bx^2)^3-a^2b^2(x^2+y^2)^2,\quad ab\neq 0$
Sluze curve	$h_6(x,y)=y^6-k(a-x)x^5,\quad ak\neq 0$
Tear Drop curve	$h_7(x,y)=16a^4y^2-(a+x)(a-x)^5,\ \ a\neq 0$ and $-a\leqslant x\leqslant a$

Our first main result is the following.

Theorem 1.1. *The global phase portraits of the planar quadratic polynomial differential systems* (1), *with the polynomials P and Q coprime, exhibiting an invariant algebraic curve of degree 6 of Table 1, are topologically equivalent to the phase portraits of the following systems*:

(i) QS with the *Kiss* invariant curve and cofactor x:

$$\dot{x}=-\frac{a^2}{6}+\frac{x^2}{6},\qquad \dot{y}=\frac{xy}{2}.$$

(ii) QS with the *Nephroid* invariant curve and cofactor x :

$$\dot{x}=-\frac{a^2}{6}+\frac{x^2}{6}-\frac{y^2}{3},\qquad \dot{y}=\frac{xy}{2}.$$

(iii) QS with the *Egg* invariant curve and cofactor y:

$$\dot{x}=\frac{xy}{4},\qquad \dot{y}=-\frac{x^2}{12}+\frac{y^2}{6}.$$

(iv) QS with the *Dipole* invariant curve and cofactor $-3y$:

$$\dot{x}=-\frac{3xy}{2},\qquad \dot{y}=x^2-\frac{y^2}{2}.$$

(v) QS with the *Radial* invariant curve and cofactor y:

$$\dot{x}=\frac{xy}{6},\qquad \dot{y}=\frac{x^2}{12}+\frac{y^2}{4}.$$

(vi) QS with the *Pearls of Sluze* invariant curve and cofactor $-5a+6x$:

$$\dot{x} = x^2 - ax, \qquad \dot{y} = xy - \frac{5ay}{6}.$$

(vii) QS with the *Tear Drop* invariant curve and cofactor $4a+6x$:

$$\dot{x} = x^2 - a^2, \qquad \dot{y} = 2ay + 3xy.$$

Theorem 1.2. *The phase portraits in the Poincaré disc of the seven systems of Theorem 1.1 are*:

1 for System (i) when $a > 0$. This phase portrait is topologically equivalent to the phase Portrait 7 of System (vi) for $a > 0$, but the invariant algebraic curves are different;

2 for System (ii) when $a > 0$;

3 for System (iii). This phase portrait is topologically equivalent to the phase Portrait 4 System (iv), but the invariant algebraic curves are different;

4 for System (iv);

5 for System (v);

6 for System (vi) when $a > 0$;

7 for System (vii) when $a > 0$.

2. Preliminaries and basic results

In this section, we present some basic results and notations which are necessary to prove our results.

As usual, we classify the singular points of a planar differential system into *hyperbolic*, *semi-hyperbolic*, *nilpotent* and *linearly zero*. The *hyperbolic* ones are the singular points such that the linear part of the differential system has eigenvalues with nonzero real parts see for instance [9, Theorem 2.15] for the classification of their local phase portraits. The *semi-hyperbolic* ones have a unique eigenvalue equal to zero, their phase portraits are characterized in [9, Theorem 2.19]. The *nilpotent* singular points have both eigenvalues zero but their linear part is not identically zero. Finally, the *linearly zero* singular points are the ones such that their linear part is identically zero, and their local phase portraits must be studied using the change of variables called Blow-ups, see for instance [9, chapters 2 and 3].

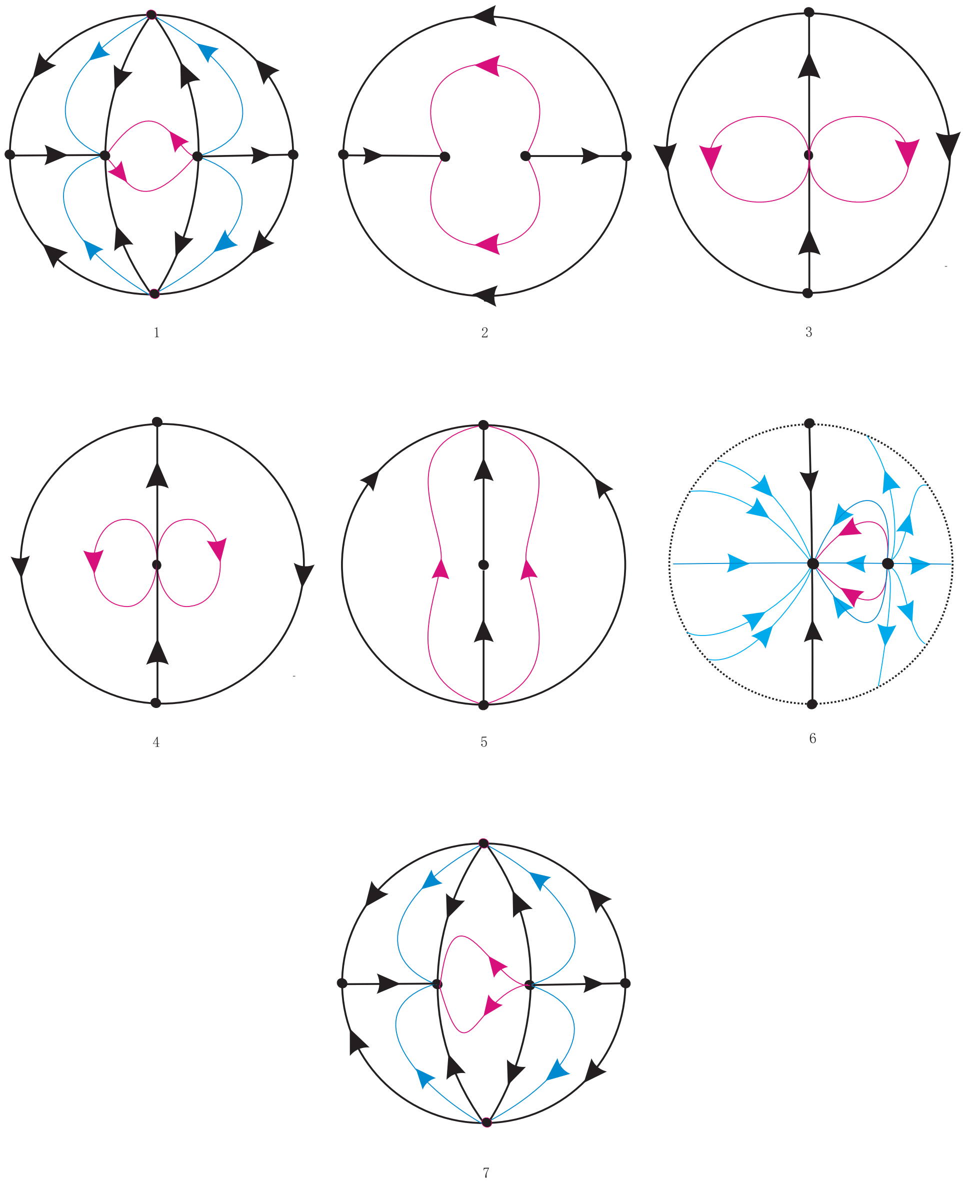

FIGURE 1. Phase portraits in the Poincaré disc. The invariant algebraic curves of degree 6 are drawn in red colour. An orbit inside the canonical region is drawn in blue. The separatrices are drawn in black.

Roughly speaking, the Poincaré disc is the closed disc centred at the origin of the plane of radio one, its interior is diffeomorphic to $\mathbb{R}^2$, and its boundary $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Any polynomial differential system in $\mathbb{R}^2$ can be extended analytically to the Poincaré disc in such a way that the circle at infinity $\mathbb{S}^1$ is invariant by the flow of the extended system, which is called the Poincaré compactification of the given polynomial differential system. The equations for studying the Poincaré compactification are well-known, see for instance [9, chapter 5]. We shall use the notation introduced in that chapter in this paper. The singular points of the Poincaré compactification in the interior of the Poincaré disc are called finite singular points and the ones which in its boundary are called infinite singular points.

Note that for studying the singular points at infinity, we only need to study the infinite singular points of the chart U_1 and the origin of the chart U_2, because the singular points at infinity appear in pairs diametrically opposite.

2.1. Phase portraits on the Poincaré disc

In this subsection, we shall see how to characterize the global phase portraits in the Poincaré disc of all the quadratic polynomial differential systems mentioned in Theorem 1.1.

A *separatrix* of $p(\mathcal{X})$ is an orbit that is either a singular point, or a limit cycle, or a trajectory that lies in the boundary of a hyperbolic sector at a singular point. Neumann [13] proved that the set formed by all separatrices of $p(\mathcal{X})$ denoted by $S\big(p(\mathcal{X})\big)$ is closed. We denote by S the number of separatrices.

The open connected components of $\mathbb{D}^2 \setminus S\big(p(\mathcal{X})\big)$ are called *canonical regions* of $p(\mathcal{X})$: We define a *separatrix configuration* as a union of $S\big(p(\mathcal{X})\big)$ plus one solution chosen from each canonical region. Two separatrix configurations $S\big(p(\mathcal{X})\big)$ and $S\big(p(\mathcal{Y})\big)$ are said to be *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S\big(p(\mathcal{X})\big)$ into the trajectories of $S\big(p(\mathcal{Y})\big)$. The following result is according to Markus [12], Neumann [13] and Peixoto [14]. We denote by R the number of canonical regions.

Theorem 2.1. *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S\big(p(\mathcal{X})\big)$ and $S\big(p(\mathcal{Y})\big)$ are topologically equivalent.*

2.2. Reduction of the parameters

Each system which was given in Theorem 1.1 is invariant by the symmetries mentioned below, so we only need to study their phase portraits for $a > 0$.

System (i): is invariant under the change $(x, y, t, a) \to (x, y, t, -a)$.

System (ii): is invariant under the change $(x, y, t, a) \to (x, y, t, -a)$.

System (v): is invariant under the change $(x, y, t, a) \to (x, y, t, -a)$.

System (vi): is invariant under the change $(x, y, t, a) \to (-x, y, -t, -a)$.

System (vii): is invariant under the change $(x, y, t, a) \to (x, -y, t, -a)$.

3. Finite and infinite singularities

To study the phase portraits of the considered systems, we shall characterize all the finite and infinite singular points of these systems with their local phase portraits.

Finite singular points

The singular points of any quadratic system mentioned in Theorem 1.1 are described in the following Proposition. Remember that when considering the specifications of the systems that were previously provided,

we will take into account their sign.

Proposition 3.1. *The following statements hold for the quadratic systems of Theorem 1.1.*

(a) *System* (i) *has two finite hyperbolic singularities, an unstable node at* $(a, 0)$ *and a stable node at* $(-a, 0)$.
(b) *System* (ii) *has two finite hyperbolic nodes, a stable one at* $(-a, 0)$ *and an unstable one at* $(a, 0)$.
(c) *System* (iii) *has one finite linearly zero singular point at the origin of coordinates and its local phase portrait consists of two elliptic sectors.*
(d) *System* (iv) *has one finite linearly zero singular point at the origin of coordinates and by doing a blow-up, we know that its local phase portrait is formed by two elliptic sectors.*
(e) *System* (v) *has one finite linearly zero singular point at the origin, where its local phase portrait is formed by two hyperbolic sectors.*
(f) *System* (vi) *has two finite hyperbolic nodes, a stable one at* $(0, 0)$ *and an unstable one at* $(a, 0)$.
(g) *System* (vii) *has two finite hyperbolic singularities, a stable node at* $(-a, 0)$ *and an unstable node at* $(a, 0)$.

Infinite singular points

We shall use the notations and definitions given in Subsection 2.2 for determining the local phase portraits at the infinite singular points in the Poincaré disc.

Proposition 3.2.

(h) *In the local chart U_1, the System* (i) has one infinite hyperbolic saddle at $(0,0)$. The origin of the local chart U_2 is a linearly zero singularity and by doing the blow-up change of variables we conclude that its local phase portrait consists of two parabolic sectors and one hyperbolic sector.

(j) In the local chart U_1, the System (ii) has one infinite hyperbolic saddle at $(0,0)$. The origin of the local chart U_2 is not a singularity.

(k) In the local chart U_1, the System (iii) has no singularity and the origin of the local chart U_2 is a hyperbolic saddle.

(l) The System (iv) has no singular points in the local chart U_1. In the local chart U_2 the origin is a hyperbolic saddle.

(m) In the local chart U_1, the System (v) has no singularity, and on the local chart U_2 the origin is a hyperbolic stable node.

(n) In the local chart U_1, the System (vi) has infinity as a line of singularity, and in the local chart U_2 the origin is a saddle.

(p) In the local chart U_1, the System (vii) has one singularity at the origin which is a saddle, and in the local chart U_2 the origin is a linearly zero singularity. By doing the blow-up change of variables we obtain that its local phase portrait consists of four parabolic and two hyperbolic sectors.

P r o o f o f P r o p o s i t i o n 3.1.

System (i): has two finite hyperbolic singularities: $(a,0)$ with eigenvalues $a/3$ and $a/2$. Hence by using Theorem 2.15 of [9], we get that it is an unstable node, and $(-a,0)$ with eigenvalues $(-a)/3$ and $(-a)/2$, then it is a stable node. This completes the proof of Statement (a).

System (ii): has two finite hyperbolic singular points, a stable node at $(-a,0)$ with eigenvalues $(-a)/2$ and $(-a)/3$, and an unstable node at $(a,0)$ with eigenvalues $a/3$ and $a/2$. Then Statement (b) holds.

System (iii) : has only one finite linearly zero singular point at the origin of coordinates. We need to do a blow-up for describing its local phase portrait. Doing the blow-up change of variables $y = zx$, and after eliminating the common factor x of $\dot{x}$, and $\dot{z}$ rescaling of the independent variable $ds = xdt$, we obtain the system

$$\dot{x} = \frac{xz}{4}, \qquad \dot{z} = \frac{1}{12} - \tfrac{z^2}{4}.$$

This system has two hyperbolic saddles $\left(0, \pm\frac{1}{\sqrt{3}}\right)$ on $x = 0$, with eigenvalues $\mp\frac{1}{2\sqrt{3}}$ and $\pm\frac{1}{4\sqrt{3}}$. Going back through the two changes of variables and taking into account the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of System (iii) is formed by two elliptic sectors. This completes the proof of Statement (c).

System (iv): has only one finite linearly zero singular point at the origin. We need to do a blow-up $x = zy$ for describing its local phase portrait. Doing the blow-up change of variables $y = zx$, and after eliminating the common factor x of $\dot{x}$ and $\dot{z}$ by doing the rescaling of the independent variable $ds = xdt$, we obtain the system

$$\dot{z} = -z - z^3, \qquad \dot{z} = yz^2 - \tfrac{y}{2}.$$

This system has one singular point on $x = 0$, which is a stable node at $(0, 0)$, with eigenvalues (-1) and $(-1/2)$. Going back through two changes of variables and taking into account the direction of the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of System (iv) is formed by two elliptic sectors. The proof of Statement (d) is done.

System (v): has only one finite singular point at the origin, which is linearly zero. Doing the blow-up $y = xz$, and after eliminating the common factor x of $\dot{x}$ and $\dot{z}$ by doing the rescaling of the independent variable $ds = xdt$, we obtain the differential system

$$\dot{x} = \frac{xz}{6}, \qquad \dot{z} = \frac{1}{12} + \frac{z^2}{4}.$$

This system has no singular points. Going back through the two changes of variables and taking into account the direction of the flow of the system on the axes, we obtain that the local phase portrait at the origin of System (vi) is formed by one hyperbolic sector. Hence Statement (e) holds.

System (vi): has two singularities, a stable node at the origin with its corresponding eigenvalues $(-a)$ and $(-5a)/6$, and an unstable node at $(a, 0)$ with eigenvalues $a/6$ and a.

System (vii): has a hyperbolic stable node at $(-a, 0)$, the eigenvalues of the linear part of the differential System (vii) are $(-2a)$ and $(-a)$, the second singularity is an unstable hyperbolic node at $(a, 0)$, and the eigenvalues of its linear part are $2a$ and $5a$. This completes the proof of Statement (g).

□

Proof of Proposition 3.2.

Statement (h): In the local chart U_1, System (i) becomes

$$\dot{u} = \frac{1}{6}u(2 + a^2v^2), \qquad \dot{v} = \frac{1}{6}v(-1 + a^2v^2).$$

This system has one hyperbolic singular point $q_1 = (0, 0)$ with eigenvalues $\frac{1}{3}$ and $-\frac{1}{6}$, then it is a saddle.

In the local chart U_2, System (i) written as

$$\dot{u} = \frac{1}{6}(-2u^2 - a^2v^2), \qquad \dot{v} = -\frac{uv}{2}. \tag{3}$$

The origin of this system is a linearly zero singularity. We need to do a blow-up $v = uw$ for describing its local phase portrait. After eliminating the common factor u of $\dot{u}$ and $\dot{w}$, rescaling of the independent variable $ds = udt$, we obtain the system

$$\dot{u} = \frac{1}{6}(-2u - a^2w^2u), \qquad \dot{v} = \frac{1}{6}w + w^3.$$

This system has three singular points on $u = 0$, a stable node at $(0, 0)$, with eigenvalues $-\frac{1}{6}$ and $-\frac{1}{3}$ and two saddles at $\left(0, \pm\frac{1}{\sqrt{6}}\right)$ with eigenvalues $\frac{1}{3}$ and $-\frac{1}{3} - \frac{a^2}{36}$. Going back through two changes of variables, $ds = udt$ and $v = uw$, and taking into account the direction of the flow on the axes, we obtain that the local phase portrait at the origin of System (3) is formed by one hyperbolic and two parabolic sectors.

Statement (j): In the local chart U_1, System (ii) is written as

$$\dot{u} = \frac{1}{6}u(2 + 2u^2 + a^2v^2), \qquad \dot{v} = \frac{1}{6}v(-1 + 2u^2 + a^2v^2).$$

This system has one hyperbolic singular point at $(0, 0)$ with eigenvalues $\frac{1}{3}$ and $-\frac{1}{6}$, then it is a saddle.

In the local chart U_2, System (ii) is

$$\dot{u} = \frac{1}{6}(-2 - 2u^2 - a^2v^2), \qquad \dot{v} = -\frac{uv}{2},$$

therefore the origin of U_2 is not a singular point.

Statement (k): In the local chart U_1, System (iii) is

$$\dot{u} = -\frac{1}{12}(1 + u^2), \qquad \dot{v} = -\frac{uv}{4}.$$

This system has no singular point.

In the local chart U_2, the system is

$$\dot{u} = \frac{1}{12}u\left(u^2 + 1\right), \qquad \dot{v} = \frac{1}{12}\left(u^2 - 2\right)v.$$

So the origin of this system is a saddle with eigenvalues $-\frac{1}{6}$ and $\frac{1}{12}$.

Statement (l): In the local chart U_1, System (iv) becomes

$$\dot{u} = 1 + u^2, \qquad \dot{v} = \frac{3uv}{2}. \tag{4}$$

So the system has no singularities in U_1.

In the local chart U_2, it is written

$$\dot{u} = -u(1 + u^2), \qquad \dot{v} = \frac{1}{2}(v - 2u^2 v).$$

Therefore, the origin of this system is a saddle with eigenvalues (–1) and $\frac{1}{2}$.

Statement (m): In the local chart U_1, System (v) is

$$\dot{u} = \frac{1}{12}(1 + u^2), \qquad \dot{v} = -\frac{uv}{6}.$$

So the system has no singular point.

In the local chart U_2 we get

$$\dot{u} = -\frac{1}{12}u(1 + u^2), \qquad \dot{v} = -\frac{1}{12}(3 + u^2)v.$$

The origin of this system is a hyperbolic stable node with eigenvalues $-\frac{1}{4}$ and $-\frac{1}{12}$.

Statement (n): In the local chart U_1, System (vi) is

$$\dot{u} = \frac{auv}{6}, \qquad \dot{v} = v(av - 1). \tag{5}$$

Then $v = 0$ is a solution of System (5), which means that the infinity is a line of singularities. We do a change of variable $vdt = ds$, the differential System (5) becomes

$$\dot{u} = \frac{au}{6}, \qquad \dot{v} = (av - 1). \tag{6}$$

This system has no singularity on $v = 0$.

In the local chart U_2, we get

$$\dot{u} = -\frac{1}{6}auv, \qquad \dot{v} = \frac{1}{6}v(5av - 6u). \tag{7}$$

We have $v = 0$ is a singularity of this system, we do a change of variable $vdt = ds$, the differential System (7) becomes

$$\dot{u} = -\frac{1}{6}au, \qquad \dot{v} = \frac{1}{6}(5av - 6u).$$

The origin of this system is a saddle with eigenvalues $(-a)/6$ and $(5a)/6$.

Statement (p): In the local chart U_1, System (vii) is

$$\dot{u} = u(a^2v^2 + 2av + 2), \qquad \dot{v} = v\left(a^2v^2 - 1\right).$$

This system has one singular point at the origin, with eigenvalues 2 and –1, hence, it is a saddle.

In the local chart U_2, we get

$$\dot{u} = -a^2v^2 - 2auv - 2u^2, \qquad \dot{v} = -v(2av + 3u). \tag{8}$$

The origin of this system is a linearly zero singular point. We do a blow-up $v = uw$ for describing its local phase portrait. After eliminating the common factor u of $\dot{u}$ and $\dot{w}$, by doing the rescaling of the independent variable $ds = udt$, we obtain the system

$$\dot{u} = -a^2uw^2 - 2auw - 2u, \qquad \dot{v} = a^2w^3 - w.$$

This system has three singular points on $u = 0$, a stable node at $(0, 0)$, with eigenvalues (-2) and (-1), a saddle at $(0, (-1)/a)$ with eigenvalues (-1) and 2 and the third singular point is a saddle at $(0, 1/a)$ with eigenvalues (-5) and 2. Going back through the two changes of variables, $ds = udt$ and $v = uw$, and taking into account the flow on the axes, we obtain that the local phase portrait at the origin of System (vii) is formed by four parabolic and two hyperbolic sectors. □

4. Local and global phase portraits

In this section, we shall see how to characterize the global phase portraits in the Poincaré disc of all our seven quadratic polynomial differential systems.

System (i): From Statement (a) of Proposition 3.1, we obtain that the system has two hyperbolic finite nodes which belong to the *Kiss* invariant curve of the system, so they connect each one to the other. The system has only one infinite saddle in the local chart U_1 at the origin, and the origin of the local chart U_2 is a linearly zero singular point, its locale phase portrait consists of one hyperbolic and two parabolic sectors. Since $\dot{x}_{|x=0} < 0$ and $\dot{y}_{|y=0} = 0$, then $y = 0$ is an invariant straight line of the system (see the local phase Portrait 1 of Figure 2). It results in the global phase Portrait 1 of Figure 1.

System (ii): From Statement (b) of Proposition 3.1, we know that the system has two finite hyperbolic nodes which belong to the *Nephroid* invariant curve of the system, which means that they connect each one to the other. From Statement (j) of Proposition 3.2 the system has only one infinite saddle in the local chart U_1 at the origin. Since $\dot{x}_{|x=0} < 0$ and $\dot{y}_{|y=0} = 0$, then $y = 0$ is an invariant straight line for the system, (see the local phase Portrait 2 of Figure 2). It results in the global phase Portrait 2 of Figure 1.

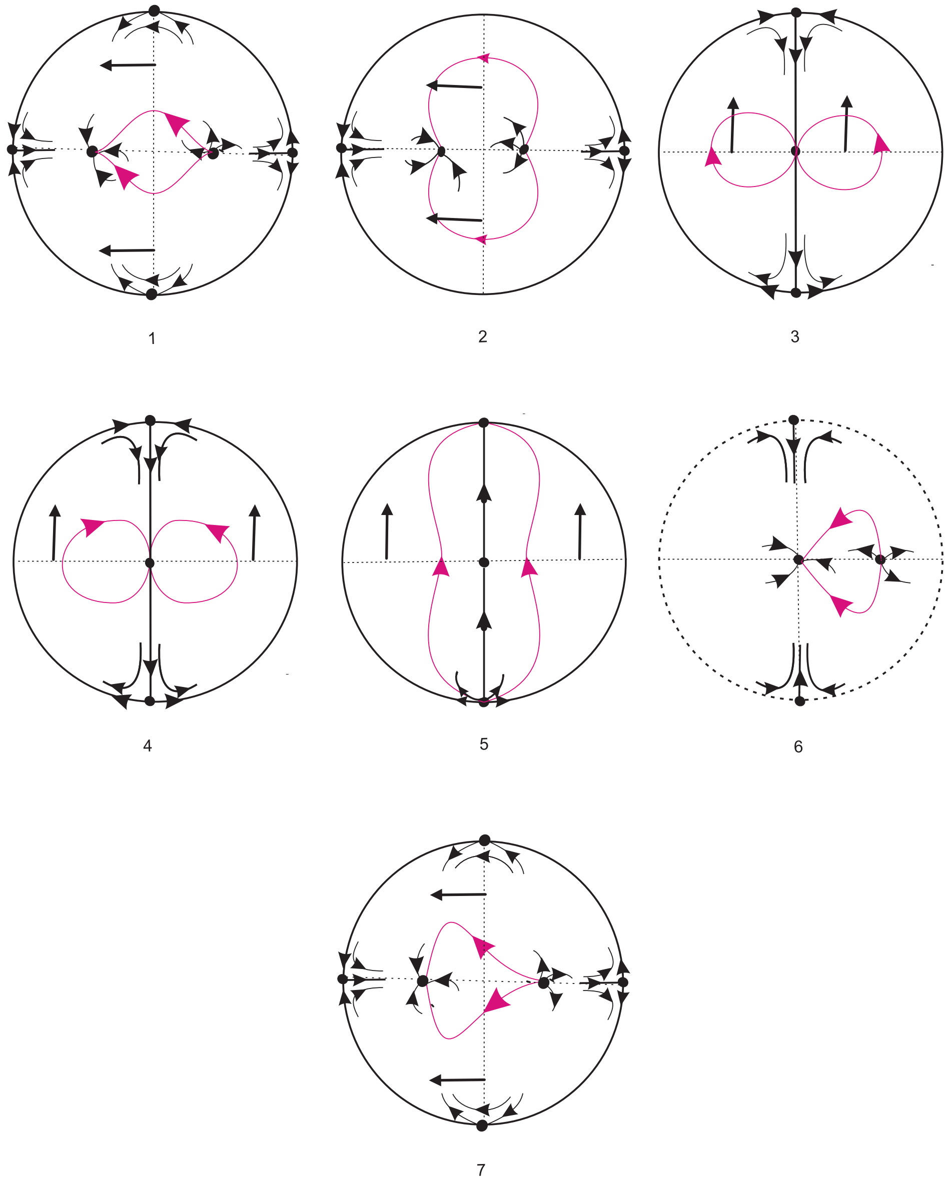

FIGURE 2. Local phase portraits at the singular points. The invariant algebraic curves of degree 6 are drawn in red colour.

System (iii): From Statement (c) of Proposition 3.1 we know that the system has one finite linearly zero singularity, such that its local phase portrait consists of two elliptic sectors, this point belongs to the *Egg* invariant curve of the system. It results from Statement (k) of Proposition 3.2 that the system has no singularity in the chart U_1, and the origin of the chart U_2 is a hyperbolic saddle. Since $x = 0$ is an invariant straight line of the system and $\dot{y}_{|y=0} > 0$ (see the local phase Portrait 3 of Figure 2). It results in the global phase Portrait 3 of Figure 1.

System (iv): From Statement (d) of Proposition 3.1, System (iv) has one linearly zero singular point at the origin, whose local phase portrait consists of one elliptic sector. This finite singular point belongs to *Dipole* invariant curve of the system. From Statement (l) of Proposition 3.2 the system has one hyperbolic saddle at the origin of the local chart U_2. Knowing that both of Dipole curve and the straight line $x = 0$ are invariant for the system, and the fact that $\dot{y}_{|y=0} > 0$, (see the local phase Portrait 4 of Figure 2), so it follows the global phase Portrait 4 of Figure 1.

System (v): has one linearly zero singular point at the origin, whose local phase portrait consists of one hyperbolic sector (see Statement (e) of Proposition 3.1). This singular point belongs to *Radial* invariant curve of the system. For the infinite ones it has one hyperbolic stable node at the origin of the local chart U_2 (see Statement (m) of Proposition 3.2). Knowing that both of *Radial* curve and the straight line $x = 0$ are invariant for the system, and the fact that $\dot{y}_{|y=0} > 0$, (see local phase Portrait 5 of Figure 2), we get the global phase Portrait 5 of Figure 1.

System (vi): has the global phase Portrait 6 of Figure 1 corresponds. Indeed, from statement (f) of Proposition 3.1 we obtain that the system has two hyperbolic nodes belonging to *Pearls of Sluze* invariant curve of the system. From statement (n) of Proposition 3.2 we obtain that the infinity is a line of singularities and on the local chart U_2 the origin is a saddle.

Since that *Pearls of Sluze* curve and the straight lines $x = 0$ and $y = 0$ are invariant for the system, and from the local phase Portrait 6 given in Figure 2, it follows the global phase Portrait 6 of Figure 1.

System (vii): has the global phase Portrait 7 of Figure 1. Indeed, from Statement (g) of Proposition 3.1 we obtain that the system has two hyperbolic nodes belonging to *Tear Drope* the invariant curve of the system. From statement (p) of Proposition 3.2 we obtain that the system has one infinite singular point in the chart U_1 which is a hyperbolic saddle and the behaviour of the system at the origin of the chart U_2 consists of four parabolic sectors and two hyperbolic sectors.

Since both of *Tear Drop* curve and the straight line $y = 0$ are invariant for the system, and from the local phase Portrait 7 given in Figure 2, it follows the global phase Portrait 7 of Figure 1.

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Received June 8, 2022

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