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SUCCESSIVE APPROXIMATIONS FOR CAPUTO-FABRIZIO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work we deal with a uniqueness result of solutions for a class of fractional differential equations involving the Caputo-Fabrizio derivative. We provide a result on the global convergence of successive approximations.

1. Introduction

Fractional differential equations have been of great interest to physics, mathematics, engineering, biology, image processing, electricity, control theory, economics, biophysics, mechanics, etc., see [7, 19–21, 24, 25, 30, 31]. For a few fundamental results in the theory of fractional differential equations see [2, 4, 5, 22, 27, 29, 32], for new kinds of the Caputo-Fabrizio derivative see [11, 23], and for several results demonstrated on Caputo-Fabrizio fractional differential equations problem see [8–10, 16, 17].

In [26], the author proved an abstract monotone iterative scheme by using a maximum principle. Many articles have been made on the convergence of successive approximations for nonlinear functional equations, and on global

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convergence of successive approximations of the problem for functional differential equations. In 1968, Browder [12] gave a brief and transparent proof of a generalization of the classical Picard-Banach contraction principle by using the convergence of successive approximations. In 1981, Chen [14] used the successive approximations method to study the existence of solutions for the functional integral equations

$$\begin{cases} x(t) = f(t), & t \in [\sigma - r, \sigma], \\ x(t) = f(t) + \int_{\sigma}^{t} g(t, s, x_s)ds, & t \in [\sigma, b]. \end{cases}$$

In [15], Czlapiiǹski studied the global convergence of successive approximations of the Darboux problem for partial functional differential equations with infinite delay, and in [18], Faina studied the generic property of global convergence of successive approximations for functional differential equations with infinite delay.

Recently, the global convergence of successive approximations has been initiated by Abbas *et al.* [1,3]. Some results about the global convergence of successive approximations for abstract semilinear differential equations are obtained in [1], and other results concern the successive approximations for the Darboux problem for implicit partial differential equations are mentioned in [3].

In this paper, we begin the study of the global convergence of successive approximations for Caputo-Fabrizio fractional differential equation(CFFDE)

$$\begin{cases} \left({}^{\mathrm{CF}}D_0^s \rho\right)(t) = \varphi\big(t, \rho(t)\big), & t \in \Upsilon := [0, \lambda], \\ \rho(0) = \rho_0. \end{cases} \tag{1}$$

Here ${}^{\mathrm{CF}}D_t^s$ is for the CFFDE, $0 < s < 1$, $\varphi : [0, \lambda] \times \mathbb{R} \to \mathbb{R}$ is continuous and $\rho_0 \in \mathbb{R}$.

2. Preliminaries

Let $\Upsilon := [0, \lambda],\ \lambda > 0$. We will denote by $C := C(\Upsilon, \mathbb{R})$ the Banach space of continuous functions from Υ into $\mathbb{R}$ with the norm

$$\|\rho\|_\infty := \sup_{t \in \Upsilon} |\rho(t)|.$$

The Banach space of the measurable function $\rho : \Upsilon \to \mathbb{R}$ integrable Lebesgue is denoted by $L^1(\Upsilon)$, equipped with the norm

$$\|\rho\|_{L^1} = \int_0^\lambda |\rho(t)|dt.$$

Now we will give results of fractional calculus.

Definition 2.1 ([13]). We define the Caputo-Fabrizio fractional integral of order $0 < s < 1$ for a function $\varphi \in L^1(\Upsilon)$ as follows

$$^{\mathrm{CF}}I^s\varphi(\tau) = \frac{2(1-s)}{M(s)(2-i)}\varphi(\tau) + \frac{2s}{M(s)(2-s)}\int_0^\tau \varphi(\eta)d\eta, \qquad \tau \geq 0,$$

where $M(s)$ is normalization constant depending on s. For $M(s) = \frac{2}{2-s}$, we obtain

$$^{\mathrm{CF}}I^s\varphi(\tau) = (1-s)\varphi(\tau) + s\int_0^\tau \varphi(\eta)d\eta, \qquad \tau \geq 0.$$

Definition 2.2 ([13]). We define the Caputo-Fabrizio fractional derivative of order $0 < s < 1$ for a function $\varphi \in AC(\Upsilon)$ as follows

$$^{\mathrm{CF}}D^s\varphi(\tau) = \frac{(2-s)M(s)}{2(1-s)}\int_0^\tau \exp\left(-\frac{s}{1-s}(\tau-\eta)\right)\varphi'(\eta)d\eta, \quad \tau \in \Upsilon.$$

We note that $(^{\mathrm{CF}}D^s)(\varphi) = 0$ if and only if φ is a constant function. If

$$M(s) = \frac{2}{2-s},$$

we have

$$^{\mathrm{CF}}D^s\varphi(\tau) = \frac{1}{1-s}\int_0^\tau \exp\left(-\frac{s}{1-s}(\tau-\eta)\right)\varphi'(\eta)d\eta, \qquad \tau \in \Upsilon.$$

Lemma 2.3 ([6,28]). *Let $\varphi \in L^1(\Upsilon)$. Then the problem*

$$\begin{cases} \left(^{\mathrm{CF}}D_0^s\rho\right)(t) = \varphi(t), & t \in \Upsilon := [0,\lambda], \\ \rho(0) = \rho_0, \end{cases} \tag{2}$$

admits a unique solution which is given by

$$\rho(t) = \rho_0 + \frac{2(1-s)}{(2-s)M(s)}\big(\varphi(t) - \varphi(0)\big) + \frac{2s}{(2-s)M(s)}\int_0^t \varphi(\tau)d\tau. \tag{3}$$

3. Successive Approximations and Uniqueness Results

The family which contains all real valued and continuous functions on the interval Υ which is a Banach space supplied with the norm

$$\|\rho\| = \sup_{t\in\Upsilon} |\rho(t)|.$$

This section is devoted to giving the main result of the global convergence of successive approximations.

Definition 3.1. The solution of the problem (1) is a continuous function $\rho \in \mathcal{C}$ which satisfies the equation $\left({}^{\mathrm{CF}}D_0^s\rho\right)(t) = \varphi\big(t, \rho(t)\big)$ on Υ and initial condition $\rho(0) = \rho_0$.

Set $\Upsilon_\varrho := [0, \varrho\lambda]$, for any $\varrho \in [0, 1]$. We are going to cite some hypotheses.

(H$_1$) The functions $\varphi : \Upsilon \times \mathbb{R} \to \mathbb{R}$ are continuous.

(H$_2$) There exist a constant $\varsigma > 0$ and a continuous function $h : \Upsilon \times [0, \varsigma] \to \mathbb{R}_+$ such that $h(t, \cdot)$ is nondecreasing for all $t \in \Upsilon$, and the inequality

$$|\varphi(t, \rho) - \varphi(t, \overline{\rho})| \le h(t, |\rho - \overline{\rho}|) \tag{4}$$

holds for all $t \in \Upsilon$ and $\rho, \overline{\rho} \in \mathbb{R}$ such that $|\rho - \overline{\rho}| \le \varsigma$.

(H$_3$) $R \equiv 0$ is the function in $C\left(\Upsilon_\delta, [0, \varsigma]\right)$ satisfying the inequality

$$R(t) \le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t, \rho)| \tag{5}$$

$$+\frac{2s}{(2-s)M(s)} \int_0^{\delta t} h\big(\tau, R(\tau)\big)\, d\tau \tag{6}$$

with $\varrho \le \delta \le 1$.

We define the successive approximations of the problem (1) as

$$\begin{aligned} \rho_0(t) &= \rho_0, \qquad t \in \Upsilon, \\ \rho_{n+1}(t) &= \rho_0 + \frac{2(1-s)}{(2-s)M(s)} \Big(\varphi\big(t, \rho_n(t)\big) - \varphi(0, \rho_0)\Big) \\ &\quad + \frac{2s}{(2-s)M(s)} \int_0^t \varphi\big(\tau, \rho_n(\tau)\big)\, d\tau, \quad t \in \Upsilon. \end{aligned}$$

Theorem 3.2. *Suppose the hypotheses* (H_1)–(H_3) *hold. Then the successive approximations* $\rho_n, n \in \mathbb{N}$ *are defined and we obtain a convergence towards the unique solution of problem* (1) *uniformly on* Υ.

P r o o f. Since ρ_n is in $\mathcal{C}$, there exist $\eta > 0$ such that

$$\|\rho_n\|_\infty \leq \eta.$$

From (H_1) the successive approximations are well-defined. Let

$$\varpi = \sup_{(t,\rho)\in\Upsilon\times[0,\eta]} |\varphi(t,\rho)|.$$

For each $t_1, t_2 \in \Upsilon$ with $t_1 < t_2$, and for all $t \in \Upsilon$,

$$\begin{aligned}
|\rho_n(t_2) - \rho_n(t_1)| \leq \Bigg| \rho_0 &+ \frac{2(1-s)}{(2-s)M(s)}\Big(\varphi\big(t_2, \rho_{n-1}(t_2)\big) - \varphi(0,\rho_0)\Big) \\
&+ \frac{2s}{(2-s)M(s)} \int_0^{t_2} \varphi\big(\tau, \rho_{n-1}(\tau)\big)\, d\tau \\
&- \Bigg[\rho_0 + \frac{2(1-s)}{(2-s)M(s)}\Big(\varphi\big(t_1, \rho_{n-1}(t_1)\big) - \varphi(0,\rho_0)\Big) \\
&+ \frac{2s}{(2-s)M(s)} \int_0^{t_1} \varphi\big(\tau, \rho_{n-1}(\tau)\big)\, d\tau\Bigg]\Bigg| \\
\leq \Bigg| &\frac{2(1-s)}{(2-s)M(s)}\Big(\varphi\big(t_2, \rho_{n-1}(t_2)\big) - \varphi\big(t_1, \rho_{n-1}(t_1)\big)\Big) \\
&+ \frac{2s}{(2-s)M(s)} \left(\int_0^{t_2} \varphi\big(\tau, \rho_{n-1}(\tau)\big)\, d\tau - \int_0^{t_1} \varphi\big(\tau, \rho_{n-1}(\tau)\big)\, d\tau \right)\Bigg| \\
\leq \Bigg| &\frac{2(1-s)}{(2-s)M(s)}\Big(\varphi\big(t_2, \rho_{n-1}(t_2)\big) - \varphi\big(t_1, \rho_{n-1}(t_1)\big)\Big) \\
&+ \frac{2s}{(2-s)M(s)} \int_{t_1}^{t_2} \varphi\big(\tau, \rho_{n-1}(\tau)\big)\, d\tau\Bigg| \\
\leq &\frac{2(1-s)}{(2-s)M(s)} \Big|\varphi\big(t_2, \rho_{n-1}(t_2)\big) - \varphi\big(t_1, \rho_{n-1}(t_1)\big)\Big| \\
&+ \frac{2s}{(2-s)M(s)} \varpi \int_{t_1}^{t_2} d\tau.
\end{aligned}$$

From the continuity of the function φ we have

$$|\rho_n(t_2) - \rho_n(t_1)| \leq \frac{2(1-s)}{(2-s)M(s)} \big|\varphi\big(t_2, \rho_{n-1}(t_2)\big) - \varphi\big(t_1, \rho_{n-1}(t_1)\big)\big| + \frac{2s}{(2-s)M(s)} \varpi(t_2 - t_1) \longrightarrow 0, \qquad \text{as } t_1 \to t_2.$$

Hence

$$\|\rho_n(t_2) - \rho_n(t_1)\| \longrightarrow 0, \quad as \ \ t_1 \to t_2,$$

and we get the equi-continuous on Υ of the sequence $\{\rho_n(t), n \in \mathbb{N}\}$.

Let

$$\nu := \sup\{\varrho \in [0,1] : \{\rho_n(t)\} \text{ converges uniformly on } \Upsilon_\varrho\}.$$

If $\nu = 1$, we obtain the global convergence of successive approximations. We will suppose that $\nu < 1$, and the sequence $\{\rho_n(t)\}$ is equi-continuous on Υ_ν, so it converges uniformly towards a function $\tilde{\rho}(t)$. If we show it, there exists $\delta \in (\nu, 1]$ such that $\{\rho_n(t)\}$ converges uniformly on Υ_δ, and we lead a contradiction.

Put

$$\rho(t) = \tilde{\rho}(t) \qquad \text{for all} \qquad t \in \Upsilon_\nu.$$

From (H_2), there exist a constant $\zeta > 0$ and a continuous function $h : \Upsilon \times [0, \zeta] \to \mathbb{R}_+$ checking inequality (4). So therefore, there exist

$$\delta \in [\nu, 1] \qquad \text{and} \qquad n_0 \in \mathbb{N},$$

such that, for all $t \in \Upsilon_\delta$ and $n, m > n_0$, we get

$$|\rho_n(t) - \rho_m(t)| \leq \zeta.$$

For all $t \in \Upsilon_\delta$, put

$$R^{(n,m)}(t) = |\rho_n(t) - \rho_m(t)|,$$

$$R_k(t) = \sup_{n,m \geq k} R^{(n,m)}(t).$$

$R_k(t)$ is the non-increasing sequence, thus it converges to a function $R(t)$ and that for all $t \in \Upsilon_\delta$. From the equi-continuity of $\{R_k(t)\}$ we have

$$\lim_{k\to\infty} R_k(t) = R(t) \quad \text{uniformly on } \Upsilon_\delta.$$

Further, for each $t \in \Upsilon_\delta$ and $n, m \geq k$, we get

$$\begin{aligned} R^{(n,m)}(t) &= |\rho_n(t) - \rho_m(t)| \\ &\leq \bigg| \rho_0 + \frac{2(1-s)}{(2-s)M(s)} \Big(\varphi\big(t, \rho_{n-1}(t)\big) - \varphi(0, \rho_0)\Big) \\ &\quad + \frac{2s}{(2-s)M(s)} \int_0^t \varphi\big(\tau, \rho_{n-1}(\tau)\big) d\tau \\ &\quad - \rho_0 - \frac{2(1-s)}{(2-s)M(s)} \Big(\varphi\big(t, \rho_{m-1}(t)\big) - \varphi(0, \rho_0)\Big) \end{aligned}$$

$$
\begin{aligned}
&- \frac{2s}{(2-s)M(s)} \int_0^t \varphi\big(\tau, \rho_{m-1}(\tau)\big)d\tau \Bigg| \\
&\le \frac{2(1-s)}{(2-s)M(s)} \big|\varphi\big(t, \rho_{n-1}(t)\big) - \varphi\big(t, \rho_{m-1}(t)\big)\big| \\
&\quad + \frac{2s}{(2-s)M(s)} \int_0^t \big|\varphi\big(\tau, \rho_{n-1}(\tau)\big) - \varphi\big(\tau, \rho_{m-1}(\tau)\big)\big| d\tau \\
&\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| \\
&\quad + \frac{2s}{(2-s)M(s)} \int_0^t \big|\varphi\big(\tau, \rho_{n-1}(\tau)\big) - \varphi\big(\tau, \rho_{m-1}(\tau)\big)\big| d\tau \\
&\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| \\
&\quad + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} \big|\varphi\big(\tau, \rho_{n-1}(\tau)\big) - \varphi\big(\tau, \rho_{m-1}(\tau)\big)\big| d\tau,
\end{aligned}
$$

therefore, by (4) we get

$$
\begin{aligned}
R^{(n,m)}(t) &\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| \\
&\quad + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h\big(\tau, |\rho_{n-1}(\tau) - \rho_{m-1}(\tau)|\big) d\tau \\
&\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| \\
&\quad + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h\big(\tau, R^{(n-1,m-1)}(\tau)\big) d\tau,
\end{aligned}
$$

then,

$$
\begin{aligned}
R_k(t) &\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| \\
&\quad + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h\big(\tau, R_{k-1}(\tau)\big) d\tau.
\end{aligned}
$$

By the Lebesgue dominated convergence theorem we have

$$R(t) \leq \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h\big(\tau, R(\tau)\big)d\tau.$$

By (H_1) and (H_3) we have $R \equiv 0$ on Υ_δ, which gives that

$$\lim_{k\to\infty} R_k(t) = 0 \quad \text{uniformly on } \Upsilon_\delta.$$

Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence on Υ_δ. So $\{\rho_k(t)\}_{k=1}^{\infty}$ is uniformly convergent on Υ_δ which gives the contradiction.

Thus $\{\rho_k(t)\}_{k=1}^{\infty}$ converges uniformly on Υ to a continuous function $\rho_*(t)$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{k\to\infty} \rho_0 &+ \frac{2(1-s)}{(2-s)M(s)}\Big(\varphi\big(t,\rho_k(t)\big) - \varphi(0,\rho_0)\Big) \\ &+ \frac{2s}{(2-s)M(s)}\int_0^t \varphi\big(\tau,\rho_k(\tau)\big)d\tau \\ = \rho_0 &+ \frac{2(1-s)}{(2-s)M(s)}\Big(\varphi\big(t,\rho_*(t)\big) - \varphi(0,\rho_0)\Big) \\ &+ \frac{2s}{(2-s)M(s)}\int_0^t \varphi\big(\tau,\rho_*(\tau)\big)d\tau,\end{aligned}$$

for all $t \in \Upsilon$. This leads to ρ_* that is a solution to the problem (1). □

Now we will demonstrate the uniqueness of the solutions of the previous problem (1). Let ρ_1 and ρ_2 be two solutions of (1). Then put

$$\nu := \sup\{\varrho \in [0,1] : \rho_1(t) = \rho_2(t) \quad \text{for } t \in \Upsilon_\varrho\},$$

and suppose that $\nu < 1$. There exist a constant $\zeta > 0$ and a function h: $\Upsilon_\nu \times [0,\zeta] \to \mathbb{R}_+$ verifying inequality (4). We will choose $\delta \in (\varrho, 1)$ such as

$$|\rho_1(t) - \rho_2(t)| \leq \zeta \quad \text{for } t \in \Upsilon_\delta.$$

Then for all $t \in \Upsilon_\delta$ we get

$$|\rho_1(t) - \rho_2(t)| \le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} \big|\varphi\big(\tau,\rho_0(\tau)\big) - \varphi\big(\tau,\rho_1(\tau)\big)\big| d\tau$$
$$\le \frac{4(1-s)}{(2-s)M(s)} \sup_{(t,\rho)\in\Upsilon_\delta\times[0,\eta]} |\varphi(t,\rho)| + \frac{2s}{(2-s)M(s)} \int_0^{\delta t} h\big(\tau, |\rho_0(\tau) - \rho_1(\tau)|\big) d\tau.$$

Again, by (H_1) and (H_3) we get $\rho_1 - \rho_2 \equiv 0$ on Υ_δ. This gives $\rho_1 = \rho_2$ on Υ_δ, which leads to a contradiction. So $\nu = 1$, the studied problem (1) admits a unique solution on Υ.

4. An Example

We consider the following Caputo-Fabrizio fractional differential Cauchy problem:

$$\begin{cases} \big({}^{\mathrm{CF}}D_0^s \rho\big)(t) = \varphi\big(t, \rho(t)\big), \quad t \in \Upsilon := [0,1],\ s \in (0,1), \\ \rho(0) = 1, \end{cases} \tag{7}$$

where

$$\varphi\big(t,\rho(t)\big) = \big(e^{t-1} + |\rho(t)|\big) \frac{t}{(1+t^2)(1+|\rho(t)|)}.$$

For each $\rho,\ \overline{\rho} \in \mathbb{R}$ and $t \in \Upsilon$ we have

$$|\varphi(t,\rho) - \varphi(t,\overline{\rho})| \le t(1+e^{t-1})|\rho - \overline{\rho}|.$$

This leads to the condition (4) which holds for each $t \in \Upsilon$, $\zeta > 0$ and the function

$$h : [0,1] \times [0,\zeta] \to [0,\infty)$$

such as

$$h(t,\rho) = t(1+e^{t-1})|\rho|.$$

Then Theorem 3.2 leads us to the successive approximations ρ_n, $n \in \mathbb{N}$, defined by

$$\rho_0(t) = 1, \qquad t \in \Upsilon,$$

$$\rho_{n+1}(t) = 1 + \frac{2(1-s)}{(2-s)M(s)}\Big(\varphi\big(t, \rho_n(t)\big) - \varphi(0,1)\Big) + \frac{2s}{(2-s)M(s)}\int_0^t \varphi\big(\tau, \rho_n(\tau)\big)d\tau, \quad t \in \Upsilon,$$

which converges uniformly on Υ to the unique solution of the problem (7).

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