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ON THE GENERALIZED INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

Mehmet Zeki Sarikaya

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, TURKEY

ABSTRACT. The aim of this paper is to establish some generalized integral inequalities for convex functions of 2-variables on the co-ordinat. Then, we will give a generalized identity and with the help of this integral identity, we will investigate some integral inequalities connected with the right hand side of the Hermite-Hadamard type inequalities involving Riemann integrals and Riemann--Liouville fractional integrals.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well-known in the literature as the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w)$$

holds for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [8]).

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A formal definition for co-ordinated convex function may be stated as follows

Definition 1.1. A function $f:\Delta\to\mathbb{R}$ will be called co-ordinated convex on Δ, for all $t,s\in[0,1]$ and $(x,y),(u,w)\in\Delta$, if the following inequality holds:

$$\begin{aligned}&f\big(tx+(1-t)\,y,su+(1-s)\,w\big)\\&\le tsf(x,u)+s(1-t)f(y,u)+t(1-s)f(x,w)+(1-t)(1-s)f(y,w).\end{aligned}\tag{1}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exists a co-ordinated convex function which is not convex (see, [8]). Dragomir first proved Hermite-Hadamard inequalities for co-ordinated convex mappings in [8]. The midpoint and trapezoid type inequalities for co-ordinated convex functions were established in the papers [12] and [16], respectively. Moreover, Sarikaya obtained Hermite-Hadamard inequalities for functions with two variables by utilizing Riemann-Liouville fractional integrals in [17]. Whereas Sarikaya gave the corresponding fractional trapezoid inequalities for co-ordinated convex functions in [17], Tunç et al. presented fractional midpoint type inequalities for co-ordinated convex functions in [23]. In the literature, there are numerous papers related to Hermite-Hadamard inequalities for several type co-ordinated convex functions. For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1], [2], [8], [9] – [12], [14] – [16], [22]).

Recently, in [8], Dragomir has established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then, one has the inequalities:*

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\&\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\&\quad\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\&\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}\tag{2}$$

The above inequalities are sharp.

Generalized double fractional integrals are given by Sarikaya et al. in [16] as follows

Definition 1.3. Let $f \in L_1([a,b] \times [c,d])$. The Riemann-Liouville integrals

$$J^{\alpha,\beta}_{a+,c+},\ J^{\alpha,\beta}_{a+,d-},\ J^{\alpha,\beta}_{b-,c+} \text{ and } J^{\alpha,\beta}_{b-,d-} \text{ of order } \alpha,\beta > 0 \text{ with } a,c \geq 0$$

are defined by:

$$J^{\alpha,\beta}_{a+,c+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt, \quad x > a,\ y > c,$$

$$J^{\alpha,\beta}_{a+,d-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt, \quad x > a,\ y < d,$$

$$J^{\alpha,\beta}_{b-,c+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt, \quad x < b,\ y > c,$$

and

$$J^{\alpha,\beta}_{b-,d-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt, \quad x < b,\ y < d,$$

respectively. Here, Γ is the Gamma function,

$$J^{0,0}_{a+,c+}f(x,y) = J^{0,0}_{a+,d-}f(x,y) = J^{0,0}_{b-,c+}f(x,y) = J^{0,0}_{b-,d-}f(x,y) = f(x,y),$$

and

$$J^{1,1}_{a+,c+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y f(t,s)dsdt.$$

For some recent results connected with fractional integral inequalities see ([3]– [7], [18]– [21]).

The aim of this paper is to establish some generalized integral inequalities for convex functions of 2-variables on the co-ordinat. Then, we will give a generalized identity and, with the help of this integral identity, we will investigate some integral inequalities connected with the right hand side of the Hermite-Hadamard type inequalities involving Riemann integrals and Riemann-Liouville fractional integrals.

2. Fractional Inequalities for co-ordinated convex functions

Throughout this section, we will use the following symbols

$$F_1(t) = h_1(t) - h_1(1-t), \quad F_2(s) = h_2(s) - h_2(1-s),$$

and

$$\begin{aligned} S(F_1, F_2) &= F_1(0) F_2(0) f(b,d) - F_1(0) F_2(1) f(b,c) \\ &- F_1(1) F_2(0) f(a,d) + F_1(1) F_2(1) f(a,c) \\ &+ \frac{F_1(0)}{(d-c)} \int_c^d F_2'\left(\frac{d-y}{d-c}\right) f(b,y)\, dy \\ &- \frac{F_1(1)}{(d-c)} \int_c^d F_2'\left(\frac{d-y}{d-c}\right) \frac{\partial f}{\partial s}(a,y)\, dy \\ &+ \frac{F_2(0)}{(b-a)} \int_a^b F_1'\left(\frac{b-x}{b-a}\right) f(x,d)\, dx \\ &- \frac{F_2(1)}{(b-a)} \int_a^b F_1'\left(\frac{b-x}{b-a}\right) f(x,c)\, dx. \end{aligned}$$

In this part, we will give the following inequalities by using convex functions of 2-variables on the co-ordinat. In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$ with $0 \le a < b$, $0 \le c < d$ and $h_1, h_2 : [0,1] \to R$ be two positive differentiable functions. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then the following equality holds:*

$$\begin{aligned} &S(F_1, F_2) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F_1'\left(\frac{b-x}{b-a}\right) F_2'\left(\frac{d-y}{d-c}\right) f(x,y)\, dy dx \\ &= (b-a)(d-c) \int_0^1 \int_0^1 F_1(t) F_2(s) \frac{\partial^2 f}{\partial t \partial s}\big(ta + (1-t)b, sc + (1-s)d\big) ds dt. \end{aligned} \tag{3}$$

P r o o f. By integration by parts, we get

$$\int_0^1 \int_0^1 F_1(t) F_2(s) \frac{\partial^2 f}{\partial t \partial s}\big(ta + (1-t)b, sc + (1-s)d\big) ds dt$$

$$= \int_0^1 F_2(s) \left\{ F_1(t) \frac{1}{a-b} \frac{\partial f}{\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)\Big|_0^1 \right.$$

$$\left. - \frac{1}{a-b} \int_0^1 F_1'(t) \frac{\partial f}{\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)dt \right\} ds$$

$$= \int_0^1 F_2(s) \left\{ \frac{F_1(0)}{b-a} \frac{\partial f}{\partial s}\big(b, sc+(1-s)d\big) - \frac{F_1(1)}{b-a} \frac{\partial f}{\partial s}\big(a, sc+(1-s)d\big) \right.$$

$$\left. + \frac{1}{b-a} \int_0^1 F_1'(t) \frac{\partial f}{\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)dt \right\} ds$$

$$= \frac{F_1(0)}{b-a} \int_0^1 F_2(s) \frac{\partial f}{\partial s}\big(b, sc+(1-s)d\big)ds - \frac{F_1(1)}{b-a} \int_0^1 F_2(s) \frac{\partial f}{\partial s}\big(a, sc+(1-s)d\big)ds$$

$$+ \frac{1}{b-a} \int_0^1 F_1'(t) \left[\int_0^1 F_2(s) \frac{\partial f}{\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)ds \right] dt$$

$$= \frac{F_1(0)}{b-a} \left\{ \frac{F_2(s)}{c-d} f\big(b, sc+(1-s)d\big)\Big|_0^1 + \frac{1}{d-c} \int_0^1 F_2'(s) f\big(b, sc+(1-s)d\big)ds \right\}$$

$$- \frac{F_1(1)}{b-a} \left\{ \frac{F_2(s)}{c-d} f\big(a, sc+(1-s)d\big)\Big|_0^1 + \frac{1}{d-c} \int_0^1 F_2'(s) \frac{\partial f}{\partial s}\big(a, sc+(1-s)d\big)ds \right\}$$

$$+ \frac{1}{b-a} \int_0^1 F_1'(t) \left\{ \frac{F_2(s)}{c-d} f\big(ta+(1-t)b, sc+(1-s)d\big)\Big|_0^1 \right.$$

$$\left. + \frac{1}{d-c} \int_0^1 F_2'(s) \frac{\partial f}{\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)ds \right\} dt$$

$$= \frac{F_1(0)}{b-a} \left[\frac{F_2(0)}{d-c} f(b,d) - \frac{F_2(1)}{d-c} f(b,c) \right] + \frac{F_1(0)}{(b-a)(d-c)} \int_0^1 F_2'(s) f\big(b, sc+(1-s)d\big)$$

$$- \frac{F_1(1)}{b-a} \left[\frac{F_2(0)}{d-c} f(a,d) - \frac{F_2(1)}{d-c} f(a,c) \right] - \frac{F_1(1)}{(b-a)(d-c)} \int_0^1 F_2'(s) \frac{\partial f}{\partial s}\big(a, sc+(1-s)d\big)ds$$

$$+ \frac{F_2(0)}{(b-a)(d-c)} \int_0^1 F_1'(t) f\big(ta+(1-t)b, d\big) - \frac{F_2(1)}{(b-a)(d-c)} \int_0^1 F_1'(t) f\big(ta+(1-t)b, c\big)$$

$$+ \frac{1}{(b-a)(d-c)} \int_0^1 F_1'(t) F_2'(s) \frac{\partial f}{\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)dsdt.$$

Thus, using the change of the variable $x = ta + (1-t)b$ and $y = sc + (1-s)d$ for $(t,s) \in [0,1] \times [0,1]$, we can write

$$\begin{aligned}
&\int_0^1 \int_0^1 F_1(t) F_2(s) \frac{\partial^2 f}{\partial t \partial s}\big(ta + (1-t)b, sc + (1-s)d\big) ds dt \\
&= \frac{F_1(0)}{b-a}\left[\frac{F_2(0)}{d-c} f(b,d) - \frac{F_2(1)}{d-c} f(b,c)\right] \\
&\quad - \frac{F_1(1)}{b-a}\left[\frac{F_2(0)}{d-c} f(a,d) - \frac{F_2(1)}{d-c} f(a,c)\right] \\
&\quad + \frac{F_1(0)}{(b-a)(d-c)^2} \int_c^d F_2'\left(\frac{d-y}{d-c}\right) f(b,y)\, dy \\
&\quad - \frac{F_1(1)}{(b-a)(d-c)^2} \int_c^d F_2'\left(\frac{d-y}{d-c}\right) f(a,y)\, dy \\
&\quad + \frac{F_2(0)}{(b-a)^2(d-c)} \int_a^b F_1'\left(\frac{b-x}{b-a}\right) f(x,d)\, dx \\
&\quad - \frac{F_2(1)}{(b-a)^2(d-c)} \int_a^b F_1'\left(\frac{b-x}{b-a}\right) f(x,c)\, dx \\
&\quad + \frac{1}{(b-a)^2(d-c)^2} \int_a^b \int_c^d F_1'\left(\frac{b-x}{b-a}\right) F_2'\left(\frac{d-y}{d-c}\right) f(x,y)\, dy dx.
\end{aligned} \tag{4}$$

Multiplying the both sides of (4) by $(b-a)(d-c)$, we obtain (3), which completes the proof. □

Remark 1. If in Lemma 2.1,

i) we choose $h_1(t) = t$, $h_2(s) = s$ on $[0,1]$, then the equality (3) becomes the equality

$$\begin{aligned}
&\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} - \frac{1}{2(d-c)} \int_c^d [f(b,y) + f(a,y)]\, dy \\
&- \frac{1}{2(b-a)} \int_a^b [f(x,d) + f(x,c)]\, dx + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy dx \\
&= \frac{(b-a)(d-c)}{4} \int_0^1 \int_0^1 (2t-1)(2s-1) \frac{\partial^2 f}{\partial t \partial s}\big(ta + (1-t)b, sc + (1-s)d\big) ds dt,
\end{aligned}$$

which is proved by Sarikaya et al. in [16].

ii) we choose $h_1(t) = t^\alpha\ (\alpha > 0)$, $h_2(s) = s^\beta\ (\beta > 0)$ on $[0,1]$, then the equality (3) becomes the fractional integral equality

$$\begin{aligned}
&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a+,c+}f(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)\right]\\
&-\frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^{\beta}_{c+}f(a,d)+J^{\beta}_{c+}f(b,d)+J^{\beta}_{d-}f(a,c)+J^{\beta}_{d-}f(b,c)\right]\\
&-\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^{\alpha}_{a+}f(b,c)+J^{\alpha}_{a+}f(b,d)+J^{\alpha}_{b-}f(a,c)+J^{\alpha}_{b-}f(a,d)\right]\\
&=\frac{(b-a)(d-c)}{4}\int_0^1\!\!\int_0^1\left[t^\alpha-(1-t)^\alpha\right]\left[s^\beta-(1-s)^\beta\right]\\
&\times\frac{\partial^2 f}{\partial t\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)dsdt,
\end{aligned}$$

which is proved by Sarikaya in [17].

Next, we start to state the first theorem containing the Hermite-Hadamard type ineqaulity for fractional integrals.

Theorem 2.2. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$ with $0\le a<b,\ 0\le c<d$ and let $h_1,h_2:[0,1]\to R$ be two positive differentiable functions. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequality:*

$$\begin{aligned}
&\left|S(F_1,F_2)+\frac{1}{(b-a)(d-c)}\int_a^b\!\!\int_c^d F_1'\left(\frac{b-x}{b-a}\right)F_2'\left(\frac{d-y}{d-c}\right)f(x,y)\,dydx\right|\\
&\le(b-a)(d-c)\int_0^1\!\!\int_0^1 ts\left\{F_1(t)F_2(s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right.\\
&+F_1(1-t)F_2(s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+F_1(t)F_2(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&\left.+F_1(1-t)F_2(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}dsdt.
\end{aligned}\tag{5}$$

P r o o f. Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is convex function on the co-ordinates on Δ, from Lemma 2.1, we have

$$\left| S(F_1,F_2) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F_1'\left(\frac{b-x}{b-a}\right) F_2'\left(\frac{d-y}{d-c}\right) f(x,y)\,dydx \right|$$
$$\leq (b-a)(d-c) \int_0^1 \int_0^1 F_1(t) F_2(s) \left| \frac{\partial^2 f}{\partial t \partial s}\big(ta+(1-t)b, sc+(1-s)d\big) \right| dsdt$$
$$\leq (b-a)(d-c) \int_0^1 \int_0^1 F_1(t) F_2(s) \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right| + s(1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right| \right.$$
$$\left. + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right| + (1-s)(1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right| \right\} dsdt.$$

By adapting the integral in above inequality, we have inequality (5). □

Remark 2. If in Theorem 2.2,

i) we choose $h_1(t) = t$, $h_2(s) = s$ on $[0,1]$, then inequality (5) becomes the inequality

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(d-c)} \int_c^d [f(b,y)+f(a,y)]\,dy \right.$$
$$\left. - \frac{1}{2(b-a)} \int_a^b [f(x,d)+f(x,c)]\,dx + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx \right|$$
$$\leq \frac{(b-a)(d-c)}{8} \left(\left| \frac{\partial^2 f}{\partial s \partial t}(a,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a,d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,d) \right| \right),$$

which is proved by Sarikaya et al. in [16].

ii) we choose $h_1(t) = t^\alpha$ $(\alpha > 0)$, $h_2(s) = s^\beta$ $(\beta > 0)$ on $[0,1]$, then inequality (5) becomes the fractional integral inequality

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right.$$
$$+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta} \Big[J^{\alpha,\beta}_{a+,c+} f(b,d) + J^{\alpha,\beta}_{a+,d-} f(b,c) +$$
$$J^{\alpha,\beta}_{b-,c+} f(a,d) + J^{\alpha,\beta}_{b-,d-} f(a,c) \Big]$$
$$- \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[J^\alpha_{a+} f(b,c) + J^\alpha_{a+} f(b,d) + J^\alpha_{b-} f(a,c) + J^\alpha_{b-} f(a,d) \right]$$
$$\left. - \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \Big[J^\beta_{c+} f(a,d) + J^\beta_{c+} f(b,d) + J^\beta_{d-} f(a,c) + J^\beta_{d-} f(b,c) \Big] \right|$$
$$\leq \frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)} \left(\left| \frac{\partial^2 f}{\partial s \partial t}(a,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a,d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,d) \right| \right),$$

which is proved by Sarikaya in [17].

Theorem 2.3. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$ with $0 \leq a < b$, $0 \leq c < d$ and let $h_1, h_2 : [0,1] \to R$ be two positive differentiable functions. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$, $q > 1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\left|S(F_1, F_2) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F_1'\left(\frac{b-x}{b-a}\right) F_2'\left(\frac{d-y}{d-c}\right) f(x,y)\, dydx\right|$$

$$\leq (b-a)(d-c) \left(\int_0^1 \int_0^1 F_1^p(t) F_2^p(s)\, dsdt\right)^{\frac{1}{p}} \tag{6}$$

$$\times \left(\frac{\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, using the well-known Hölder's inequality for double integrals and by co-ordinates convexity function of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ on Δ, we have

$$\left|S(F_1, F_2) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F_1'\left(\frac{b-x}{b-a}\right) F_2'\left(\frac{d-y}{d-c}\right) f(x,y)\, dydx\right|$$

$$\leq (b-a)(d-c) \int_0^1 \int_0^1 F_1(t) F_2(s) \left|\frac{\partial^2 f}{\partial t \partial s}\big(ta + (1-t)b, sc + (1-s)d\big)\right| dsdt$$

$$\leq (b-a)(d-c) \left(\int_0^1 \int_0^1 F_1^p(t) F_2^p(s)\, dsdt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \int_0^1 \left|\frac{\partial^2 f}{\partial t \partial s}\big(ta + (1-t)b, sc + (1-s)d\big)\right|^q dsdt\right)^{\frac{1}{q}}$$

$$\leq (b-a)(d-c) \left(\int_0^1 \int_0^1 F_1^p(t) F_2^p(s)\, dsdt\right)^{\frac{1}{p}}$$

$$\times \left(\frac{\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},$$

which completes the proof. □

Remark 3. If in Theorem 2.3,

i) we choose $h_1(t) = t$, $h_2(s) = s$ on $[0,1]$, then inequality (6) becomes the inequality

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{2(d-c)}\int_c^d [f(b,y)+f(a,y)]\,dy \right.$$

$$\left. -\frac{1}{2(b-a)}\int_a^b [f(x,d)+f(x,c)]\,dx + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \right|$$

$$\leq \frac{(b-a)(d-c)}{4^{\frac{1}{q}}\left[(p+1)(p+1)\right]^{\frac{1}{p}}}$$

$$\times \left(\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \right)^{\frac{1}{q}},$$

which is proved by Sarikaya et al. in [16].

ii) we choose $h_1(t) = t^\alpha\ (\alpha > 0)$, $h_2(s) = s^\beta\ (\beta > 0)$ on $[0,1]$, then inequality (6) becomes the fractional integral inequality

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right.$$

$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b+a)^\alpha (d+c)^\beta}$$

$$\times \left[J^{\alpha,\beta}_{a+,c+} f(b,d) + J^{\alpha,\beta}_{a+,d-} f(b,c) + J^{\alpha,\beta}_{b-,c+} f(a,d) + J^{\alpha,\beta}_{b-,d-} f(a,c) \right]$$

$$-\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a+} f(b,c) + J^\alpha_{a+} f(b,d) + J^\alpha_{b-} f(a,c) + J^\alpha_{b-} f(a,d)\right]$$

$$\left. -\frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c+} f(a,d) + J^\beta_{c+} f(b,d) + J^\beta_{d-} f(a,c) + J^\beta_{d-} f(b,c)\right] \right|$$

$$\leq \frac{(b-a)(d-c)}{4^{\frac{1}{q}}\left[(\alpha p+1)(\beta p+1)\right]^{\frac{1}{p}}}$$

$$\times \left(\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \right)^{\frac{1}{q}},$$

which is proved by Sarikaya in [17].

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Mehmet Zeki Sarıkaya
Department of Mathematics
Faculty of Science and Arts
Düzce University
Konuralp Campus
81100-Düzce
TURKEY
E-mail: sarikayamz@gmail.com