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SELECTIVE VERSION OF STAR-SEMI-LINDELÖFNESS IN (a) TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we deal with the properties $(a)R$-star-semi-Lindelöf and $(a)M$-star-semi-Lindelöf in (a)topological spaces. These properties are interesting as every $(a)R^s$-separable space is $(a)R$-star-semi-Lindelöf and every $(a)^s$-semi-Lindelöf space is $(a)R$-star-semi-Lindelöf but not every $(a)R$-star-semi--Lindelöf space is $(a)R^s$-separable or $(a)^s$-semi-Lindelöf. It is shown that if an (a)topological space X is the union of countably many (a)-open and $(a)R$star--semi-Lindelöf subspaces, then X is $(a)R$-star-semi-Lindelöf. Similar results are obtained in the context of $(a)M$-star-semi-Lindelöf spaces. Further, suitable and required counterexamples are given.

1. Introduction

Several covering properties are studied extensively in the literature of topological spaces [12, 13, 27]. All these covering properties can be described in terms of selection principles [18, 20, 29, 31, 34]. Selection principles, introduced by Scheepers [14, 30], have received much attention in last few years. Also, the notion of selective separability can be expressed in terms of selection principles. Initially, Scheepers introduced the property of selective separability (M-separability) and R-separability in topological space (see [3]) and Van Douwen introduced the property of star-Lindelöfness and star-compactness in [10].

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Recently, Kočinac [16, 17, 19–21] studied a lot of selection principles by using separability properties and star operator. Motivated by the work of Kočinac, Bal et al. [1, 2] introduced and studied the notion of R-star-Lindelöfness and M-star-Lindelöfness in topological spaces. Our aim is to extend this concept to (a)topological spaces which is a more general structure than bitopological spaces [15], (ω)topological spaces [4, 5, 7] and $(\aleph_0)$topological spaces [6]. Recently, Luthra et al. [24] studied the notion of selective version of separability in (a)topological spaces by using the notion of semi-closure, pre-closure, α-closure, β-closure and δ-closure. Also, covering properties by using the notion of semi-open sets in (a)topological spaces are studied in [25]. Motivated by Luthra et al. [24,25] and Bal et al. [1,2], in this paper, we study the properties $(a)R$-star-semi-Lindelöf and $(a)M$-star-semi-Lindelöf in (a)topological spaces by using the notion of semi-closure and respective density.

Let X be a non-empty set and $< \tau_n : n \in \mathbb{N} >$ be a sequence of topologies on X. Then, the pair $(X, \{\tau_n\})$ is called an (a)topological space (in short, (a)space) [8]. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Let F be a subset of X and $\mathcal{G}$ be a collection of subsets of X. Then:

- $\mathcal{S}t(F, \mathcal{G}) = \mathcal{S}t^1(F, \mathcal{G}) = \bigcup\{G \in \mathcal{G} : G \cap F \neq \emptyset\}$.
- For $k \in \mathbb{N}$, $\mathcal{S}t^{k+1}(F, \mathcal{G}) = \mathcal{S}t\big(\mathcal{S}t^k(F, \mathcal{G}), \mathcal{G}\big)$.
- $F \in \mathcal{S}t[\mathcal{A}]$ means that $\mathcal{S}t(F, \mathcal{A}) = X$.
- $F \in \mathcal{S}t^{k+1}[\mathcal{A}]$ means that $\mathcal{S}t^{k+1}(F, \mathcal{A}) = X$.
- $R_{\mathcal{A}}(\mathcal{B}, \mathcal{S}t[\mathcal{A}])$ (resp. $R_{\mathcal{A}}(\mathcal{B}, \mathcal{S}t^k[\mathcal{A}])$) denotes the selection principle:
  For each member of $\mathcal{A}$ and for every sequence $< B_n : n \in \mathbb{N} >$ of elements of $\mathcal{B}$, there exist points $x_n \in B_n$ (for each $n \in \mathbb{N}$) such that $\cup_{n\in\mathbb{N}}\{x_n\} \in \mathcal{S}t[\mathcal{A}]$ (resp. $\cup_{n\in\mathbb{N}}\{x_n\} \in \mathcal{S}t^k[\mathcal{A}]$).
- $M_{\mathcal{A}}(\mathcal{B}, \mathcal{S}t[\mathcal{A}])$ (resp. $M_{\mathcal{A}}(\mathcal{B}, \mathcal{S}t^k[\mathcal{A}])$) denotes the selection principle:
  For each member of $\mathcal{A}$ and for every sequence $< B_n : n \in \mathbb{N} >$ of elements of $\mathcal{B}$, there exist finite sets $b_n \subseteq B_n$ (for each $n \in \mathbb{N}$) such that $\cup_{n\in\mathbb{N}}\, b_n \in \mathcal{S}t[\mathcal{A}]$ (resp. $\cup_{n\in\mathbb{N}}\, b_n \in \mathcal{S}t^k[\mathcal{A}]$).

Throughout this paper, $(X, \{\tau_n\})$ or X, in short, denotes an (a)topological space on which no separation axioms are assumed unless explicitly stated. For $Y \subseteq X$, we say $(Y, \{\tau_{nY}\})$ is an (a)subspace of X, where τ_{nY} is the induced subspace topology on Y inherited from τ_n. For a subset A of X, the (τ_n)closure (resp. (τ_n)interior) of A is denoted by τ_n-cl(A) (resp. τ_n-int(A)) and the (m,n)-semi-closure (resp. (m,n)-semi-interior) of A is denoted by $\tau_{(m,n)}$-cl$_s(A)$ (resp. $\tau_{(m,n)}$-int$_s(A)$). For a subset A of (a)subspace Y of X, the (τ_n)closure (resp. (τ_n)interior) of A in Y is denoted by τ_n-cl$^Y(A)$ (resp. τ_n-int$^Y(A)$) and the (m,n)-semi-closure (resp. (m,n)-semi-interior) of A in Y is denoted by $\tau_{(m,n)}$-cl$_s^Y(A)$ (resp. $\tau_{(m,n)}$-int$_s^Y(A)$). $P(X)$ denotes the power set of X and $\mathbb{N}$ denotes the set of natural numbers. For more literature of selection principles and undefined notions, one can see [9, 11, 17, 22, 26, 28].

The layout of this paper is as follows: Section 2 contains basic notions and terminology used in this paper. All the preliminary results required for this manuscript are also covered in Section 2. Section 3 introduces the $(a)R$-star-semi-Lindelöf and $(a)M$-star-semi-Lindelöf (a)topological spaces. It is shown that every $(a)R^s$-separable space is $(a)R$-star-semi-Lindelöf and every $(a)^s$-semi-Lindelöf space is $(a)R$-star-semi-Lindelöf but not every $(a)R$-star-semi-Lindelöf space is $(a)R^s$-separable or $(a)^s$-semi-Lindelöf. It is observed that the properties $(a)R$-star-semi-Lindelöf and $(a)M$-star-semi-Lindelöf are not hereditary. Further, it is shown that an (a)subspace of $(a)R$-star-semi-Lindelöf (resp. $(a)M$-star-semi-Lindelöf) space is $(a)R$-star-semi-Lindelöf (resp. $(a)M$-star-semi-Lindelöf) if it is (a)-open and (a)-semi-closed. The properties $(a)R$-star-semi-Lindelöf and $(a)M$-star-semi-Lindelöf are preserved under those maps which are $(a)^s$-irresolute and $(a)^s$-semi-open. Further, it is shown that if an (a)space X is the union of countably many (a)-open and $(a)R$-star-semi-Lindelöf subspaces, then X is $(a)R$-star-semi-Lindelöf. Similar results are obtained in the case of $(a)M$-star-semi-Lindelöf. In the last section, relationships among other properties have been investigated and the properties of $(a)R$-k-star-semi-Lindelöf and $(a)M$-k-star-semi-Lindelöf spaces have been discussed.

2. Preliminaries

In this section, we collect some results which are necessary for this paper.

Definition 2.1. A subset A of $(X, \{\tau_n\})$ is said to be:

(1) [35] (m,n)-semi-open $\big((m,n)$-s-open$\big)$ if $A \subseteq \tau_m\text{-cl}\big(\tau_n\text{-int}(A)\big)$, or equivalently, if there exists a τ_n-open set U such that $U \subseteq A \subseteq \tau_m\text{-cl}(U)$.

(2) [35] (a)-semi-open if A is (m,n)-semi-open for all $m \neq n$.

(3) [24] (a)-open if A is τ_n-open for all $n \in \mathbb{N}$.

The collection of all (a)-open (resp. (a)-semi-open, (m,n)-semi-open) subsets of X is denoted by $O(X)$ (resp. $SO(X), SO(m,n)X$). The complement of (a)-open (resp. (a)-semi-open, (m,n)-semi-open) set is called (a)-closed (resp. (a)-semi-closed, (m,n)-semi-closed).

Proposition 2.2. *Let $(X,\{\tau_n\})$ be an (a)space. If $A \in SO(m,n)X$ and $G \in \tau_m \cap \tau_n$, then $A \cap G \in SO(m,n)X$.*

Proof. Let $A \in SO(m,n)X$ and $G \in \tau_m \cap \tau_n$. By Definition 2.1, $A \subseteq \tau_m$-cl$\big(\tau_n$-int$(A)\big)$. It follows that $A \cap G \subseteq \tau_m$-cl$\big(\tau_n$-int$(A)\big) \cap G$. Since G is τ_m-open, τ_m-cl$\big(\tau_n$-int$(A)\big) \cap G \subseteq \tau_m$-cl$(\tau_n$-int$(A) \cap G) = \tau_m$-cl$\big(\tau_n$-int$(A) \cap \tau_n$-int$(G)\big) = \tau_m$-cl$\big(\tau_n$-int$(A \cap G)\big)$. Therefore, $A \cap G \subseteq \tau_m$-cl$\big(\tau_n$-int$(A \cap G)\big)$. Thus, $A \cap G$ is (m,n)-semi-open in X. □

Proposition 2.3. *Let $A \subseteq Y \subseteq X$. If A is (m,n)-semi-open in X, then A is (m,n)-semi-open in Y.*

Proof. Let $A \subseteq Y \subseteq X$ and A be (m,n)-semi-open in X. Therefore, $A \subseteq \tau_m$-cl$\big(\tau_n$-int$(A)\big)$. It follows that τ_m-cl$^Y\big(\tau_n$-int$^Y(A)\big) \supseteq \tau_m$-cl$^Y\big(\tau_n$-int$(A) \cap Y\big) = \tau_m$-cl$^Y\big(\tau_n$-int$(A)\big) = \tau_m$-cl$\big(\tau_n$-int$(A)\big) \cap Y \supseteq A \cap Y = A$. Thus, A is (m,n)-semi-open in Y. □

Definition 2.4 ([25])**.** A function $f\colon (X,\{\tau_n\}) \to (Y,\{\sigma_n\})$ is said to be:

(1) $(a)^s$-irresolute if the preimage of every (m,n)-semi-open set in Y is (m,n)-semi-open in X for all $m \neq n$.

(2) $(a)^s$-semi-closed (resp. $(a)^s$-semi-open) if image of each (m,n)-semi-closed (resp. (m,n)-semi-open) set in X is (m,n)-semi-closed (resp. (m,n)-semi-open) in Y for all $m \neq n$.

Definition 2.5. A function $f\colon (X,\{\tau_n\}) \to (Y,\{\sigma_n\})$ is (τ_n,σ_m)-continuous (resp. (τ_n,σ_m)-open) if $f\colon (X,\tau_n) \to (Y,\sigma_m)$ is continuous (resp. open).

Theorem 2.6. *If $f\colon (X,\{\tau_n\}) \to (Y,\{\sigma_n\})$ is a (τ_m,σ_m)-continuous and (τ_n,σ_n)-open map, then f preserves (m,n)-semi-open sets.*

Proof. Let $f\colon (X,\{\tau_n\}) \to (Y,\{\sigma_n\})$ be a (τ_m,σ_m)-continuous and (τ_n,σ_n)-open map and G be an (m,n)-semi-open set in X. Then, $G \subseteq \tau_m$-cl$\big(\tau_n$-int$(G)\big)$. It follows that $f(G) \subseteq f\big(\tau_m$-cl$(\tau_n$-int$(G))\big)$. Since f is (τ_m,σ_m)-continuous, $f\big(\tau_m$-cl$(\tau_n$-int$(G))\big) \subseteq \sigma_m$-cl$\big[f\big(\tau_n$-int$(G)\big)\big]$. By (τ_n,σ_n)-openness of f, $f\big(\tau_n$-int$(G)\big) \subseteq \sigma_n$-int$\big(f(G)\big)$ and therefore, $f(G) \subseteq \sigma_m$-cl$\big(\sigma_n$-int$(f(G))\big)$. Thus, $f(G)$ is (m,n)-semi-open in Y. □

Proposition 2.7. *Let $(X, \{\tau_n\})$ be an (a)space and $Y \in \tau_m \cap \tau_n$. A subset A of Y is (m, n)-semi-open in Y if and only if $A = G \cap Y$, where G is (m, n)-semi-open in X.*

Proof. Let $Y \in \tau_m \cap \tau_n$ and G be an (m, n)-semi-open set in X. By Proposition 2.2, $G \cap Y$ is (m, n)-semi-open in X. But $G \cap Y \subseteq Y \subseteq X$, so, by Proposition 2.3, $G \cap Y$ is (m, n)-semi-open in Y. Conversely, let A be an (m, n)-semi-open subset of Y. It suffices to show that A is (m, n)-semi-open in X. Let $f\colon (Y, \{\tau_{nY}\}) \to (X, \{\tau_n\})$ be the inclusion map. It is obvious that f is (τ_{mY}, τ_m)-continuous. Also, f is (τ_{nY}, τ_n)-open as Y is τ_n-open. Therefore, from Theorem 2.6, $f(A) = A$ is (m, n)-semi-open in X. □

Theorem 2.8. *Let $(X, \{\tau_n\})$ be an (a)space and $Y \in \tau_m \cap \tau_n$. A subset A of Y is (m, n)-semi-open in Y if and only if A is (m, n)-semi-open in X.*

Proof. The proof follows from Proposition 2.3 and Proposition 2.7. □

Proposition 2.9 ([24]). *For any subset A of an (a)space $(X, \{\tau_n\})$, the (m, n)-semi closure of A, denoted by $\tau_{(m,n)}$-$cl_s(A)$, is $A \cup \tau_m$-$int\big(\tau_n$-$cl(A)\big)$.*

Definition 2.10 ([24]). In an (a)space $(X, \{\tau_n\})$, a set $A \subseteq X$ is said to be:

(1) τ_n-dense in X if A is dense in (X, τ_n).

(2) s-(m, n)-dense if $\tau_{(m,n)}$-$\mathrm{cl}_s(A) = X$.

(3) dense in X if A is dense in (X, τ_n) for all $n \in \mathbb{N}$.

$D^s(m, n)$ denotes the collection of all s-(m, n)-dense subsets of $(X, \{\tau_n\})$.

Definition 2.11 ([24]). An (a)space $(X, \{\tau_n\})$ is said to be:

(1) separable if there exists a countable subset of X which is dense in X.

(2) $(a)R^s$-separable if, for all $m \neq n$, for each sequence $< D_n \colon n \in \mathbb{N} >$ of s-(m, n)-dense subsets of X there exist points $x_n \in D_n$ (for each $n \in \mathbb{N}$) such that $\{x_n \colon n \in \mathbb{N}\}$ is s-(m, n)-dense in X.

(3) $(a)M^s$-separable if, for all $m \neq n$, for each sequence $< D_n \colon n \in \mathbb{N} >$ of s-(m, n)-dense subsets of X there exist finite sets $F_n \subseteq D_n$ (for each $n \in \mathbb{N}$) such that $\cup_{n\in\mathbb{N}} F_n$ is s-(m, n)-dense in X.

Recall that a family $\mathcal{B}$ of non-empty open subsets of a topological space (X, τ) is said to be π-base of X if for each non-empty open subset, say $G \subseteq X$, there exists a $U \in \mathcal{B}$ such that $U \subseteq G$.

Theorem 2.12 ([24]). *If (X, τ_n) has a countable π-base for all $n \in \mathbb{N}$, then $(X, \{\tau_n\})$ is $(a)R^s$-separable.*

Definition 2.13 ([24]). Let $(X, \{\tau_n\})$ be an (a)space. Then:

(1) X has countable (m, n)-s-fan tightness $(m \neq n)$ if for each $x \in X$ and each sequence $< A_k : k \in \mathbb{N} >$ of subsets of X such that $x \in \tau_{(m,n)}\text{-cl}_s(A_k)$ for each k, there are finite sets $F_k \subseteq A_k$ such that $x \in \tau_{(m,n)}\text{-cl}_s(\cup_{k\in\mathbb{N}} F_k)$.

(2) X has countable (m, n)-s-strong fan tightness $(m \neq n)$ if for each $x \in X$ and each sequence $< A_k : k \in \mathbb{N} >$ of subsets of X such that $x \in \tau_{(m,n)}\text{-cl}_s(A_k)$ for each k, there are points $x_k \in A_k$ such that $x \in \tau_{(m,n)}\text{-cl}_s(\{x_k : k \in \mathbb{N}\})$.

Theorem 2.14 ([24]). *Let X be a separable (a)space. Then, the following holds:*

(1) *If X has countable (m, n)-s-strong fan tightness for all $m \neq n$, then X is $(a)R^s$-separable.*

(2) *If X has countable (m, n)-s-fan tightness for all $m \neq n$, then X is $(a)M^s$-separable.*

Recall that a cover $\mathcal{G}$ of an (a)space X is said to be (m, n)-semi-cover [25] if each $G \in \mathcal{G}$ is (m, n)-semi-open subset of X. The set of all (m, n)-semi-covers of X is denoted by $\mathcal{S}(m, n)X$. If there is no scope of confusion, we will write only $\mathcal{S}(m, n)$ in place of $\mathcal{S}(m, n)X$.

Definition 2.15 ([25]). An (a)space $(X, \{\tau_n\})$ is said to be $(a)^s$-semi-Menger if, for all $m \neq n$, for every sequence $< \mathcal{U}_k : k \in \mathbb{N} >$ of (m, n)-semi-covers of X there exists a finite set $\mathcal{V}_k \subseteq \mathcal{U}_k$ (for each $k \in \mathbb{N}$) such that $\bigcup\big(\bigcup_{k\in\mathbb{N}} \mathcal{V}_k\big) = X$.

Definition 2.16. An (a)space $(X, \{\tau_n\})$ is said to be $(a)^s$-semi-Rothberger if, for all $m \neq n$, for every sequence $< \mathcal{U}_k : k \in \mathbb{N} >$ of (m, n)-semi-covers of X there exist $U_k \in \mathcal{U}_k$ (for each $k \in \mathbb{N}$) such that $\bigcup\big(\bigcup_{k\in\mathbb{N}} U_k\big) = X$.

Following facts are trivial and readily follows from the definition of star operator.

Proposition 2.17. *For families $\mathcal{U}$ and $\mathcal{V}$ of subsets of an infinite set X and for subsets A and B of X, the following facts hold:*

(1) $\mathcal{S}t(A,\mathcal{U}) \subseteq \mathcal{S}t(A, \mathcal{V})$ *if* $\mathcal{U} \subseteq \mathcal{V}$;

(2) $\mathcal{S}t(A,\mathcal{U}) \subseteq \mathcal{S}t(B,\mathcal{U})$ *if* $A \subseteq B$;

(3) $\mathcal{S}t(A,\mathcal{U}) \cup \mathcal{S}t(B, \mathcal{V}) \subseteq \mathcal{S}t(A \cup B,\mathcal{U} \cup \mathcal{V})$; *equality holds if* $\mathcal{S}t(A, \mathcal{V}) = \emptyset$ *and* $\mathcal{S}t(B,\mathcal{U}) = \emptyset$.

3. Selectively star-semi-Lindelöf (a)spaces

In this section, we discuss the notion of $(a)R$-star-semi-Lindelöf spaces and $(a)M$-star-semi-Lindelöf spaces. We study their relationships with $(a)R^s$-separable spaces, $(a)M^s$-separable spaces and $(a)^s$-semi-Lindelöf spaces. Further, we investigate these spaces in detail.

Definition 3.1. An (a)space $(X, \{\tau_n\})$ is said to be:

(1) $(a)R$-star-semi-Lindelöf if X satisfies $R_{\mathcal{S}(m,n)}\big(D^s(m,n), \mathcal{S}t[\mathcal{S}(m,n)]\big)$ for all $m \neq n$.

(2) $(a)M$-star-semi-Lindelöf if X satisfies $M_{\mathcal{S}(m,n)}\big(D^s(m,n), \mathcal{S}t[\mathcal{S}(m,n)]\big)$ for all $m \neq n$.

From Definition 3.1, it is clear that every $(a)R$-star-semi-Lindelöf space is $(a)M$-star-semi-Lindelöf.

Theorem 3.2. *Every $(a)^s$-semi-Lindelöf space is $(a)R$-star-semi-Lindelöf.*

Proof. Let $(X, \{\tau_n\})$ be an $(a)^s$-semi-Lindelöf space. For any $m \neq n$, let $\{D_i \colon i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of X and $\mathcal{U}$ be an (m,n)-semi-cover of X. Since X is $(a)^s$-semi-Lindelöf, there exists a countable subset $\mathcal{V} = \{U_k \colon k \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\bigcup\{U_k \colon k \in \mathbb{N}\} = X$. Let $x_i \in D_i \cap U_i$ for each $i \in \mathbb{N}$. Now, $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{x_k\}, \mathcal{U}) \supseteq \mathcal{S}t(\cup_{k\in\mathbb{N}}\{x_k\}, \mathcal{V}) = \bigcup\{U \in \mathcal{V} \colon U \cap (\cup_{k\in\mathbb{N}}\{x_k\}) \neq \emptyset\} = \bigcup\{U_k \colon k \in \mathbb{N}\} = X$. Hence, X is $(a)R$-star-semi-Lindelöf. $\square$

The converse of Theorem 3.2 need not be true.

Example 1. Let $X = \mathbb{R}$ and F be an uncountable subset of X such that $X - F$ is uncountable. Let τ_m be the cocountable topology if m is odd and $\tau_m = \{\emptyset, F, X\}$ if m is even. It is observed that the collection $\{F \cup \{q\} \colon q \in X - F\}$ is a $(1,2)$-semi-cover of X which is not reducible to any countable subcover. Therefore, X is not $(a)^s$-semi-Lindelöf. Claim that X is $(a)R$-star-semi-Lindelöf. Let $\{D_i \colon i \in \mathbb{N}\}$ be a sequence of s-$(1,2)$-dense (resp. s-$(2,2)$-dense) subsets of X and $\mathcal{U}$ be a $(1,2)$-semi-cover (resp. $(2,2)$-semi-cover) of X. It is observed that $SO(1,2)X = SO(2,2)X = \{A \subseteq X \colon A \supseteq F\}$. Also, every s-$(1,2)$-dense (resp. s-$(2,2)$-dense) subset of X contains a member of F. Let $x_i \in D_i \cap F$ for each $i \in \mathbb{N}$. It is clear that $\mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U}) = X$. Indeed, each $x \in X$ belongs to some $U \in \mathcal{U}$. Since $F \subseteq U$, $\{x_i \colon i \in \mathbb{N}\} \cap U \neq \emptyset$. Therefore, $x \in \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U})$. Now, $SO(2,1)X = SO(1,1)X = \tau_1$. Let $\{D_i \colon i \in \mathbb{N}\}$ be a sequence of s-$(2,1)$-dense (resp. s-$(1,1)$-dense) subsets of X and $\mathcal{U}$ be a $(2,1)$-semi-cover (resp. $(1,1)$-semi-cover) of X. Since (X, τ_1) is Lindelöf, there exists a countable subset $\mathcal{V} = \{U_k \colon k \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\bigcup \mathcal{V} = X$. Let $x_i \in D_i \cap U_i$ for each $i \in \mathbb{N}$. Then, $\mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U}) = X$. Indeed, each $x \in X$ belongs to U_k for some $k \in \mathbb{N}$.

But $x_k \in U_k$, therefore $\{x_i \colon i \in \mathbb{N}\} \cap U_k \neq \emptyset$. Thus, $x \in \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{V})$. Hence, X is $(a)R$-star-semi-Lindelöf.

Theorem 3.3. *Every $(a)^s$-semi-Rothberger space is $(a)R$-star-semi-Lindelöf.*

Proof. The proof follows from Theorem 3.2 as every $(a)^s$-semi-Rothberger space is $(a)^s$-semi-Lindelöf. □

Theorem 3.4. *Every $(a)^s$-semi-Menger space is $(a)M$-star-semi-Lindelöf.*

Proof. The proof follows from Theorem 3.2 as every $(a)^s$-semi-Menger space is $(a)^s$-semi-Lindelöf and every $(a)R$-star-semi-Lindelöf space is $(a)M$-star-semi--Lindelöf. □

The following example shows that the converse of Theorem 3.3 and Theorem 3.4 is not true.

Example 2. Let $X = \mathbb{R}$ and $\tau_m = \{\emptyset\} \cup \{\mathbb{R}\} \cup \{(-\infty, x) \colon x \in \mathbb{R}\}$ if m is odd and $\tau_m = \{G \subseteq X \colon G = \emptyset \text{ or } 2 \in G\}$ if m is even. Since (X, τ_n) has a countable π-base for all $n \in \mathbb{N}$, therefore X is $(a)R$-star-semi-Lindelöf (see Corollary 3.7). But $\{(-\infty, 1) \cup \{x\} \colon x \notin (-\infty, 1)\}$ is a $(1,1)$-semi-cover of X which is not reducible to any countable subcover. Therefore, X is not $(a)^s$-semi-Lindelöf. Thus, X is neither $(a)^s$-semi-Rothberger nor $(a)^s$-semi-Menger.

Theorem 3.5. *Every $(a)R^s$-separable space is $(a)R$-star-semi-Lindelöf.*

Proof. Let $(X, \{\tau_n\})$ be an $(a)R^s$-separable space. For any $m \neq n$, let $\{D_i \colon i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of X and $\mathcal{U}$ be an (m,n)-semi--cover of X. Since X is $(a)R^s$-separable, there exist points $x_i \in D_i$ such that $\{x_i \colon i \in \mathbb{N}\}$ is s-(m,n)-dense in X. Now, $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{x_k\}, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \colon U \cap (\cup_{k\in\mathbb{N}}\{x_k\}) \neq \emptyset\} = \bigcup \mathcal{U} = X$. Hence, X is $(a)R$-star-semi-Lindelöf. □

The following theorem can be proved similarly.

Theorem 3.6. *Every $(a)M^s$-separable space is $(a)M$-star-semi-Lindelöf.*

Example 3. The (a)space considered in Example 1 is $(a)R$-star-semi-Lindelöf. But $SO(1,1)X = \tau_1$ and no countable set can be dense in (X, τ_1), therefore X is not $(a)M^s$-separable and hence, not $(a)R^s$-separable.

Corollary 3.7. *If (X, τ_n) has a countbale π-base for all $n \in \mathbb{N}$, then $(X, \{\tau_n\})$ is $(a)R$-star-semi-Lindelöf.*

Proof. The proof follows by Theorem 2.12 and Theorem 3.5. □

Example 4. The (a)space considered in Example 1 is $(a)R$-star-semi-Lindelöf. But (X, τ_1) do not have any countable π-base.

Corollary 3.8. *Let $(X, \{\tau_n\})$ be a separable (a)space. If X has countable (m, n)-s-strong fan tightness for all $m \neq n$, then X is $(a)R$-star-semi-Lindelöf.*

Proof. The proof follows by Theorem 2.14 and Theorem 3.5. □

Corollary 3.9. *Let $(X, \{\tau_n\})$ be a separable (a)space. If X has countable (m, n)-s-fan tightness for all $m \neq n$, then X is $(a)M$-star-semi-Lindelöf.*

Proof. The proof follows by Theorem 2.14 and Theorem 3.6. □

Theorem 3.10. *An (a)space $(X, \{\tau_n\})$ is $(a)M$-star-semi-Lindelöf if and only if, for all $m \neq n$, for every decreasing sequence $< D_i : i \in \mathbb{N} >$ of s-(m, n)-dense subsets of X and for every (m, n)-semi-cover $\mathcal{U}$ of X, there is a sequence $< F_i : i \in \mathbb{N} >$ of finite sets such that $F_i \subseteq D_i$ for all $i \in \mathbb{N}$ and $\mathcal{S}t(\cup_{i\in\mathbb{N}} F_i, \mathcal{U}) = X$.*

Proof. The necessity is obvious. Conversely, for any $m \neq n$, let $\{D_i : i \in \mathbb{N}\}$ be a sequence of s-(m, n)-dense subsets of X and $\mathcal{U}$ be an (m, n)-semi-cover of X. For each $j \in \mathbb{N}$, let $B_j = D_j \cup D_{j+1} \cup D_{j+2} \cup \cdots = \cup_{i\geq j} D_i$. It is clear that $B_j \supseteq B_{j+1}$ for all $j \in \mathbb{N}$ and $\tau_{(m,n)}\text{-cl}_s(B_j) \supseteq \tau_{(m,n)}\text{-cl}_s(D_j) = X$. Therefore, $\{B_j : j \in \mathbb{N}\}$ is a decreasing sequence of s-(m, n)-dense subsets of X. By given hypothesis, there exist finite sets $F_j \subseteq B_j$ such that $X = \mathcal{S}t(\cup_{j\in\mathbb{N}} F_j, \mathcal{U})$. Let $E_j = (\cup_{i\leq j} F_i) \cap D_j$ for each $j \in \mathbb{N}$. Then, $\cup_{i\in\mathbb{N}} F_i = \cup_{j\in\mathbb{N}} E_j$. Therefore, $X = \mathcal{S}t(\cup_{j\in\mathbb{N}} E_j, \mathcal{U})$. □

The properties $(a)R$-star-semi-Lindelöfness and $(a)M$-star-semi-Lindelöfness are not hereditary.

Example 5. Consider the topological space (X, τ^*) as in [33, Example 78]. Let $X = P \cup L$, where $P = \{(x, y) : x, y \in \mathbb{R}, y > 0\}$ is an open upper half plane with the euclidean topology τ and L is a real axis. Let τ^* be the topology on X by adding to τ all sets of the form $\{x\} \cup (P \cap U)$ where $x \in L$, and U is a euclidean neighborhood of x in the plane. Let $\tau_m = \tau^*$ if m is odd and τ_m be the indiscrete topology if m is even. Since (X, τ_m) has a countable π-base for all $m \in \mathbb{N}$, so $(X, \{\tau_m\})$ is $(a)R$-star-semi-Lindelöf and hence, $(a)M$-star-semi-Lindelöf. It is observed that the real axis L is an uncountable (a)subspace of $(X, \{\tau_m\})$ with subspace topology σ_m as the discrete topology when m is odd and indiscrete topology when m is even. For $(1, 1)$-semi-cover $\mathcal{U} = \{\{x\} : x \in L\}$ of L, there does not exist any finite set $F_i \subseteq D_i$ for any sequence $\{D_i : i \in \mathbb{N}\}$ of s-$(1, 1)$-dense subsets of X satisfying $\mathcal{S}t(\cup_{i\in\mathbb{N}} F_i, \mathcal{U}) = L$. Therefore, the (a)subspace $(L, \{\sigma_m\})$ is not $(a)M$-star-semi-Lindelöf and hence, not $(a)R$-star-semi-Lindelöf.

Theorem 3.11. *Let Y be an (a)-open and (a)-semi-closed subset of an (a)space $(X,\{\tau_n\})$. If X is $(a)R$-star-semi-Lindelöf, then Y is $(a)R$-star-semi-Lindelöf.*

Proof. Let $(X,\{\tau_n\})$ be an (a)space and Y be an (a)-open and (a)-semi-closed subset of X. For any $m \neq n$, let $\{B_i : i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of Y and $\mathcal{V}$ be an (m,n)-semi-cover of Y. For each $i \in \mathbb{N}$, let $D_i = B_i \cup (X-Y)$. Claim that each D_i is s-(m,n)-dense in X. Let $k \in \mathbb{N}$ be arbitrary but fixed. Since $\tau_{(m,n)}$-$\mathrm{cl}_s^Y(B_k) = Y$ and $\tau_{(m,n)}$-$\mathrm{cl}_s^Y(A) \subseteq \tau_n$-$\mathrm{cl}^Y(A)$ for all $A \subseteq Y$, so $Y = \tau_n$-$\mathrm{cl}^Y(B_k) \subseteq \tau_n$-$\mathrm{cl}(B_k)$. Also Y is (a)-open, therefore $Y \subseteq \tau_m$-$\mathrm{int}\big(\tau_n$--$\mathrm{cl}(B_k)\big)$. Thus, $Y \subseteq \tau_{(m,n)}$-$\mathrm{cl}_s(B_k)$ and hence, $X = Y \cup (X-Y) \subseteq \tau_{(m,n)}$--$\mathrm{cl}_s(B_k) \cup (X-Y) = \tau_{(m,n)}$-$\mathrm{cl}_s(B_k) \cup \tau_{(m,n)}$-$\mathrm{cl}_s(X-Y) = \tau_{(m,n)}$-$\mathrm{cl}_s(D_k)$. Thus, $\{D_i : i \in \mathbb{N}\}$ is a sequence of s-(m,n)-dense subsets of X. Also, each $V \in \mathcal{V}$ is (m,n)-semi-open in X as Y is (a)-open (see Theorem 2.8). Therefore, $\mathcal{U} = \mathcal{V} \cup \{(X-Y)\}$ is an (m,n)-semi-cover of X as Y is (a)-semi-closed in X. By given hypothesis, there exist $x_i \in D_i$ such that $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{x_k\},\mathcal{U}) = X$. This emphasizes that $\{x_i : i \in \mathbb{N}\} \cap B_i \neq \emptyset$ for some values of i. The set $\{x_k : k \in \mathbb{N}\}$ may intersect those B_i in finitely or infinitely many points. Let $z_i \in \{x_k : k \in \mathbb{N}\} \cap B_i$ whenever $\{x_k : k \in \mathbb{N}\} \cap B_i$ is non-empty. Define the sequence $< y_i : i \in \mathbb{N} >$ as follows: $y_i = z_i$ whenever $\{x_k : k \in \mathbb{N}\} \cap B_i$ is non-empty; $y_i = a$ otherwise, where a is one of $z_i's$. Clearly, $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{y_k\},\mathcal{V}) = Y$. Indeed, each $y \in Y$ belongs to U for some $U \in \mathcal{V}$ with $U \cap \{x_k : k \in \mathbb{N}\} \neq \emptyset$. Therefore, $y \in U$ for some $U \in \mathcal{V}$ with $U \cap \{y_k : k \in \mathbb{N}\} \neq \emptyset$. □

Similarly, we have the following result.

Theorem 3.12. *Let Y be an (a)-open and (a)-semi-closed subset of an (a)space $(X,\{\tau_n\})$. If X is $(a)M$-star-semi-Lindelöf, then Y is $(a)M$-star-semi-Lindelöf.*

Theorem 3.13. *Let $f\colon (X,\{\tau_n\}) \to (Y,\{\sigma_n\})$ be an $(a)^s$-semi-open and $(a)^s$-irresolute surjection. If X is $(a)R$-star-semi-Lindelöf, then Y is so.*

Proof. For any $m \neq n$, let $\{B_i : i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of Y and $\mathcal{V}$ be an (m,n)-semi-cover of Y. Since f is $(a)^s$-semi-open, $\{f^{-1}(B_i) : i \in \mathbb{N}\}$ is a sequence of s-(m,n)-dense subsets of X. Put $f^{-1}(B_i) = D_i$ for each $i \in \mathbb{N}$. Since f is $(a)^s$-irresolute, $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$ is an (m,n)-semi-cover of X. By given hypothesis, there exist points $x_i \in D_i$ such that $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{x_k\},\mathcal{U}) = X$. So, $\bigcup\{U \in \mathcal{U} : U \cap (\cup_{k\in\mathbb{N}}\{x_k\}) \neq \emptyset\} = X$. For each $i \in \mathbb{N}$, let $y_i = f(x_i) \in B_i$. Claim that $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{y_k\},\mathcal{V}) = Y$. For $y \in Y$, there exists $x \in X$ such that $y = f(x)$. But $x \in U$ for some $U \in \mathcal{U}$ with $U \cap \{x_k : k \in \mathbb{N}\} \neq \emptyset$. Therefore, $y \in V$ for some $V \in \mathcal{V}$ such that $V \cap \{y_k : k \in \mathbb{N}\} \neq \emptyset$. □

The following theorem can be proved similarly.

Theorem 3.14. *Let $f\colon (X,\{\tau_n\}) \to (Y,\{\sigma_n\})$ be an $(a)^s$-semi-open and $(a)^s$-irresolute surjection. If X is $(a)M$-star-semi-Lindelöf, then Y is so.*

The condition of $(a)^s$-irresolute cannot be relaxed in Theorem 3.13 and Theorem 3.14. It can be seen in the following example:

Example 6. Let $X = \mathbb{R}$ and τ_m be the cocountable topology if m is odd and $\tau_m = \{G \subseteq X : G = \emptyset$ or $2 \in G\}$ if m is even. Let $Y = \mathbb{R}$ and σ_m be the discrete topology if m is odd and $\sigma_m = \{G \subseteq X : G = \emptyset$ or $2 \in G\}$ if m is even. Let $f : (X, \{\tau_m\}) \to (Y, \{\sigma_m\})$ be defined by $f(x) = x$ for all $x \in X$. It is clear that f is a surjection map. It is observed that $SO(1,1)Y = SO(2,1)Y = P(Y)$, $SO(1,2)X = SO(2,2)X = \tau_2$, $SO(2,1)X = SO(1,1)X = \tau_1$ and $SO(2,2)X = SO(2,2)Y = SO(1,2)X = SO(1,2)Y$. Therefore, f is $(a)^s$-semi-open but not $(a)^s$-irresolute. Claim that X is $(a)R$-star-semi-Lindelöf and Y is not $(a)M$-star-semi-Lindelöf. Let $\{D_i : i \in \mathbb{N}\}$ be a sequence of s-$(2,1)$-dense (resp. s-$(1,1)$-dense) subsets of X and $\mathcal{U}$ be a $(2,1)$-semi-cover (resp. $(1,1)$-semi-cover) of X. Since (X, τ_1) is Lindelöf, there exists a countable subset $\mathcal{V} = \{U_k : k \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\bigcup \mathcal{V} = X$. Let $x_i \in D_i \cap U_i$ for all $i \in \mathbb{N}$. Claim that $\mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U}) = X$. Let $x \in X$. Then, $x \in U_k$ for some $k \in \mathbb{N}$. But $x_k \in U_k$, therefore $\{x_i : i \in \mathbb{N}\} \cap U_k \neq \emptyset$ and thus, $x \in \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{V})$. Hence, $\mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U}) \supseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{V}) = X$. Now, let $\{D_i : i \in \mathbb{N}\}$ be a sequence of s-$(1,2)$-dense (resp. s-$(2,2)$-dense) subsets of X and $\mathcal{U}$ be a $(1,2)$-semi-cover (resp. $(2,2)$-semi-cover) of X. Since $SO(1,2)X = SO(2,2)X = \tau_2$, $2 \in D_i$ for all $i \in \mathbb{N}$ such that $\mathcal{S}t(\{2\}, \mathcal{U}) = X$. Thus, X is $(a)R$-star-semi-Lindelöf. Now, $\mathcal{G} = \{\{x\} : x \in Y\}$ is a $(1,1)$-semi-cover of Y. For any sequence $\{D_i : i \in \mathbb{N}\}$ of s-$(1,1)$-dense subsets of Y, no finite sets $F_i \subseteq D_i$ can be chosen so that $\mathcal{S}t(\cup_{i\in\mathbb{N}}F_i, \mathcal{G}) = Y$. Therefore, Y is not $(a)M$-star-semi-Lindelöf and hence, not $(a)R$-star-semi-Lindelöf.

Definition 3.15. A subset Y of an (a)space $(X, \{\tau_n\})$ is said to be:

(1) $(a)R$-star-semi-Lindelöf with respect to X if for every sequence $\{D_i : i \in \mathbb{N}\}$ of subsets of X with $Y \subseteq \tau_{(m,n)}\text{-cl}_s(D_i)$ for all $i \in \mathbb{N}$ and for each cover $\mathcal{U}$ of Y by (m,n)-semi-open subsets of X, there exist points $x_i \in D_i$ such that $Y \subseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U})$ for all $m \neq n$.

(2) $(a)M$-star-semi-Lindelöf with respect to X if for every sequence $\{D_i : i \in \mathbb{N}\}$ of subsets of X with $Y \subseteq \tau_{(m,n)}\text{-cl}_s(D_i)$ for all $i \in \mathbb{N}$ and for each cover $\mathcal{U}$ of Y by (m,n)-semi-open subsets of X, there exist finite sets $F_i \subseteq D_i$ such that $Y \subseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}F_i, \mathcal{U})$ for all $m \neq n$.

Theorem 3.16. *Let Y be an (a)-open subspace of (a)space $(X, \{\tau_n\})$. If Y is $(a)R$-star-semi-Lindelöf subspace of X, then Y is $(a)R$-star-semi-Lindelöf with respect to X.*

Proof. To prove this theorem, we will use the following two facts: (1). In topological space (X, τ), for $A \subseteq X$, $\tau\text{-cl}(A \cap Y) = \tau\text{-cl}[\tau\text{-cl}(A) \cap Y]$ whenever $Y \in \tau$ and (2). For all $m \neq n$, $\tau_{(m,n)}\text{-cl}_s(A) \subseteq \tau_n\text{-cl}(A)$. For any $m \neq n$, let $\mathcal{U}$ be a cover of Y by (m,n)-semi-open sets in X and $\{D_i : i \in \mathbb{N}\}$ be a sequence of subsets of X with $Y \subseteq \tau_{(m,n)}\text{-cl}_s(D_i)$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $B_i = D_i \cap Y$. Now, $\tau_{(m,n)}\text{-cl}_s^Y(B_i) = B_i \cup \tau_m\text{-int}^Y\big(\tau_n\text{-cl}^Y(B_i)\big) = B_i \cup \tau_m\text{-int}^Y[\tau_n\text{-cl}(B_i) \cap Y] = B_i \cup \tau_m\text{-int}^Y[\tau_n\text{-cl}(D_i \cap Y) \cap Y] = B_i \cup \tau_m\text{-int}^Y[\tau_n\text{-cl}[\tau_n\text{-cl}(D_i) \cap Y] \cap Y] = B_i \cup \tau_m\text{-int}^Y[\tau_n\text{-cl}(Y) \cap Y)] = B_i \cup \tau_m\text{-int}^Y(Y) = B_i \cup Y = Y$. Thus, $\{B_i : i \in \mathbb{N}\}$ is a sequence of s-(m,n)-dense subsets of Y. Let $\mathcal{G} = \{U \cap Y : U \in \mathcal{U}\}$. Since Y is (a)-open, each $U \cap Y$ is (m,n)-semi-open in Y whenever U is (m,n)-semi-open in X (see Proposition 2.7). Therefore, we get an (m,n)-semi-cover $\mathcal{G}$ of Y. By given hypothesis, Y is $(a)R$-star-semi-Lindelöf subspace of X, so there exist points $x_i \in B_i$ such that $Y = \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{G})$. It follows that $x_i \in D_i$ such that $Y \subseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U})$. □

Similarly, the following result holds.

Theorem 3.17. *Let Y be an (a)-open subspace of (a)space $(X, \{\tau_n\})$. If Y is $(a)M$-star-semi-Lindelöf subspace of X, then Y is $(a)M$-star-semi-Lindelöf with respect to X.*

Theorem 3.18. *Let $X = \cup_{n\in\mathbb{N}}X_n$, each X_n is (a)-open and $(a)R$-star-semi-Lindelöf subspace of X. Then, X is $(a)R$-star-semi-Lindelöf.*

Proof. For any $m \neq n$, let $\{D_i : i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of X and $\mathcal{U}$ be an (m,n)-semi-cover of X. Let $< N_k : k \in \mathbb{N} >$ be a sequence of infinite subsets of $\mathbb{N}$ such that $N_i \cap N_j = \emptyset$ for all $i \neq j$ and $\mathbb{N} = \cup_{l\in\mathbb{N}}N_l$. For each $l \in \mathbb{N}$, consider a sequence $< D_i \cap X_l : i \in N_l >$ of subsets of X. Now, $\tau_{(m,n)}\text{-cl}_s(D_i \cap X_l) = (D_i \cap X_l) \cup \tau_m\text{-int}\big(\tau_n\text{-cl}(D_i \cap X_l)\big) = (D_i \cap X_l) \cup \tau_m\text{-int}(\tau_n\text{-cl}[\tau_n\text{-cl}(D_i) \cap X_l]) = (D_i \cap X_l) \cup \tau_m\text{-int}[\tau_n\text{-cl}(X_l)] \supseteq (D_i \cap X_l) \cup \tau_m\text{-int}(X_l) = (D_i \cap X_l) \cup X_l = X_l$. Therefore, $\{D_i \cap X_l : i \in N_l\}$ is a sequence of subsets of X such that $X_l \subseteq \tau_{(m,n)}\text{-cl}_s(D_i \cap X_l)$. Since each X_i is (a)-open and $(a)R$-star-semi-Lindelöf subspace of X, so, by Theorem 3.16, there exist points $x_i \in D_i \cap X_l$ such that $X_l \subseteq \mathcal{S}t(\cup_{i\in N_l}\{x_i\}, \mathcal{U})$ for each $l \in \mathbb{N}$. Thus, $X = \cup_{l\in\mathbb{N}}X_l \subseteq \cup_{l\in\mathbb{N}}\mathcal{S}t(\cup_{i\in N_l}\{x_i\}, \mathcal{U}) \subseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\}, \mathcal{U})$. Hence, X is $(a)R$-star-semi-Lindelöf. □

Theorem 3.19. *Let $X = \cup_{n\in\mathbb{N}}X_n$, each X_n is (a)-open and $(a)M$-star-semi-Lindelöf subspace of X. Then, X is $(a)M$-star-semi-Lindelöf.*

Proof. It can be proved in a manner similar to Theorem 3.18. □

Theorem 3.20. *Let Y be an (a)-semi-open and (a)-semi-closed subset of an (a)space $(X, \{\tau_n\})$. If X is $(a)R$-star-semi-Lindelöf, then Y is $(a)R$-star-semi-Lindelöf with respect to X.*

Proof. Let $(X, \{\tau_n\})$ be an (a)space and Y be an (a)-semi-open and (a)-semi-closed subset of X. For any $m \neq n$, let $\mathcal{V}$ be a cover of Y by (m,n)-semi-open subsets of X and $\{B_i : i \in \mathbb{N}\}$ be a sequence of subsets of X with $Y \subseteq \tau_{(m,n)}$-$\mathrm{cl}_s(B_i)$. It is clear that $X = Y \cup (X - Y) \subseteq \tau_{(m,n)}$-$\mathrm{cl}_s(B_i) \cup (X - Y) = \tau_{(m,n)}$-$\mathrm{cl}_s[B_i \cup (X - Y)]$. Thus, $\{D_i : i \in \mathbb{N}\}$ is a sequence of s-(m,n)-dense subsets of X, where $D_i = B_i \cup (X - Y)$. Also, $\mathcal{U} = \mathcal{V} \cup \{(X - Y)\}$ is an (m,n)-semi-cover of X as Y is (a)-semi-closed in X. By given hypothesis, there exist $x_i \in D_i$ such that $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{x_k\}, \mathcal{U}) = X$. This emphasizes that $\{x_i : i \in \mathbb{N}\} \cap B_i \neq \emptyset$ for some values of i. Let $z_i \in \{x_k : k \in \mathbb{N}\} \cap B_i$ whenever $\{x_k : k \in \mathbb{N}\} \cap B_i$ is non-empty. Define sequence $< y_i : i \in \mathbb{N} >$ as follows: $y_i = z_i$ whenever $\{x_k : k \in \mathbb{N}\} \cap B_i$ is non-empty; $y_i = a$ otherwise, where a is one of z_i's. It is clear that $\mathcal{S}t(\cup_{k\in\mathbb{N}}\{y_k\}, \mathcal{V}) = Y$. Indeed, for $y \in Y$, $y \in U$ for some $U \in \mathcal{V}$ with $U \cap \{x_k : k \in \mathbb{N}\} \neq \emptyset$. Therefore, $y \in U$ for some $U \in \mathcal{V}$ with $U \cap \{y_k : k \in \mathbb{N}\} \neq \emptyset$. □

Problem 1. Find an $(a)M$-star-semi-Lindelöf space which is not $(a)R$-star-semi-Lindelöf.

4. Selectively k-star-semi-Lindelöf (a)spaces

This section is devoted to the study of $(a)R$-k-star-semi-Lindelöf spaces and $(a)M$-k-star-semi-Lindelöf spaces. We investigate their relationships with $(a)^s$-star-semi-Lindelöf spaces, $(a)^s$-absolutely-star-semi-Lindelöf spaces. Now, we begin this section with some definitions we will do with.

Definition 4.1. An (a)space X is said to be:

(1) $(a)R$-k-star-semi-Lindelöf if X satisfies $R_{\mathcal{S}(m,n)}\big(D^s(m,n), \mathcal{S}t^k[\mathcal{S}(m,n)]\big)$ for all $m \neq n$.

(2) $(a)M$-k-star-semi-Lindelöf if X satisfies $M_{\mathcal{S}(m,n)}\big(D^s(m,n), \mathcal{S}t^k[\mathcal{S}(m,n)]\big)$ for all $m \neq n$.

(3) $(a)^s$-star-semi-Lindelöf if, for all $m \neq n$, for each (m,n)-semi-cover $\mathcal{U}$ of X, there exists a countable set $F \subseteq X$ such that $\mathcal{S}t(F, \mathcal{U}) = X$.

(4) $(a)^s$-absolutely-star-semi-Lindelöf if, for all $m \neq n$, for each (m,n)-semi-cover $\mathcal{U}$ of X and each s-(m,n)-dense subset D of X, there exists a countable set $F \subseteq D$ such that $\mathcal{S}t(F, \mathcal{U}) = X$.

It is clear that every $(a)^s$-absolutely-star-semi-Lindelöf space is $(a)^s$-star-semi-Lindelöf and every $(a)R$-k-star-semi-Lindelöf space is $(a)R$-$(k+1)$-star-semi-Lindelöf.

Proposition 4.2. *Every* $(a)M$*-star-semi-Lindelöf space is* $(a)^s$*-absolutely-star--semi-Lindelöf.*

Proof. Let X be an $(a)M$-star-semi-Lindelöf space. For any $m \neq n$, let $\mathcal{U}$ be any (m,n)-semi-cover of X and D be any s-(m,n)-dense subset of X. For each $k \in \mathbb{N}$, let $D_k = D$. Since X is $(a)M$-star-semi-Lindelöf, there exist finite sets $F_i \subseteq D_i$ for each $i \in \mathbb{N}$ such that $\mathcal{S}t(\cup_{k\in\mathbb{N}} F_k, \mathcal{U}) = X$. The proof follows. □

Corollary 4.3. *Every* $(a)M$*-star-semi-Lindelöf space is* $(a)^s$*-star-semi--Lindelöf.*

Proposition 4.4. *Let* X *be an* (a)*space. If, for all* $m \neq n$*, for each* (m,n)*--semi-cover* $\mathcal{U}$ *of* X *and each* s*-*(m,n)*-dense subset* D *of* X*, there exists a finite set* $F \subseteq D$ *such that* $\mathcal{S}t(F,\mathcal{U}) = X$*, then* X *is* $(a)M$*-star-semi-Lindelöf.*

Proof. Let X be an (a)space. For any $m \neq n$, let $\{D_i : i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of X and $\mathcal{U}$ be an (m,n)-semi-cover of X. By given hypothesis, for each $i \in \mathbb{N}$ there exist finite sets $F_i \subseteq D_i$ such that $\mathcal{S}t(F_i,\mathcal{U}) = X$. Therefore, $X = \bigcup_i \mathcal{S}t(F_i,\mathcal{U}) \subseteq \mathcal{S}t(\bigcup_i F_i,\mathcal{U})$. □

Theorem 4.5. *Let* X *be an* $(a)^s$*-star-semi-Lindelöf space. If* X *has countable* (m,n)*-*s*-fan tightness for all* $m \neq n$*, then* X *is* $(a)M$*-star-semi-Lindelöf.*

Proof. For any $m \neq n$, let $\{D_i : i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of X and $\mathcal{U}$ be an (m,n)-semi-cover of X. Since X is $(a)^s$-star-semi-Lindelöf, there exists a countable set $F = \{x_i : i \in \mathbb{N}\} \subseteq X$ such that $\mathcal{S}t(F,\mathcal{U}) = X$. Let $< N_k : k \in \mathbb{N} >$ be a sequence of infinite subsets of $\mathbb{N}$ such that $N_i \cap N_j = \emptyset$ for all $i \neq j$ and $\mathbb{N} = \cup_{l\in\mathbb{N}} N_l$. For each $l \in \mathbb{N}$, $x_l \in \cap_{i\in N_l} \tau_{(m,n)}\text{-cl}_s(D_i)$ and X has countable (m,n)-s-fan tightness for all $m \neq n$, there exist finite sets $F_i \subseteq D_i$ for each $i \in N_l$ such that $x_l \in \tau_{(m,n)}\text{-cl}_s(\cup\{F_i : i \in N_l\})$. Hence, $F_i \subseteq D_i$ for each $i \in \mathbb{N}$ such that $\mathcal{S}t(\cup_{i\in\mathbb{N}} F_i,\mathcal{U}) = X$. Indeed, for any $x \in X$, there is a $U \in \mathcal{U}$ such that $x \in U$ and $x_k \in U$ for some $k \in \mathbb{N}$. But $x_k \in \tau_{(m,n)}\text{-cl}_s(\cup\{F_i : i \in N_k\})$, therefore $U \cap \cup\{F_i : i \in N_k\} \neq \emptyset$. Thus, $U \cap \cup\{F_i : i \in \mathbb{N}\} \neq \emptyset$ and hence, $x \in \mathcal{S}t(\bigcup_i F_i,\mathcal{U})$. □

Theorem 4.6. *If* X *is an* $(a)^s$*-star-semi-Lindelöf space and* X *has countable* (m,n)*-*s*-strong fan tightness for all* $m \neq n$*, then* X *is* $(a)R$*-star-semi-Lindelöf.*

Theorem 4.7. *Let* X *be an* $(a)^s$*-star-semi-Lindelöf space. If for every* (m,n)*--semi-cover* $\mathcal{U}$ *of* X*, for each* $x \in X$ *and for any sequence* $< A_i : n \in \mathbb{N} >$ *of subsets of* X *satisfying* $x \in \cap_{i\in\mathbb{N}} \tau_{(m,n)}\text{-}cl_s(A_i)$ *for all* $m \neq n$*, there exist points* $x_i \in A_i$ *such that* $x \in \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\},\mathcal{U})$*, then* X *is* $(a)R$*-2-star-semi-Lindelöf.*

Proof. Let X be an $(a)^s$-star-semi-Lindelöf space. For any $m \neq n$, let $\{D_i \colon i \in \mathbb{N}\}$ be a sequence of s-(m,n)-dense subsets of X and $\mathcal{U}$ be an (m,n)--semi-cover of X. Since X is $(a)^s$-star-semi-Lindelöf, there exists a countable set $F = \{x_i \colon i \in \mathbb{N}\} \subseteq X$ such that $\mathcal{S}t(F,\mathcal{U}) = X$. Let $< N_k \colon k \in \mathbb{N} >$ be a sequence of infinite subsets of $\mathbb{N}$ such that $N_i \cap N_j = \emptyset$ for all $i \neq j$ and $\mathbb{N} = \cup_{l\in\mathbb{N}} N_l$. For each $l \in \mathbb{N}$, $x_l \in \cap_{i\in N_l} \tau_{(m,n)}\text{-cl}_s(D_i)$. By given hypothesis, there exist points $y_i \in D_i$ for each $i \in N_l$ such that $x_l \in \mathcal{S}t(\cup_{i\in N_l}\{y_i\},\mathcal{U})$. Claim that $\mathcal{S}t^2(\cup_{i\in\mathbb{N}}\{y_i\},\mathcal{U}) = X$. Let $x \in X$. Since $X = \mathcal{S}t(F,\mathcal{U})$, there exists some $U \in \mathcal{U}$ such that $x \in U$ and $x_n \in U$ for some $n \in \mathbb{N}$. Also, $x_n \in \mathcal{S}t(\cup_{i\in N_n}\{y_i\},\mathcal{U})$, so there exists $V \in \mathcal{U}$ such that $x_n \in V$ and $y_k \in V$ for some $k \in N_n$. Thus, $V \cap U \neq \emptyset$ and $V \subseteq \mathcal{S}t(\cup_{i\in N_n}\{y_i\},\mathcal{U}) \subseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}\{y_i\},\mathcal{U})$. Therefore, $\mathcal{S}t(\cup_{i\in\mathbb{N}}\{y_i\},\mathcal{U}) \cap U \neq \emptyset$ and hence, $x \in \mathcal{S}t^2(\cup_{i\in\mathbb{N}}\{y_i\},\mathcal{U})$. □

The following theorem can be proved similarly.

Theorem 4.8. *Let X be an $(a)^s$-star-semi-Lindelöf space. If for every (m,n)--semi-cover $\mathcal{U}$ of X, for each $x \in X$ and for any sequence $< A_i \colon n \in \mathbb{N} >$ of subsets of X satisfying $x \in \cap_{i\in\mathbb{N}} \tau_{(m,n)}$-$cl_s(A_i)$ for all $m \neq n$, there exist finite sets $F_i \subseteq A_i$ such that $x \in \mathcal{S}t(\cup_{i\in\mathbb{N}} F_i,\mathcal{U})$, then X is $(a)M$-2-star-semi-Lindelöf.*

Definition 4.9. A subset Y of an (a)space $(X,\{\tau_n\})$ is said to be:

(1) $(a)R$-k-star-semi-Lindelöf with respect to X if for every sequence $\{D_i \colon i \in \mathbb{N}\}$ of subsets of X with $Y \subseteq \tau_{(m,n)}\text{-cl}_s(D_i)$ for all $i \in \mathbb{N}$ and for each cover $\mathcal{U}$ of Y by (m,n)-semi-open subsets of X, there exist points $x_i \in D_i$ such that $Y \subseteq \mathcal{S}t^k(\cup_{i\in\mathbb{N}}\{x_i\},\mathcal{U})$ for all $m \neq n$.

(2) $(a)M$-k-star-semi-Lindelöf with respect to X if for every sequence $\{D_i \colon i \in \mathbb{N}\}$ of subsets of X with $Y \subseteq \tau_{(m,n)}\text{-cl}_s(D_i)$ for all $i \in \mathbb{N}$ and for each cover $\mathcal{U}$ of Y by (m,n)-semi-open subsets of X, there exist finite sets $F_i \subseteq D_i$ such that $Y \subseteq \mathcal{S}t^k(\cup_{i\in\mathbb{N}} F_i,\mathcal{U})$ for all $m \neq n$.

Theorem 4.10. *Let X be an (a)space and $A \subseteq B \subseteq \tau_{(m,n)}$-$cl_s(A) \subseteq X$ for all $m \neq n$. If A is $(a)R$-star-semi-Lindelöf with respect to X, then B is $(a)R$-2--star-semi-Lindelöf with respect to X.*

Proof. For any $m \neq n$, let $A \subseteq B \subseteq \tau_{(m,n)}\text{-cl}_s(A) \subseteq X$ and $\mathcal{U}$ be a cover of B by (m,n)-semi-open subsets of X and $\{D_i \colon i \in \mathbb{N}\}$ be a sequence of subsets of X such that $B \subseteq \tau_{(m,n)}\text{-cl}_s(D_i)$ for each $i \in \mathbb{N}$. It is obvious that $A \subseteq \tau_{(m,n)}\text{-cl}_s(D_i)$ for each $i \in \mathbb{N}$ and $\mathcal{U}$ is a cover of A by (m,n)-semi-open subsets of X. Since A is $(a)R$-star-semi-Lindelöf with respect to X, there exist points $x_i \in D_i$ such that $A \subseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\},\mathcal{U})$. Claim that $B \subseteq \mathcal{S}t^2(\cup_{i\in\mathbb{N}}\{x_i\},\mathcal{U})$. Let $x \in B \subseteq \tau_{(m,n)}$--$\text{cl}_s(A)$. For each $U \in \mathcal{U}$ with $x \in U$, $U \cap A \neq \emptyset$. Let V be one such element of $\mathcal{U}$. Let $y \in V \cap A$. Since $y \in A \subseteq \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\},\mathcal{U})$, there exists $U \in \mathcal{U}$ with $y \in U$ and $x_k \in U$ for some $k \in \mathbb{N}$. Thus, $V \cap U \neq \emptyset$ and therefore, $V \cap \mathcal{S}t(\cup_{i\in\mathbb{N}}\{x_i\},\mathcal{U}) \neq \emptyset$. Hence, $x \in \mathcal{S}t^2(\cup_{i\in\mathbb{N}}\{x_i\},\mathcal{U})$. □

Similarly, the following result holds.

Theorem 4.11. *Let X be an (a)space and $A \subseteq B \subseteq \tau_{(m,n)}\text{-}cl_s(A) \subseteq X$ for all $m \neq n$. If A is $(a)M$-star-semi-Lindelöf with respect to X, then B is $(a)M$-2-star-semi-Lindelöf with respect to X.*

Problem 2. Find an $(a)R$-$(k+1)$-star-semi-Lindelöf space which is not $(a)R$-k-star-semi-Lindelöf.

5. Conclusion

This study is devoted to investigating the concept of selectively star-semi-Lindelöfness in (a)topological spaces. We used the notion of selection principles and star operator to study this concept. We discussed under what conditions the notion of selectively-star-semi-Lindelöfness are preserved. Also, a few open problems are posed to create research interest in this field. It is observed that the concept of selectively star-semi-Lindelöfness can be adapted in bitopological spaces also and almost all results of this paper can be done in a similar manner by using suitable notations.

6. Future Direction

It would be interesting to study more about the concept of selectively-k-star-semi-Lindelöf (a)topological spaces, $k \geq 2$. Selection principles in the soft topological spaces are being explored by Kočinac, Al-Shami and Çetkin in [23, 32]. It will be very useful if these concepts would be investigated in the soft topological spaces on the path of [23, 32].

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