sciendo

Tatra Mountains
*Mathematical Publications*
DOI: 10.2478/tmmp-2022-0006
Tatra Mt. Math. Publ. **81** (2022), 89–92

# REMARK ON A THEOREM OF TONELLI

Władysław Wilczyński

Department of Real Functions Faculty of Mathematics and Computer Science University, Łódź, POLAND

ABSTRACT. It is well known that if the surface $f : [-1,1] \times [-1,1] \to \mathbb{R}$ has a finite area, then the total variations of both sections $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$ of $f$ are finite almost everywhere in $[-1,1]$. In the note it is proved that these variations can be infinite on residual subsets of $[-1,1]$.

A continuous function $f : [-1,1] \times [-1,1] \to \mathbb{R}$ is of bounded variation (in the Tonelli sense) if the integrals $\int_{-1}^{1} V(x)dx$ and $\int_{-1}^{1} W(x)dx$ are both finite, where $V(x)$ for $x \in [-1,1]$ is the total variation of $f$ on $[-1,1]$ treated as a function of $y$ and $W$ for $y \in [-1,1]$ is the total variation of $f$ on $[-1,1]$ treated as a function of $x$. $V$ and $W$ are lower semicontinuous functions of $x$ and $y$, respectively, since $f$ is continuous. Theorem of Tonelli [1] or [2] (see also [3, p. 181]) says that in order that a continuous surface

$$z = f(x,y), \quad (x,y) \in [-1,1] \times [-1,1]$$

have a finite area it is necessary and sufficient that the function $f$ be of bounded variation on

$$[-1,1] \times [-1,1].$$

From this theorem it follows immediately that both sets

$$\{x \in [-1,1] : V(x) = +\infty\} \quad \text{and} \quad \{y \in [-1,1] : W(y) = +\infty\}$$

are of Lebesgue measure zero.

The aim of this note is to show that the kind of the category analogue of this theorem does not hold. In the sequel, $K((x,y),r)$ will denote the circle on the plane with center $(x,y)$ and radius $r$.

© 2022 Mathematical Institute, Slovak Academy of Sciences.
2020 Mathematics Subject Classification: 28A75, 26B15, 26A45.
Keywords: surface area, bounded variation, Baire category.
Licensed under the Creative Commons BY-NC-ND 4.0 International Public License.

**Theorem 1.** *There exists a continuous function* $f : [-1,1]\times[-1,1] \to \mathbb{R}$ *such that the area of the surface* $f$ *is finite and the sets* $\{x \in [-1,1] : V(x) = +\infty\}$ *and* $\{y \in [-1,1] : W(y) = +\infty\}$ *are residual in* $(-1,1)$.

Proof. Let $\varphi(x) = \operatorname{dist}(x, N \cup \{0\})$ for $x \in [0,+\infty)$, where $N$ is the set of positive integers.

Put

$$\varphi_n(x) = \begin{cases} \frac{1}{2n(n+1)} \cdot \varphi\Big(\big(2n(n+1)\big)^3 \cdot x\Big) & \text{for} \quad x \in \big[0, \frac{1}{(2n(n+1))^2}\big], \\ 0 & \text{for} \quad x > \frac{1}{(2n(n+1))^2} \end{cases}$$

for each $n \in N$. Then $\bigvee_0^{\frac{1}{(2n(n+1))^2}} \varphi_n = 1$, where $\bigvee_a^b \varphi$ means the total variation of $\varphi$ on the interval $[a,b]$ and $0 \le \varphi_n(x) \le \frac{1}{4n(n+1)}$ for each $x \in [0,+\infty)$.

Put

$$\Psi_n(x,y) = \begin{cases} \varphi_n(\sqrt{x^2+y^2}) & \text{for} \quad (x,y) \in K\Big((0,0), \frac{1}{(2n(n+1))^2}\Big), \\ 0 & \text{for remaining} \quad (x,y) \text{ in the plane } \mathbb{R}^2. \end{cases}$$

For each $n \in N$ and $k \in \{0,1,\dots,n-1\}$ let

$$x_{n,k} = \left(1 - \frac{1}{n}\right)\cos\frac{2k\pi}{n}, \qquad y_{n,k} = \left(1 - \frac{1}{n}\right)\sin\frac{2k\pi}{n}.$$

Put

$$f_{n,k}(x,y) = \Psi_n\big(x - x_{n,k}, y - y_{n,k}\big)$$

for $n \in N$, $k \in \{0,1,\dots,n-1\}$ and $(x,y) \in [-1,1]\times[-1,1]$. Observe that for each $n \in N$ and $k \in \{0,1,\dots,n-1\}$ the set

$$\{(x,y) : f_{n,k}(x,y) \neq 0\}$$

is included in the ball $K\big((x_{n,k}, y_{n,k}), \frac{1}{(2n(n+1))^2}\big)$ and the family

$$\left\{K(x_{n,k}, y_{n,k}), \frac{1}{(2n(n+1))^2}\right\} : n \in N,\ k \in \{0,1,\dots,n-1\}$$

consists of disjoint balls.

At last let

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} f_{n,k}(x,y) \quad \text{for } (x,y) \in [-1,1]\times[-1,1].$$

It is not difficult to see that $f$ is continuous on $[-1,1]\times[-1,1]$. We shall show that the area of the graph of $f$ is finite.

For fixed $x \in (-1,1)$ let $V_n(x)$ be a total variation of $\Psi_n$ treated as a function of $y$ on the interval $[-\sqrt{1-x^2}, \sqrt{1-x^2}]$ (or on the interval $[-1,1]$). Then

$$V_n(x) = 0 \quad \text{for } \; x \in (-1,1) \setminus \left[-\frac{1}{(2n(n+1))^2}, \frac{1}{(2n(n+1))^2}\right],$$

$$V_n(0) = 2 \quad \text{and} \quad V_n(x) \le 2 \quad \text{for } \; x \in \left[-\frac{1}{(2n(n+1))^2}, \frac{1}{(2n(n+1))^2}\right].$$

Hence

$$\int_{-1}^{1} V_n(x)dx \le 2 \cdot \frac{2}{(2n(n+1))^2} = \frac{1}{(n(n+1))^2}.$$

Observe that

$$\int_{-1}^{1} V_{n,k}(x)dx = \int_{-1}^{1} V_n(x)dx \quad \text{for each} \quad k \in \{0,1,\dots,n-1\},$$

where $V_{n,k}(x)$ obviously is a total variation of $f_{n,k}$ treated a function of $y$ on the interval $[-1,1]$. If $V(x)$ is a total variation of $f$ treated as a function of $y$ on the interval $[-1,1]$, then we have

$$\int_{-1}^{1} V(x)dx = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \int_{-1}^{1} V_{n,k}(x)dx = \sum_{n=1}^{\infty} n \cdot \int_{-1}^{1} V_n(x)dx \le \sum_{n=1}^{\infty} n \cdot \frac{1}{(n(n+1))^2} < +\infty.$$

Similarly, $\int_{-1}^{1} W(y)dy < \infty$, where $W(y)$ is a total variation of $f$ treated as a function of $x$ on the interval $[-1,1]$. From the theorem of L. Tonelli (see [1], [2] or [3, Chapter V.8]), the area of $f$ is finite.

Observe now that since $V_{n,k}(x_{n,k}) = 2$ and the total variation is semicontinuous from below, there exists an open interval $I_{n,k}$ such that

$$x_{n,k} \in I_{n,k} \quad \text{and} \quad V_{n,k}(x) \ge 1 \quad \text{for each } \; x \in I_{n,k}.$$

From the construction it follows that for each $m$ the set $\bigcup_{n=m}^{\infty} \bigcup_{k=0}^{n-1} I_{n,k}$ is open and dense in $(-1,1)$. Hence the set

$$E = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k=0}^{n-1} I_{n,k} \qquad \text{is residual in} \quad (-1,1).$$

For each $x \in E$ we have $V(x) = +\infty$ (it follows from the fact that the family of supports of $\{f_{n,k}\}_{n \in N,\ k \in \{0,1,\dots,n-1\}}$ is disjoint). ($E$ has Lebesgue measure zero). □

The same reasoning works in any direction of coordinate axes in the plane, in particular for $V(y)$. In the next theorem it will be more convenient to construct a function defined on $[0,1]\times[0,1]$, so $V(x)$ and $W(y)$ will denote total variation in the direction of $O_y$ and $O_x$, respectively, on the interval $[0,1]$.

**Theorem 2.** *There exists a continuous function $f:[0,1]\times[0,1]\to\mathbb{R}$ such that the area of the surface $f$ is finite and the set $\{x\in[0,1]:V(x)=+\infty\}$ is residual in $(0,1)$ and the set $\{y\in[0,1]:W(y)=+\infty\}$ is empty.*

Proof. Let $\Psi_n$ be the same function as in the proof of Theorem 1 for $n\in N$. For each $n\in N$ and $k\in\{1,2,\dots,n\}$ let $x_{n,k}=\frac{k}{n+1}$ and $y_{n,k}=\frac{1}{n}$.

Put

$$f_{n,k}(x,y)=\Psi_n(x-x_{n,k},y-y_{n,k})$$

for $n\in N$ and $k\in\{1,2,\cdots,n\}$ and $(x,y)\in[0,1]\times[0,1]$. At last let

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{k=1}^{n}f_{n,k}(x,y).$$

It is easy to see that

$$\{y\in[0,1]:W(y)=+\infty\}=\emptyset.$$

The proof of the fact that $\{x\in[0,1]:V(x)=+\infty\}$ is residual in $(0,1)$ and that the area of the surface $f$ is finite is exactly the same as in the proof of Theorem 1. □

## REFERENCES

[1] TONELLI, L.: *Sur la quadrature des surfaces*, C. R. Math. Acad. Sci. Paris **182** (1926), 1198–1200.

[2] TONELLI, L.: *Sulla quadratura delle superficie*, Atti Accad. Naz. Lincei, **6** (1926), no. 3 357–363, 445–450, 633–638.

[3] SAKS, S.: *Theory of the Integral*, Monografie Matematyczne, Tom VII G. E. Stechert & Co. vi, New York, 1937.

Received April 20, 2022

*Władysław Wilczyński*
*Department of Real Functions*
*Faculty of Mathematics and Computer Science*
*University of Łódź*
*Banacha 22*
*90–238 Łódź*
*POLAND*
*E-mail*: wladyslaw.wilczynski@wmii.lodz.pl