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STABILITIES AND NON-STABILITIES OF A NEW RECIPROCAL FUNCTIONAL EQUATION

B. V. Senthil Kumar*[1] — J. M. Rassias[2] — S. Idir[3] — S. Sabarinathan[4]

[1]University of Technology and Applied Sciences, Nizwa, OMAN

[2]National and Capodistrian University of Athens, Athens, GREECE

[3]University of Mouloud Mammeri, Tizi-Ouzou, ALGERIA

[4]SRM Institute of Science & Technology, Kattankulathur, Chennai, INDIA

ABSTRACT. The intention of this study is to present some stronger results by investigating Ulam-JRassias product stability and Ulam-JRassias mixed-type sum-product stability of a new reciprocal functional equation. Also, a suitable counter-example is presented to show the failure of stability result for the singular case.

1. Introduction

The investigation of classical stability results of various forms of functional equations is a trending research by many mathematicians. The following fascinating question connected with the stability of group homomorphisms is the foundation laid down to the development of stability theory of mathematical equations [27]: Suppose A_1 and A_2 are a group and a metric group with a metric $d(\cdot,\cdot)$, and let $\rho > 0$. Is it attainable to exist a positive real number μ such that if a mapping $\phi : A_1 \longrightarrow A_2$ satisfies the inequality $d\big(\phi(m,n), \phi(m)\phi(n)\big) < \mu$ for all $m, n \in A_1$, then a group homomorphism $\Phi : A_1 \longrightarrow A_2$ exists with the condition $d\big(\phi(m), \Phi(m)\big) < \rho$ for all $m, n \in A_1$? For this interesting query, the foremost partial response was provided in [8] and then this issue was dealt in different versions in [7] and [16] by considering a general control function and

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* Corresponding author.

sum of powers of norms as upper bounds. There are several other versions of the stability results investigated in [**?**, 1, 3, 9, 11, 12]. The analogous stronger results for investigating stability results were discussed in [14] and [17] with product of powers of norms and mixed-type sum-product of powers of norms as upper bounds. A collection of stability results in different normed spaces and applications of various forms of functional equations is available in [15].

The instigating and interesting results of [18] pertinent to the pioneer rational type functional equation

$$r(a_1 + a_2) = \frac{r(a_1)r(a_2)}{r(a_1) + r(a_2)} \tag{1}$$

has brought out a lot of attention to the researchers in this field. There are many papers published on stability results of various rational form of functional equations and their role in different disciplines such as electric circuit theory, electromagnetism, optics, physics discussed in [2, 4–6, 13, 19–26].

In this study, we give attention to the following new reciprocal type functional equation introduced in [10]:

$$g(2r+s) + g\left(\frac{r+s}{2}\right) = \frac{2g(r)g(s)}{g(r)+g(s)} + \frac{2g(r+s)g(s-r)}{3g(s-r) - g(r+s)}. \tag{2}$$

The general solution of equation (2) is $g(r) = \frac{\alpha}{r+\beta}$, where $\alpha, \beta (\neq 0)$ are constants and

$$g(r) \neq 0 \quad \text{for all } \ r \in A, \quad \text{with } \ A = \mathbb{R} - \{-\beta\}.$$

The generalized Hyers-Ulam-Rassias stability is obtained in [10].

In this paper, we further investigate the stronger results pertinent to Ulam--JRassias product and Ulam-JRassias sum-product stability results of equation (2). We also present a suitable counter-example to disprove the validity of stability result for the singular case.

Let $\mathbb{R}$ denote the set of all real numbers. In this entire study, we will consider that $A = \mathbb{R} - \{0\}$. For the sake of achieving our main results, let us define the difference operator $Dg : A \times A \longrightarrow \mathbb{R}$ via

$$Dg(r,s) = \frac{1}{g(2r+s)} - \frac{1}{\frac{2g(r)g(s)}{g(r)+g(s)} + \frac{2g(r+s)g(s-r)}{3g(s-r)-g(r+s)} - g\left(\frac{r+s}{2}\right)}$$

for all $r, s \in A$. Throughout this paper, let $g : A \longrightarrow \mathbb{R}$ be a mapping.

2. Product and mixed sum-product stabilities of equation (2)

In the following theorems, we attain the Ulam-JRassias (product) and Ulam-JRassias mixed-type (sum-product) stabilities of equation (2). For the sake of comparing the results obtained in this paper, we present here the stability result of equation (2) involving sum of powers of norms, investigated in [10].

Theorem 2.1 ([10], Corollary 1). *Let $g : A \longrightarrow \mathbb{R}$ be a mapping such that*

$$|Dg(r,s)| \leq \theta \left(|r|^q + |s|^q\right) \tag{3}$$

for all $r, s \in A$, where $\theta \geq 0$ and $q \leq 0$. Then, there exists a unique mapping $G : A \longrightarrow \mathbb{R}$ which satisfies equation (2) and the inequality

$$\left|\frac{1}{G(r)} - \frac{1}{g(r)} + \frac{1}{g(0)}\right| \leq \frac{2\theta}{3\left(2^{q-1} - 3^{q-1}\right)}|r|^q \tag{4}$$

for all $r \in A$.

Theorem 2.2. *Let the mapping g satisfy the following approximation with $k_1 \geq 0$ and $q \leq 0$,*

$$|Dg(r,s)| \leq k_1 |r|^{q/2}|s|^{q/2} \tag{5}$$

for all $r, s \in A$. Then, a unique reciprocal mapping $G : A \longrightarrow \mathbb{R}$ exists and satisfies (2) with the condition

$$\left|\frac{1}{G(r)} - \frac{1}{g(r)} + \frac{1}{g(0)}\right| \leq \frac{k_1}{3\left(2^{q-1} - 3^{q-1}\right)}|r|^q \tag{6}$$

for all $r \in A$.

Proof. First, switching (r,s) to $\left(\frac{r}{2}, \frac{r}{2}\right)$ in (5) and then multiplying by $2/3$, we obtain that

$$\left|\frac{2}{3g\left(\frac{3}{2}r\right)} - \frac{1}{g(r)} + \frac{1}{3g(0)}\right| \leq \frac{2k_1}{3 \cdot 2^q}|r|^q \tag{7}$$

for all $r \in A$. Now, letting r to $\frac{3r}{2}$ in (7), multiplying by $2/3$, and then adding the resultant with (7), we get

$$\left|\frac{4}{9g\left(\frac{9}{4}r\right)} - \frac{1}{g(r)} + \frac{1}{3g(0)}\sum_{k=0}^{1}\left(\frac{2}{3}\right)^k\right| \leq \frac{k_1}{2^q}\sum_{k=0}^{1}\left(\frac{2}{3}\right)^{k+1}\left(\frac{3}{2}\right)^{kq}|r|^q$$

for all $r \in A$. Continuing with the similar arguments and employing the induction method for an integer $m > 0$, we obtain

$$\left|\frac{2^m}{3^m g\left(\left(\frac{3}{2}\right)^m r\right)} - \frac{1}{g(r)} + \frac{1}{3g(0)}\sum_{k=0}^{m-1}\left(\frac{2}{3}\right)^k\right| \leq \frac{k_1}{2^q}\sum_{k=0}^{m-1}\left(\frac{2}{3}\right)^{k+1}\left(\frac{3}{2}\right)^{kq}|r|^q \tag{8}$$

for all $r \in A$. Now, reinstating r by $\left(\frac{3}{2}\right)^\ell r$, $\ell \in \mathbb{N}$ and multiplying by $\left(\frac{2}{3}\right)^\ell$, we obtain

$$\left|\frac{2^{m+\ell}}{3^{m+\ell}g\left(\left(\frac{3}{2}\right)^{m+\ell}r\right)} - \frac{1}{g\left(\left(\frac{3}{2}\right)^\ell r\right)} + \frac{1}{3g(0)}\sum_{k=0}^{m-1}\left(\frac{2}{3}\right)^{k+\ell}\right| \leq$$

$$\left(\frac{2}{3}\right)^{(1-q)(\ell+1)} k_1 \sum_{k=0}^{m-1}\left(\frac{2}{3}\right)^{(1-q)k}|r|^q \qquad \text{for all } r \in A. \tag{9}$$

Taking limit $\ell \to \infty$ in (9), we observe that the right-hand side of the above inequality (9) tends to 0, which indicates that the sequence $\left\{\frac{2^m}{3^m g\left(\left(\frac{3}{2}\right)^m r\right)}\right\}$ is Cauchy. Hence, we can define a mapping

$$\frac{1}{G(r)} = \lim_{m\to\infty} \frac{2^m}{3^m g\left(\left(\frac{3}{2}\right)^m r\right)}$$

for all $r \in A$. Allowing m to ∞ in (8), we find that the inequality (6) holds. Now, let us prove that g satisfies (2). For this, let us consider (r, s) as $\left(\left(\frac{3}{2}\right)^\ell r, \left(\frac{3}{2}\right)^\ell s\right)$ in (5) and multiplying by $\left(\frac{2}{3}\right)^\ell$, we find that

$$\left(\frac{2}{3}\right)^\ell \left| Dg\left(\left(\frac{3}{2}\right)^\ell r, \left(\frac{3}{2}\right)^\ell s\right)\right| \leq \left(\frac{2}{3}\right)^\ell k_1 \left(\frac{3}{2}\right)^{\ell q} |r|^{q/2}|s|^{q/2} \tag{10}$$

for all $r, s \in A$. In view of the definition of G and allowing $m \to \infty$ in (10), we notice that G satisfies (2). In order to show that the mapping G is unique satisfying (2), let us consider that there exists another mapping $G' : A \longrightarrow \mathbb{R}$ which satisfies (2) and inequality (6). Then, it implies that

$$\begin{aligned}
\left|\frac{1}{G(r)} - \frac{1}{G'(r)}\right| &= \left(\frac{2}{3}\right)^m \left|\frac{1}{G\left(\left(\frac{3}{2}\right)^m r\right)} - \frac{1}{G'\left(\left(\frac{3}{2}\right)^m r\right)}\right| \\
&\leq \left(\frac{2}{3}\right)^m \left(\left|\frac{1}{G\left(\left(\frac{3}{2}\right)^m r\right)} - \frac{1}{g\left(\left(\frac{3}{2}\right)^m r\right)}\right| + \left|\frac{1}{g\left(\left(\frac{3}{2}\right)^m r\right)} - \frac{1}{G'\left(\left(\frac{3}{2}\right)^m r\right)}\right|\right) \\
&\leq \left(\frac{2}{3}\right)^m \left(\frac{2k_1}{3\left(2^{q-1} - 3^{q-1}\right)}\right) |r|^q
\end{aligned} \tag{11}$$

for all $r \in A$. Letting $m \to \infty$ in (11) confirms that G is unique, which completes the proof. ☐

Theorem 2.3. *Let the mapping g satisfy the following inequality with $k_2 \geq 0$ and $q \leq 1$,*

$$|Dg(r, s)| \leq k_2 \left((|r|^q + |s|^q) + |r|^{q/2}|s|^{q/2}\right) \tag{12}$$

for all $r, s \in A$. Then, a unique reciprocal mapping $G : A \longrightarrow \mathbb{R}$ exists and satisfies (2). Also, G satisfies the following approximation

$$\left|\frac{1}{G(r)} - \frac{1}{g(r)} + \frac{1}{g(0)}\right| \leq \frac{k_2}{\left(2^{q-1} - 3^{q-1}\right)}|r|^q \tag{13}$$

for all $r \in A$.

Proof. The proof is obtained analogous to the arguments as in Theorem 2.2 and hence we omit the proof. ☐

3. Counter-example

In the sequel, we present a counter-example to prove the non-stability of equation (2) when $q = 1$ in Theorem 2.1. Consider the function

$$\frac{1}{\chi(r)} = \begin{cases} cr, & \text{for} \quad |r| < 1, \\ c, & \text{for} \quad |r| \geq 1, \end{cases} \tag{14}$$

where $\chi : A \longrightarrow \mathbb{R}$. Then, the function defined in (14) serves as a suitable illustration to prove that (2) is unstable for the singular case $q = 1$ in the following theorem.

Theorem 3.1. *Let a mapping $g : A \longrightarrow \mathbb{R}$ be defined as*

$$g(r) = \sum_{m=0}^{\infty} \frac{(2/3)^m}{\chi\left(\frac{3^m r}{2^m}\right)} \tag{15}$$

for all $r \in A$. Suppose the mapping g defined in (15) *satisfies*

$$|Dg(r,s)| \leq 3c\,(|r| + |s|) \tag{16}$$

for all $r, s \in A$. Then, there do not exist a reciprocal type mapping $G : A \longrightarrow \mathbb{R}$ and a constant $\mu > 0$ such that

$$|g(r) - G(r)| \leq \mu\,|r| \tag{17}$$

for all $r \in A$.

Proof. Foremostly, we shall prove that g satisfies (16). Using (14), we have

$$|g(r)| = \left| \sum_{m=0}^{\infty} \frac{(2/3)^m}{\chi\left(\frac{3^m r}{2^m}\right)} \right| \leq \sum_{m=0}^{\infty} c(2/3)^m = 3c.$$

The above inequality indicates that g is bounded above by $3c$ on $\mathbb{R}$. Suppose $|r| + |s| \geq 1$. Then, it is clear that the left-hand side of (16) is less than $3c$. Now, suppose that $0 < |r| + |s| < 1$. Hence, we can find an integer $m > 0$ so that

$$\frac{1}{(3/2)^{m+1}} \leq |r| + |s| < \frac{1}{(3/2)^m}. \tag{18}$$

The inequality (18) produces

$$(3/2)^m\,(|r| + |s|) < 1, \quad \text{or equivalently,} \quad (3/2)^m r < 1, \quad (3/2)^m s < 1.$$

Hence, the last inequalities imply

$$(3/2)^{m-1} r < \tfrac{2}{3} < 1, \quad (3/2)^{m-1} s < \tfrac{2}{3} < 1$$

and as a result, we find

$$(3/2)^{m-1}(r) < 1, \quad (3/2)^{m-1}(r+s) < 1, \quad (3/2)^{m-1}(2r+s) > 1,$$
$$(3/2)^{m-1}(s) < 1, \quad (3/2)^{m-1}(s-r) < 1, \quad (3/2)^{m-1}\left(\tfrac{r+s}{2}\right) < 1,$$

and

$$Dg\big((3/2)^n r, (3/2)^n s\big) = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots, n-1.$$

$$\begin{aligned}\frac{|Dg(r,s)|}{(|r|+|s|)} &\leq \sum_{m=n}^{\infty} \frac{c}{(3/2)^m\,(|r|+|s|)} \\ &\leq \sum_{k=0}^{\infty} \frac{c}{(3/2)^k (3/2)^n\,(|r|+|s|)} \leq \sum_{k=0}^{\infty} \frac{c}{(3/2)^k} = 3c\end{aligned}$$

for all $r, s \in A$. The above inequality indicates that (16) holds good. We claim that (4) is unstable for $q = 1$ in Theorem 2.2. Assume that there exists a reciprocal mapping $G : A \longrightarrow \mathbb{R}$ satisfying (16). Hence, we obtain

$$\frac{1}{|G(r)|} \leq (|c|+1)|r|. \tag{19}$$

On the contrary, it is possible to choose an integer $m > 0$ with $m+1 > |c|+1$. If $r \in (1, (2/3)^{-m})$, then $(2/3)^m r \in (0,1)$ for all $m = 0, 1, 2, \ldots, n-1$ and thus

$$\begin{aligned}\frac{1}{|G(r)|} &= \sum_{m=0}^{\infty} \frac{(2/3)^m}{\chi((3/2)^m r)} \\ &\geq \sum_{m=0}^{n-1} (2/3)^m \left((3/2)^m r\right) = (m+1)r > (|c|+1)r\end{aligned}$$

which contradicts (19). Hence, equation (2) is not stable for $q = 1$ in Theorem 2.1. □

Remark 1. A similar counter-example can be demonstrated to disprove the validity of stability result of equation in Theorem 2.3.

4. Conclusion

In this study, we have established some stronger results of stabilities connected with the reciprocal type functional equation (2). The stability with upper bound as product of powers of norms obtained in this investigation is better approximation than the stability involving upper bound as sum of powers of norms obtained in [10]. Also, we have extended the stability result of equation (2) with sum-product of powers of norms. An appropriate example is illustrated to prove the failure of stability result of equation (2) for a singular case.

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Department of Information Technology
University of Technology and Applied Sciences
Nizwa–611
OMAN
E-mail: bvskumarmaths@gmail.com

Pedagogical Department E.E.
National and Capodistrian University of Athens
Attikis–15342
GREECE
E-mail: jrassias@primedu.uoa.gr

Department of Mathematics
University of Mouloud Mammeri
Tizi-Ouzou–15000
ALGERIA
E-mail: sadani.idir@yahoo.fr

Department of Mathematics
SRM Institute of Science & Technology
Kattankulathur - 603 203, Chennai
INDIA
E-mail: ssabarimaths@gmail.com