SECRET SHARING ON INFINITE GRAPHS

LÁSZLÓ CSIRMAZ

ABSTRACT. We extend the notion of perfect secret sharing scheme for access structures with infinitely many participants. In particular we investigate cases when the participants are the vertices of an (infinite) graph, and the minimal qualified sets are the edges. The (worst case) information ratio of an access structure is the largest lower bound on the amount of information some participant must remember for each bit in the secret—just the inverse of the information rate. We determine this value for several infinite graphs: infinite path, two-dimensional square and honeycomb lattices; and give upper and lower bounds on the ratio for the triangular lattice.

It is also shown that the information ratio is not necessarily local, i.e., all finite spanned subgraphs have strictly smaller ratio than the whole graph. We conclude the paper by posing several open problems.

1. Introduction

A secret sharing scheme is a method of distributing secret data among a set of participants so that only specified qualified subsets of participants are able to recover the secret. In addition, if the unqualified subsets collectively yield no extra information, i.e., the joint shares are statistically independent of the secret, then the scheme is called perfect. The description of qualified subsets among all possible subsets of participants is the access structure. In this paper only perfect secret sharing schemes are considered; when we speak about a secret sharing scheme, it is assumed to be perfect.

The most frequently investigated property is the efficiency of the system: how many bits of information the participants must remember for each bit of the secret in the worst case. This amount is the (worst case) information ratio of the system, which is just the inverse of the most commonly used information rate. (The name comes from the analogy to noisy channels.) Next to the worst case, the average is also a good measure of the efficiency; in this paper by information ratio we mean the worst case one.

2000 Mathematics Subject Classification: 68P25, 94A60.
Keywords: secret sharing scheme, information theory, infinite graph, lattice.
Determining the (worst case or average) information ratio, even in special cases, turned out to be extremely hard, both theoretically and technically. Several particular classes of access structures were investigated separately: access structures on four or five element sets \([16, 20]\), access structures with three or four minimal sets \([17]\), but most importantly access structures where the minimal qualified subsets are of size two—the so-called graph access structures \([2, 4, 5, 8, 12, 13, 21]\).

Here we start the research in a new direction: we consider access structures with infinitely many participants. In the exposition we restrict ourselves to graph access structures. Our definitions, and some of the results, generalize easily for arbitrary access structures. Secret sharing systems on infinite domain with finite access structures were investigated in \([6]\).

We determine the exact value of the (worst case) information ratio for a couple of infinite graphs. This part uses a new result on the exact information ratio for a particular family of graphs (Theorem 4.2). We also show that the information ratio is not necessarily local, i.e., there are cases when this number for the whole graph is larger than that of any of its finite subgraph (Corollary 4.6).

The paper is organized as follows. In the next section we recall the definition of a perfect secret sharing system, define the worst case and average information ratio, and introduce the so-called entropy method \([4, 5, 7, 20]\). Section 3 defines secret sharing systems on infinite structures. We introduce a generalization of the decomposition technique for the infinite case (Theorem 3.6), and show that Stinson’s celebrated bound works in the infinite case as well (Corollary 3.9). In Section 4 we determine the exact information ratio of a particular family of graphs. It is used to prove the optimality of several constructions in Section 5. Finally Section 6 concludes the paper, and lists some problems. For undefined notions and for an introduction to secret sharing schemes see \([2]\) or \([8]\); for those in information theory consult \([11]\).

All logarithms in this paper are of base 2.

2. Definitions

This section defines some of the most important notions which are used in the paper. First we recall some graph properties, then give a formal definition of a (finite) perfect secret sharing scheme based on graphs. Finally we connect perfect secret sharing schemes to certain submodular functions.

Let \(G = (V, E)\) be a (finite or infinite) graph with vertex set \(V\) and edge set \(E\). A subset \(A\) of \(V\) is independent if there is no edge between vertices in \(A\). A covering of the graph \(G\) is a collection of subgraphs of \(G\) such that every edge is contained on one of the (not necessarily spanned) subgraphs in the collection.
The collection is \( k \)-covering if every edge of \( G \) is covered exactly \( k \) times. For subsets of vertices we usually omit the \( \cup \) sign, and write \( AB \) for \( A \cup B \). Also, if \( v \in V \) is a vertex, then \( Av \) denotes \( A \cup \{ v \} \).

A perfect secret sharing scheme \( S \) for a finite graph \( G \) is a collection of random variables \( \xi_v \) for each \( v \in V \) and a \( \xi_s \) (the secret) with a joint distribution so that

(i) if \( vw \) is an edge in \( G \), then \( \xi_v \) and \( \xi_w \) together determine the value of \( \xi_s \);

(ii) if \( A \) is an independent set, then \( \xi_s \) and the collection \( \{ \xi_v : v \in A \} \) are statistically independent.

The size of the discrete random variable \( \xi \) is measured by its entropy, or information content, and is denoted by \( H(\xi) \), see [11]. This amount has to be well defined and finite, consequently all random variables in this paper are assumed to be finite, i.e., they can take only finitely many different values with positive probability. This is the main obstacle one has to overcome when defining a secret scheme on infinite domain.

The information ratio for a vertex (or participant) \( v \in G \) is \( H(\xi_v)/H(\xi_s) \). This value tells how many bits of information \( v \) must remember for each bit in the secret. The worst case (or average) information ratio of \( S \) is the highest (resp. average) information ratio among all participants.

Given a graph \( G \) its information ratio is the infimum of the corresponding value for all perfect secret sharing schemes \( S \) defined on \( G \).

**Definition 2.1.** The information ratio of the (finite) graph \( G \), denoted as \( R(G) \), is defined as

\[
R(G) = \inf_{S} \max_{v \in V} \frac{H(\xi_v)}{H(\xi_s)}.
\]

The widely used information rate is the inverse of this value.

While the “information rate” is the customary measure in the literature, cf. [2, 4, 5, 7, 12, 20, 21], we found its inverse, the ratio, to be more intuitive, furthermore certain expressions are easier to write and understand using the ratio.

As it has been pointed out in [1], it is not evident that the “infimum” in Definition 2.1 should actually be taken by some scheme \( S \), i.e., whether the infimum is always a minimum. In [1] there is presented a general access structure where the infimum is not taken by any scheme. For access structures based on graphs the question whether inf = min is an open problem.

Let \( S \) be a perfect secret sharing scheme based on the (finite) graph \( G \) with the random variable acting \( \xi_s \) as secret, and \( \xi_v \) for \( v \in V \) acting as shares. For each subset \( A \) of the vertices one can define the real-valued function \( f \) as

\[
f(A) \overset{\text{def}}{=} \frac{H(\{\xi_v : v \in A\})}{H(\xi_s)}. \tag{1}
\]
Clearly, the information ratio of $S$ is the maximal value in $\{f(v) : v \in V\}$, while the average information ratio is the average of these values. Using standard properties of the entropy function, cf. [11], following inequalities hold for all subsets $A, B$ of the participants:

(a) $f(\emptyset) = 0$, and in general $f(A) \geq 0$ (positivity);
(b) if $A \subseteq B \subseteq V$, then $f(A) \leq f(B)$ (monotonicity);
(c) $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ (submodularity).

It is well known that for two (finite) random variables $\eta$ and $\xi$, the value of $\eta$ determines the value of $\xi$ iff $H(\eta \xi) = H(\eta)$, moreover $\eta$ and $\xi$ are (statistically) independent iff $H(\eta \xi) = H(\eta) + H(\xi)$. Using these facts and the definition of the perfect secret sharing scheme, we also have

(d) if $A \subseteq B$, $A$ is an independent set and $B$ is not, then $f(A) + 1 \leq f(B)$ (strong monotonicity);
(e) if neither $A$ nor $B$ is independent but $A \cap B$ is so, then $f(A) + f(B) \geq 1 + f(A \cap B) + f(A \cup B)$ (strong submodularity).

The entropy method, see, e.g., [2], can be rephrased as follows. Prove that for any real-valued function $f$ satisfying properties (a)–(e), for some vertex $v \in G$, $f(v) \geq r$. Then, as functions coming from secret sharing schemes also satisfy these properties, conclude, that the (worst case) information ratio of $G$ is also at least $r$.

Note that this method is not necessarily universal, as properties (a)–(e) are too weak to capture exactly the functions coming from entropy see [18]. However, for graphs all existing lower bound proofs use the entropy method, and no example is known where the entropy method would not work.

3. The case of infinite graphs

Trying to define secret sharing on an infinite object one faces several problems. As there are infinitely many participants, one has to define infinitely many random variables with a joint distribution. But infinitely many pairwise random bits (probably needed for any construction) require infinite event space where the standard entropy function does not exist. We used entropy as a tool to define the relative size of a share compared to the secret, but even finding such a weaker notion is problematic; see Problem 6.1.

Rather than defining the information ratio directly, we choose an indirect way. In case of graphs the set of participants might be infinite, but the minimal qualified subsets are finite, namely pairs. Thus it seems quite natural to consider finite restrictions. Our starting point is the following easy, but very useful fact
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about secret sharing schemes on finite graphs. The fact generalizes easily to other access structures as well.

**Fact 3.1.** Suppose $G'$ is a spanned subgraph of $G$. The information ratio of $G'$ is at most as large as the information ratio of $G$, i.e., $R(G') \leq R(G)$.

In general, this claim is not true for arbitrary subgraphs. By Shamir’s result in [19], $R(K_n) = 1$ where $K_n$ is the complete graph on $n$ vertices, while by [8], there is a graph $G' \subseteq K_n$ where $R(G') \geq 0.25 \log_2 n$.

Looking at an infinite graph as the “limit” of its finite spanned subgraphs, Fact 3.1 suggests the following definition:

**Definition 3.2.** The information ratio $R(G)$ for the infinite graph $G$ is $R(G) = \sup \{ R(G') : G' \text{ is a finite, spanned subgraph of } G \}$.

By Fact 3.1 this is a sound definition, and applying to a finite $G$ gives back the original value.

If the (finite) graph $G$ is the disjoint union of $G_1$ and $G_2$, i.e., there are no cross edges between $G_1$ and $G_2$, then any secret sharing scheme on $G$ trivially splits into a secret sharing scheme on $G_1$, and another one on $G_2$.

**Claim 3.3.** If $G$ has several connected components, then $R(G) = \sup \{ R(G') : G' \text{ is a connected component of } G \}$.

Consequently, in Definition 3.2 it is enough to consider connected finite subgraphs of $G$ only.

We have defined the information ratio of an infinite graph as a supremum. It is a natural question whether this value is actually taken, or it is a proper one.

**Definition 3.4.** The graph $G$ is local if there is a finite spanned subgraph $G'$ of $G$ such that $R(G) = R(G')$. Otherwise $G$ is not local.

Of course, when $R(G)$ is infinite, then $G$ cannot be local as no finite graph has infinite information ratio. Locality is interesting only when $R(G)$ is finite.

When constructing secret sharing schemes the most frequently used tool is Stinson’s decomposition technique from [21]. For our case it can be worded as

**Theorem 3.5 (Stinson).** Let $G_i \subseteq G$ be arbitrary subgraphs of $G$, and assume that each edge of $G$ is in at least $k$ of the subgraphs. Let $S_i$ be a perfect secret sharing scheme on $G_i$ such that $S_i$ assigns $S_i(v)$ bits to $v \in G$ for each bit in the secret ($S_i(v) = 0$ if $v \notin G_i$). Then there is a scheme $S$ on $G$ which assigns

$$S(v) = \frac{1}{k} \sum S_i(v)$$

bits to $v$ for each bit in the secret.
This theorem is meaningful for finite graphs only. The following generalization, however, holds for infinite graphs as well.

**Theorem 3.6.** Let $G_i \subseteq G$ be arbitrary (finite or infinite) subgraphs of $G$, and assume that each edge of $G$ is in at least $k$ of the subgraphs. For a vertex $v \in G$ define $r_i(v) = 0$ if $v \notin G_i$, and $r_i(v) = R(G_i)$, i.e., the information ratio of $G_i$ otherwise. Then

$$R(G) \leq \sup_{v \in G} \frac{\sum r_i(v)}{k}.$$ 

**Proof.** Let us denote the value of the sup on the right hand side by $r$; we may assume that $r$ is finite otherwise there is nothing to prove. Let $G' \subseteq G$ be a finite spanned subgraph of $G$. According to Definition 3.2, we need to check that there is a perfect secret sharing scheme $S$ on $G'$ which assigns at most $r_i$ bits to each vertex of $G'$ for each bit in the secret.

As $G'$ has finitely many edges, we can choose a finite set $I$ of the indices of the subgraphs $G_i$ such that each edge of $G'$ is in at least $k$ of the subgraphs in the family $\{G_i : i \in I\}$. For $i \in I$ let $G'_i$ be the spanned subgraph of $G_i$ restricted to the vertices of $G'$. As $R(G_i) = r_i$ and $G'_i$ is a spanned subgraph of $G_i$, by Fact 3.1 there is a secret sharing scheme $S_i$ on $G'_i$ which assigns at most $r_i$ bits to all $v \in G'_i$ for each bit in the secret. By Theorem 3.5 there exists a scheme $S$ which assigns

$$S(v) = \frac{1}{k} \sum_{i \in I} S_i(v) \leq \frac{1}{k} \sum_{i \in I} r_i(v) \leq \frac{\sum r_i(v)}{k} \leq r$$

bits to $v \in G'$, which was wanted. □

We close this section by a generalization of Stinson’s result [21]. For the proof we need some well-known facts. The first statement is a folklore, the proof is an easy application of the entropy method, see, e.g., [2], or the results in Section 4.

**Claim 3.7.** If $G$ is empty (independent), then $R(G) = 0$. Otherwise $R(G) \geq 1$.

The complete graph on (countably) infinitely many points is denoted by $K_\infty$, and the (infinite) graph where one point is connected to an infinite independent set is denoted as Star$_\infty$.

**Claim 3.8.** $R(K_\infty) = R(Star_\infty) = 1$.

**Proof.** All finite spanned subgraph of $K_\infty$ is the complete graph. By Shami r’s result in [19] all of them have ratio 1, thus their sup is also 1.

As for the other graph, $R(Star_\infty) \geq 1$ by Claim 3.7. We show that this value is also $\leq 1$. Each finite, connected spanned subgraph of Star$_\infty$ is a (finite) star. Let the secret be the random bit $s \in \{0, 1\}$, and let $r \in \{0, 1\}$ be selected randomly.
and independently from $s$. The center of the star will get $s \oplus r$, and all other vertices get $r$. This is a perfect secret sharing scheme; all participants get 1 bit, and the secret is 1 bit as well (see Figure 1). Thus the ratio in this case is 1 as well.

**Corollary 3.9.** If the maximal degree of $G$ is $d$, then $R(G) \leq (d + 1)/2$.

**Proof.** For each vertex $v$ in $G$ consider the star $G_v$ with center $v$ and all edges outgoing from $v$ as rays. These subgraphs $G_v$ cover all edges twice, and each vertex is in at most $d + 1$ of these subgraphs (once as center, and $d$ times as endpoint of a ray). Now $R(G_v) = 1$ by Claim 3.8, and apply Theorem 3.6. □

### 4. Lower bounds: the comb

Almost all lower bounds use the entropy method, see [4, 5, 7, 20] which has been outlined at the end of Section 2. We shall use this method for a particular family of graphs, which will then be used to determine the exact ratio of several infinite graphs as well.

![Figure 2. The graphs Comb_k and Comb_2.](image)

**Definition 4.1.** For $k \geq 2$ Comb$_k$ is the graph on $2k$ vertices as indicated on Figure 2, in particular Comb$_2$ is the path of length 3. Comb$_\infty$ is the infinite comb with no 2-degree vertex.
The main result of this section is

**Theorem 4.2.** For \( k \geq 2 \), \( R(\text{Comb}_k) = 2 - 1/k \).

**Proof.** First, by using Stinson’s decomposition method, we show that the ratio is \( \leq 2 - 1/k \). Figure 3 indicates the first two of the \( k \) different covers of \( \text{Comb}_k \); each component in a cover is a star on two, three or four vertices, thus has information ratio 1. The numbers below the vertices indicate the number of bits they receive using this cover. Putting together all \( k \) covers each edge is covered \( k \) times, the bottom vertices receive a total of \( 2k - 1 \) bits, while the top vertices receive \( k \) bits. Using Theorem 3.5 we conclude that \( R(\text{Comb}_k) \leq (2k - 1)/k \).

For the other direction we use the entropy method. Label the bottom vertices of \( \text{Comb}_k \) from left to right as \( A_1, A_2, \ldots, A_k \), the top vertices as \( a_1, \ldots, a_k \) so that \( a_i \) is connected to \( A_i \) only. Given any secret sharing scheme on \( \text{Comb}_k \), define the real-valued function \( f \) as in (1). For a subset \( A \) of the vertices let

\[
f(A) = \frac{H(\{\xi_v : v \in A\})}{H(\xi_A)}.
\]

This function satisfies properties (a)–(e) enlisted in Section 2. We claim that for any such function \( f \) we have

\[
\sum_{i=1}^{k} f(A_i) \geq 2k - 1.
\]

(2)

Showing this we are done. Indeed, the sum of these \( k \) terms is at least \( 2k - 1 \), thus at least one of them is \( \geq (2k - 1)/k \). Consequently for at least one vertex \( A_i \) we have

\[
f(A_i) = \frac{H(\xi_A)}{H(\xi_A)} = \frac{H(\xi_A)}{H(\xi_A)} \geq \frac{2k - 1}{k},
\]

i.e., \( A_i \) must remember at least \( 2 - 1/k \) bits for each bit in the secret.

To finish the proof, we state and prove Lemmas 4.3 and 4.4. Inequality (2) is just the sum of the claims of the lemmas. \( \square \)
**Lemma 4.3.** $\sum_{i=1}^{k} f(A_i) \geq f(A_1 A_2 \ldots A_k) + k - 2$.

**Proof.** In general for any $2 \leq \ell \leq k$, $\sum_{i=1}^{\ell} f(A_i) \geq f(A_1 A_2 \ldots A_{\ell}) + \ell - 2$ which we will prove by induction on $\ell$. When $\ell = 2$ the claim is $f(A_1) + f(A_2) \geq f(A_1 A_2)$, which is just the submodularity (c).

Now suppose we know the claim to be true for $\ell - 1$; to conclude it for $\ell$ it is enough to check that whenever $\ell \geq 3$ then

$$f(A_1 A_2 \ldots A_{\ell-1}) + f(A_\ell) \geq f(A_1 \ldots A_\ell) + 1.$$  \hspace{1cm} (3)

As $\ell \geq 3$, both $A_1 \ldots A_{\ell-1}$ and $A_{\ell-1} A_\ell$ contain edge, i.e., they are a qualified sets. Then property (e) says that

$$f(A_1 A_2 \ldots A_{\ell-1}) + f(A_{\ell-1} A_\ell) \geq f(A_1 \ldots A_\ell) + f(A_{\ell-1}) + 1$$

as the singleton $A_{\ell-1}$ is not qualified. By the submodularity (c) we have

$$f(A_{\ell-1}) + f(A_\ell) \geq f(A_{\ell-1} A_\ell).$$

Adding up the last two inequalities we get (3), as required. \hfill \Box

**Lemma 4.4.** $f(A_1 A_2 \ldots A_k) \geq k + 1$.

**Proof.** Let $X = \{A_1 A_2 \ldots A_k\}$, and consider the differences

$$d_i = f(Xa_1 \ldots a_i) - f(a_1 \ldots a_i).$$

As $f(\emptyset) = 0$, the value we are interested in is $d_0$. The trick is to consider these differences in reverse order. As $\{Xa_1 \ldots a_k\}$ is qualified, while $\{a_1 \ldots a_k\}$ is not, condition (d) gives $1 \leq d_k$. Furthermore, for all $1 \leq i \leq k$, $d_i + 1 \leq d_{i-1}$ which implies $k + 1 \leq d_0$ as the lemma states.

Now both $\{A_i a_1 \ldots a_i\}$ and $\{Xa_1 \ldots a_{i-1}\}$ are qualified, their intersection, which is $\{A_i a_1 \ldots a_{i-1}\}$, is not, thus (e) gives

$$f(A_i a_1 \ldots a_i) + f(Xa_1 \ldots a_{i-1}) \geq f(A_i a_1 \ldots a_{i-1}) + f(Xa_1 \ldots a_i) + 1.$$  

Furthermore the submodularity (c) tells

$$f(a_1 \ldots a_i) + f(A_i a_1 \ldots a_{i-1}) \geq f(A_i a_1 \ldots a_i) + f(a_1 \ldots a_{i-1}).$$

Adding these up and rearranging we get $d_i + 1 \leq d_{i-1}$, as needed. \hfill \Box

**Theorem 4.5.** The information ratio of $\text{Comb}_\infty$ is 2.

**Proof.** First we show that 2 is an upper bound. To this end let $G$ be an arbitrary finite spanned subgraph of $\text{Comb}_\infty$. Then, for some $k$, $G$ is a spanned subgraph of an isomorphic copy of $\text{Comb}_k$, therefore $R(G) \leq R(\text{Comb}_k) < 2$. Consequently, $R(\text{Comb}_\infty)$ as the sup of $R(G)$ for these $G$’s, is $\leq 2$, as was claimed.

On the other hand, for each natural number $k$, $\text{Comb}_k$ is a spanned subgraph of $\text{Comb}_\infty$, thus $R(\text{Comb}_\infty)$ is at least as large as $R(\text{Comb}_k) = 2 - 1/k$. Consequently, $R(\text{Comb}_\infty)$ cannot be smaller than 2. \hfill \Box
Corollary 4.6. Comb\(_\infty\) is not local, i.e., all of its finite spanned subgraphs have smaller information ratio.

5. Examples

The path of length \(n\) (with \(n + 1\) vertices) is denoted as \(P_n\), and \(P_\infty\) is the infinite path, or the 1-dimensional lattice.

The honeycomb is the two-dimensional tiling of the plane with regular hexagons. The two-dimensional (square) lattice is the usual checkered paper like tiling (see Fig. 4), and the triagonal tiling is the tiling with regular triangles (Fig. 7).

Example 5.1. \(R(P_\infty) = 3/2\).

Proof. We have seen that Comb\(_2\) is the same graph as the path of length 3, thus by Theorem 4.2 \(R(\text{Comb}_2) = R(P_3) = 3/2\). (For other proofs see, e.g., [2, 4, 7], or the Appendix.) As \(P_3\) is a spanned subgraph of \(P_\infty\), we have \(R(P_3) = 3/2 \leq R(P_\infty)\). For the other direction we use Theorem 3.6 for the 2-cover indicated on Figure 5. All subgraphs in the cover are stars having ratio 1; each edge is covered twice and each vertex gets 3 bits. Thus \(R(P_\infty) \leq 3/2\). □

Example 5.2. \(R(\text{honeycomb}) = 2\).
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Proof. As each vertex has degree 3, Corollary 3.9 says that the ratio is at most 2. On the other hand, the honeycomb contains the infinite comb as a spanned subgraph (left picture on Figure 6). Consequently, \( 2 = R(\text{Comb}_\infty) \leq R(\text{honeycomb}) \). \( \square \)

**Example 5.3.** \( R(2\text{-lattice}) = 2 \).

Proof. Here each vertex has degree 4, thus Corollary 3.9 gives only \( 5/2 \). We could, however, apply Theorem 3.6 directly for the four-cycles \( C_4 \) indicated on Figure 4. Each edge is covered once, and each vertex is in two cycles. As \( C_4 \) has information ratio 1 we proved that the information ratio for the 2-lattice is \( \leq 2 \). The statement on \( C_4 \) can be shown as follows: let the random bit \( s \in \{0,1\} \) be the secret, and pick \( r \in \{0,1\} \) randomly and independently from \( s \). Give \( r \) to the first and third node in \( C_4 \), and \( r \oplus s \) to the two other nodes.

The lower bound follows from the fact that the infinite comb can be embedded to the 2-lattice as a spanned subgraph, see Figure 6. \( \square \)

![Figure 6. The comb as spanned subgraph.](image)

The proof given here does not tell whether the 2-lattice is local or not. In the Appendix we show that a particular graph on 8 vertices has information ratio 2. That graph is a spanned subgraph of the 2-lattice, thus the 2-lattice is not local.

**Example 5.4.** \( 2 \leq R(\text{triangle lattice}) \leq 12/5 \).

Proof. The lower bound follows again from the fact that the comb can be embedded into this lattice as well, see Figure 7.

![Figure 7. The triangle lattice.](image)
The construction which gives the upper bound 12/5 is due to Péter Gergely [14]. Consider the diamond-shaped graph on four vertices at the right hand side of Figure 7. This graph has ratio 1, which can be shown as follows: pick the secret $s$ from $\{0,1,2\}$ uniformly, and pick a random $r$ from the same set. Give $r$ to nodes on the left and on the right, give $s + r \mod 3$ to the top vertex, and $s - r \mod 3$ to the bottom vertex. Any two connected vertices can recover the secret $s$, moreover each assigned number is independent of $s$. The two unconnected vertices receive the same share, thus their joint information is independent of the secret as well. Both the secret and the shares have entropy $\log 3$, thus the ratio is 1.

The cover consists of all spanned subgraphs of the triangle lattice isomorphic to this graph. It is easy to check that this is a 5-cover (each edge is in 5 of such subgraphs), and each vertex is covered 12 times. By Theorem 3.6 this gives 12/5 as an upper bound.

**Problem 5.5.** Determine the exact information ratio of the triangle lattice.

The universal or random graph turns up in many branches of mathematics, see, e.g., [3]. It has several equivalent definitions, one of them can be rephrased as follows. The universal graph is the (up to isomorphism) unique graph on countably many vertices which has the following property. Picking finitely many vertices $v_i$ and numbers $\varepsilon_i \in \{0,1\}$, there exists a vertex in the graph which is connected to $v_i$ just in case $\varepsilon_i = 1$.

**Example 5.6.** The information ratio for the universal graph is $\infty$.

**Proof.** As we have remarked, there is a graph of $n$ vertices with information ratio $\geq 0.25 \log n$, see [8]. As all finite graphs can be embedded into the universal graph as spanned subgraphs, its information ratio is at least $0.25 \log n$, i.e., not bounded.

**Example 5.7.** The information ratio for the infinite binary tree is 2.

**Proof.** Lower bound: for each $k$ the graph $\text{Comb}_k$ can be embedded into this graph, consequently $R \geq 2$.

Upper bound: we show that each tree has information ratio $\leq 2$ by using the Theorem 3.6. We may assume that the tree is connected. Pick any vertex and consider it as “root.” Direct the edges recursively away from the root. When all edges have been directed, each node has one invertex except for the root which has none. Consider the stars with center at a vertex consisting of all outgoing edges. Each edge is covered exactly once, and each vertex gets one or two bits (one bit for the root and leaves, and two bits for all other vertices).

In [10] it has been proved that all finite trees have information ratio $2 - 1/k$ for some integer $k \geq 1$. Therefore the information ratio of an infinite tree is the sup
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Figure 8. The ladder and a cover.

of numbers of this form. If $T$ is an infinite tree, then either $R(T) = 2$, and then $T$ is not local (as, e.g., is the case for the complete binary tree in Example 5.7), or $R(T) = 2 - 1/k$ for some integer $k \geq 1$, and then $T$ is local. The next example shows that there is an infinite graph with ratio strictly between $3/2$ and 2.

**Example 5.8.** There is an infinite graph with information ratio $5/3$.

**Proof.** Consider the (rooted) complete binary tree from Example 5.7, and insert a new vertex at the midpoint of every edge. This will be our graph $T$.

As three-tooth comb can be embedded into $T$, its information ratio is $R(T) \geq R(\text{Comb}_3) = 2 - 1/3$. The upper bound comes from Theorem 3.6: we define a 3-cover by stars so that each vertex gets 5 bits; this gives $R(T) \leq 5/3$. First take the stars with center at the inserted new vertices and both edges as rays. Second take stars at the old vertices with all incident edges as rays, but take them twice. That way an old vertex gets $2 \cdot 1 + 3$ bits (two from the double star, and one from each new neighbor), while a new vertex gets $2 \cdot 2 + 1$ bits, as required. □

**Example 5.9.** The infinite ladder $L$ on Figure 8 has information ratio $10/6 \leq R(L) \leq 11/6$.

**Proof.** The upper bound comes from the following construction (see Figure 8). The 1-cover on the right has period 6. It uses stars and $C_4$, both assigns a single bit for each bit in the secret. In a period all vertices get 2 bits, except for one in the top, and one in the bottom which get only 1 bit. Shifting this cover by 1, . . . , 5 we get a 6-cover, and each vertex gets a total of 11 bits. By Theorem 3.6 the upper bound follows.

For the lower bound one can observe that the infinite path is a spanned subgraph—this gives $9/6$ as a lower bound. For the missing $1/6$ we prove in the Appendix that the graph $G_1$ on Figure 9 has information ratio $10/6$. As this is a spanned subgraph of the ladder, we are done. □

We remark that the ladder of width 2 has information ratio 2. It is a spanned subgraph of the 2-lattice ($\leq 2$), and contains $\text{Comb}_{\infty}$ as a spanned subgraph ($\geq 2$).
6. Conclusion and further problems

Determining the exact amount of information a participant must remember in a perfect secret sharing scheme is an important problem both from theoretical and practical point of view. Access structures based on graphs pose special challenges. They are easier to define, have a transparent, and sometimes trivial structure.

In this paper we extended the definition of secret sharing for infinitely many participants, and gave a definition for its information ratio. We consider this extension to be an important contribution, and hope to see further, interesting applications.

We have used a compactness-type definition to overcome the difficulty of infinite entropy: the information ratio is defined as the sup of the ratio for the finite embedded structures. The first problem is to find an appropriate definition of the “relative information content” for arbitrary random variables.

**Problem 6.1.** Given two random variables \( \xi \) and \( \eta \), define their relative size, which is the analog of \( H(\xi)/H(\eta) \) when both \( \xi \) and \( \eta \) are finite.

In [6] the authors show that the secret and shares, as random variables, cannot be based on a countable domain, even if the number of participants is finite. The paper also contains the following example using reals: let the secret \( \xi \) be uniform in \([0,1)\), then choose the shares \( \xi_i \) of the first \( k-1 \) of the participants uniformly and independently in \([0,1)\), and choose \( \xi_k \in [0,1) \) so that

\[
\xi_s = \xi_1 + \cdots + \xi_{k-1} + \xi_k \pmod{1}
\]

It is easy to check that \( \xi_s \) is independent of any set of \( k-1 \) shares, and, of course, all shares determine the secret uniquely. Just as in the finite case, this scheme can be used as a building block to create a perfect secret sharing when all qualified subsets are finite: simply distribute \( \xi_s \) for each qualified subset independently. In this case each participant will receive as many shares as many minimal qualified sets are in. In particular, if the scheme is based on a graph, then this number will be the degree of the vertex.

In case of finite complete graphs the above construction has extremely high information ratio. Sh a m i r ‘s construction from [19] is more efficient, it has the lowest possible information ratio 1. Can Shamir’s construction be generalized for the infinite case? The beginning is easy. Pick elements \( x_i \) of a field for each participant, and pick the point \( x_s \) for the secret (these values are public). Choose a secret linear function \( p(x) = ax + b \) according to a certain distribution. The value of the secret is \( \xi_s = p(x_s) \); and the shares are \( \xi_i = p(x_i) \). Clearly all pair of shares determine the function \( p \), thus the value \( \xi_s \). What is not clear, it is why \( \xi_s \) and \( \xi_i \) should be independent.
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Problem 6.2. Does there exist a distribution on the linear functions (a) over the reals, (b) over some appropriately chosen infinite field, so that \( \xi_s \) and \( \xi_i \) are independent random variables?

By the result of [6] the field cannot be countable. In the next problems “determines” can (and should) mean that \( \xi_s \) is determined uniquely with probability 1.

The existence of an “finite/infinite” threshold scheme seems to be a pure probability theoretical question.

Problem 6.3. Does there exist a perfect scheme where the secret is independent of any finite collection of the shares, but is determined by infinitely many of them? Or, at least, does there exist a ramp scheme where the secret is independent of any finite collection of the shares, but is determined by any cofinite collection (i.e., all but finitely many) of shares?

We return to schemes based on graphs. In almost all examples we have used Theorem 3.6, the generalization of Stinson’s decomposition theorem. Does it generalize for infinite schemes as well?

Problem 6.4. Suppose that \( G_i \) are (arbitrary) subgraphs of \( G \), and \( S_i \) is a perfect secret sharing scheme on \( G_i \). Moreover, assume that all edges of \( G \) are contained in at least one of the subgraphs. Does it follow that there is perfect secret sharing scheme on \( G \)?

The problem is that the secret in \( S_i \) might have arbitrary distribution, and it is not clear how to combine those distributions into a single variable.

We have seen two examples for non-local graphs: the comb, and the complete binary tree. Both of them have information ratio 2, but every finite spanned subgraph has smaller information ratio.

Problem 6.5. Is there any non-local graph with information ratio strictly below 2?

In fact, is the following stronger conjecture true:

Problem 6.6. Is it true that if \( R(G) < 2 \), then \( R(G) = 2 - 1/k \) for some integer \( k \)?

REFERENCES


Appendix

We determine the exact information ratio for the graphs $G_1$ and $G_2$ of Figure 9. $G_1$ is a spanned subgraph of the infinite ladder in Example 5.9 while $G_2$ is a spanned subgraph of the 2-lattice (but not of the ladder).

![Graphs G1 and G2](image)

Figure 9. The graphs $G_1$ and $G_2$. 
The information ratio of $G_1$ is $5/3$.

Proof. For the upper bound consider the following 3 covers of $G_1$. The first cover consists of the cycle $BbcCB$ (assigning 1 bit for each secret bit), plus the two edges $ab$ and $cd$. Using this cover nodes $b$ and $c$ get two bits, all other nodes get one bit. The second cover contains the star with rays $ba$, $bB$, $bc$, the path $Ccd$ and the edge $BC$. Here $a$, $b$, and $d$ get one bit, all other nodes get two bits. The third cover is the mirror image of the second one: the star $cb$, $cC$, $cD$, the path $abB$, and the edge $BC$. Using all three covers all edges are covered three times, and every node gets either three ($a$ and $d$), or five bits (all the rest).

For the lower bound we use the entropy method. Assume $f$ satisfies properties (a)–(e) listed at the end of Section 2. We claim that

$$f(b) + f(c) + f(C) \geq 5, \quad (4)$$

i.e., at least one of $b$, $c$ and $C$ gets $5/3$ bits for each bit in the secret.

First we give a strengthening of the usual proof that the information ratio of the path of length 3 is at least $3/2$. That proof goes by showing that $f(bc) \geq f(abcd) - f(ad) + 2$. As $abcd$ is qualified and $ad$ is not, $f(abcd) - f(ad) \geq 1$. That is, $f(b) + f(c) \geq f(bc) \geq 3$, therefore either $f(b)$ or $f(c)$ is $\geq 3/2$. Here we show that in this inequality $f(abcd)$ can be replaced by $f(acd)$:

\[
\begin{align*}
    f(a) + f(b) & \geq f(ab) \\
    f(ab) + f(bc) & \geq f(b) + f(abe) + 1 \\
    f(abe) & \geq f(ac) + 1 \\
    f(ac) + f(ad) & \geq f(a) + f(acd) \\
    f(bc) + f(ad) & \geq f(acd) + 2
\end{align*}
\]

Second, we take into account the vertices $B$ and $C$ as well:

\[
\begin{align*}
    f(c) + f(C) & \geq f(cC) \\
    f(cd) + f(cC) & \geq f(c) + f(cdC) + 1 \\
    f(acd) + f(cdC) & \geq f(cd) + f(acdC) \\
    f(acdC) & \geq f(adC) + 1 \\
    f(adC) + f(adB) & \geq f(ad) + f(adBC) \\
    f(adBC) & \geq f(adB) + 1 \\
    f(C) + f(acd) & \geq f(ad) + 3
\end{align*}
\]

As $f(b) + f(c) \geq f(bc)$, the sum of the two inequalities gives (4). □

Claim 6.8. $G_2$ has information ratio 2.
Proof. $R(G_2) \leq 2$ as $G_2$ is a spanned subgraph of the 2-lattice, and the 2-lattice has information ratio 2. On the other hand, let $f$ be again any function satisfying (a)–(e); we claim that

$$f(bc) + f(BC) \geq 8. \quad (5)$$

As $f(b) + b(c) + f(B) + f(C) \geq f(bc) + f(BC) \geq 8$, the lower bound 2 follows.

Each of the inequalities below is instances of one of the properties (a)–(e) of the function $f$:

- $f(a) + f(b) \geq f(ab)$
- $f(ab) + f(bc) \geq 1 + f(b) + f(abc)$
- $f(acAC) - f(acA) \geq f(acACD) - f(acAD) \geq 1$
- $f(acABC) - f(acAC) \geq 1$
- $f(ac) - f(a) \geq f(acB) - f(aB)$
- $f(acB) - f(aB) \geq 1 + f(acABC) - f(aABC)$
- $f(abc) - f(ac) \geq f(abcA) - f(acA)$
- $f(bc) \geq 4 + f(abcA) - f(aABC)$

Swapping lower case and upper case letters leaves the graph unchanged, thus we also have the “swapped” instance:

$$f(BC) \geq 4 + f(aABC) - f(abcA).$$

Adding these latter two inequalities we get (5), as required. □

Received July 31, 2007

Computer and Statistic Center
Central European University
Nádor útca 9
H–1051 Budapest
HUNGARY
E-mail: csirmaz@renyi.hu
Corrigendum to
Secret sharing on infinite graphs

László Csirmaz*
Central European University

Abstract
The proof of Claim 6.8 in the Appendix of [1] is incorrect. Here we give a new (and hopefully
correct) proof.

Key words. Secret sharing scheme, information theory, infinite graph, lattice.

1 Introduction
The proof of Claim 6.8 in the Appendix of [1] is incorrect. I am indebted to Prof. Hamiredza
Maimani [2] who called my attention to the error.

2 The new proof

Claim 2.1 The information ratio of the graph $G$ depicted on figure 1 is 2.

Figure 1: The graph $G$

Proof The proof of the first part of the claim, namely that $R(G) \leq 2$ was correct. $G$ is a spanned
subgraph of the 2-lattice, and the 2-lattice has information ratio 2. For proving the lower bound we
use the method outlined in the paper [1]. Let $f$ be any function satisfying the Shannon inequalities
(a)–(e) enlisted there, we claim that

$$f(bc) + f(BC) \geq 8.$$  \hspace{1cm} (1)

As $f(b) + b(c) + f(B) + f(C) \geq f(bc) + f(BC) \geq 8$, at least one of $f(b), f(c), f(B), \text{ and } f(C)$
must be $\geq 2$, thus the lower bound 2 follows.

To get inequality (1) we use instances of the Shannon inequalities (a)–(e) as follows:

$$f(a) + f(b) \geq f(ab)$$

$$f(ab) + f(bc) \geq 1 + f(b) + f(abc)$$

$$f(acBD) - f(acD) \geq f(acABD) - f(acAD) \geq 1$$

$$f(acBCD) - f(acBD) \geq 1$$

$$f(ac) - f(a) \geq f(acC) - f(aC)$$

$$f(acC) - f(aC) \geq 1 + f(acBCD) - f(aBCD)$$

$$f(abc) - f(ac) \geq f(abcD) - f(acD)$$

$$f(bc) \geq 4 + f(abcD) - f(aBCD).$$

*The author can be reached at csirmaz AT renyi DOT hu
Now the graph $G$ is invariant under the following permutation of the vertices: $a \leftrightarrow D$, $b \leftrightarrow C$, $c \leftrightarrow D$, $d \leftrightarrow A$, thus applying this transformation to the above inequality we get another valid inequality for our graph:

$$f(CB) \geq 4 + f(DCBA) - f(Dcba).$$

Adding these latter two inequalities we get (1), as required. □

References