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A NOTE TO THE SIERPIŃSKI FIRST CLASS OF FUNCTIONS

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ABSTRACT. The purpose of this paper is to establish some theorems concerning approximation and representation of a function of the Sierpiński first class by Darboux function of the Sierpiński first class.

In recent years, a number of articles have appeared dealing with a problem of approximating or representing arbitrary real valued functions of real variable by functions with the Darboux (intermediate value) property. In some cases, only functions belonging to a restricted class of functions such as the Baire one class [2], [3], [13] or the class of semicontinuous functions [6], [10], [12] were considered. We deal with the classes of real functions defined on the interval $I = [0,1]$. First, we introduce all relevant notations. Let C, B_1, D, Q, $\acute{S}_s$, lsc, and usc stand for the class of continuous, Baire one, Darboux, quasi-continuous, strong Świątkowski, lower and upper semicontinuous functions, respectively. The intersection $D \cap lsc$ will be denoted by $Dlsc$, and applying the same principle, we will use notations $DQlsc$, $\acute{S}_s lsc$, too. Let C_f be a set of all points of continuity of the function f, and D_f be a set of all points of discontinuity of the function f.

A point x is said to be a bilateral $\mathfrak{c}$-point of a set A if and only if $(x; x+\delta) \cap A$ and $(x-\delta; x) \cap A$ have the cardinality continuum for every $\delta > 0$, i.e.,

$$\operatorname{card}\big((x; x+\delta) \cap A\big) = \operatorname{card}\big((x-\delta; x) \cap A\big) = \mathfrak{c}.$$

The set A is said to be bilaterally $\mathfrak{c}$-dense in the set B, $B \subset_{\mathfrak{c}} A$, if and only if each point $x \in B$ is a bilateral $\mathfrak{c}$-point of the set A.

If a function $f\colon I \to \mathbb{R}$ maps connected sets onto connected sets, then it is said to be *Darboux*.

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Let us recall the notion of quasi-continuity which was introduced by S. Kempisty in 1932 [4]. A function f is said to be *quasi-continuous* at a point x_0 if for each open set $U \ni x_0$ and each open set $V \ni f(x_0)$ there exists a nonempty open set $W \subset U$ such that $f(W) \subset V$.

According to [8], a function f is *strong Świątkowski* if whenever $\alpha, \beta \in I$, $\alpha < \beta$ and $y \in \big(f(\alpha), f(\beta)\big)$, there exists a point $x_0 \in (\alpha, \beta) \cap C_f$ such that $f(x_0) = y$.

Now, we define the Sierpiński first class S_1 of functions [15]:

$$S_1 = \left\{ \sum_{n=1}^{\infty} f_n : \sum_{n=1}^{\infty} |f_n(x)| < \infty \text{ for every } x \in [0,1], \text{ with each } f_n \in C \right\}.$$

According to [16], the Sierpiński first class coincides with the class of sums of lower semicontinuous and upper semicontinuous functions. For this reason, the Sierpiński first class will be denoted either by S_1 or, in accordance with [11], by $lsc - lsc$. It is evident that $S_1 \subset B_1$ but $S_1 \subsetneqq B_1$ (see [11], [16]). To prove the main results of the paper, we will use of [10, Theorem 3] on approximation of semicontinuous function by Darboux semicontinuous function.

Theorem 1. *Let f be a function such that $f \in lsc$, and let E be an arbitrary F_σ set which is bilaterally $\mathfrak{c}$-dense in itself. If the set E is bilaterally $\mathfrak{c}$-dense in the set of points of discontinuity of the function f, then there exists a function $g \in Dlsc$ such that*

$$g(x) < f(x) \qquad \text{for} \quad x \in E,$$
$$g(x) = f(x) \qquad \text{for} \quad x \in I \setminus E.$$

Now, we prove the theorem on approximation by Darboux function in the Sierpiński first class.

Theorem 2. *Let $f \in S_1$, $f = l + u$, where $l \in lsc$ and $u \in usc$. If a Borel set E is bilaterally $\mathfrak{c}$-dense in the set of points of discontinuity of functions l and u, then there exists a function $g \in DS_1$ such that $\{x \colon f(x) \neq g(x)\} \subset E$.*

Proof. We denote $D^* = D_l \cup D_u$. The set D^* is of type F_σ, of first category, that is, $D^* = \bigcup_{n=1}^{\infty} D_n$, where $D_1 \subset D_2 \subset D_3 \subset \cdots$ are closed nowhere dense sets and, of course, $D_f \subset D^*$. Additionally, according to [13, Lemma 7], it can be assumed that the set E is of type F_σ, of the first category, bilaterally $\mathfrak{c}$-dense in itself and $\bigcup_{n=1}^{\infty} D_n \subset_{\mathfrak{c}} E$. Otherwise, the set E can be replaced with its subset having these properties. Thus, let

$$E = \bigcup_{n=1}^{\infty} F_n,$$

where each F_n is a nowhere dense closed set. Without loss of generality, it can be assumed that $F_1 \subset F_2 \subset \cdots$ As $l \in lsc$, there exists a sequence of continuous functions

$$l_1 < l_2 < l_3 < \cdots$$

which pointwise converges to the function l and, analogously, as $u \in usc$, there exists a sequence of continuous functions

$$u_1 > u_2 > u_3 > \cdots$$

which pointwise converges to the function u. Evidently, the sequence

$$f_n = l_n + u_n, \qquad n \in \mathbb{N},$$

pointwise converges to the function f. Moreover, let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$. Functions l_n, u_n, f_n are uniformly continuous on $[0,1]$. Thus, the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ determines a sequence of positive numbers δ_n such that for every $x_1, x_2 \in [0,1]$

$$\begin{aligned} |x_1 - x_2| < \delta_n \Rightarrow\ & |\, l_n(x_1) - l_n(x_2)| < \varepsilon_n, \\ & |u_n(x_1) - u_n(x_2)| < \varepsilon_n, \\ & |f_n(x_1) - f_n(x_2)| < \varepsilon_n. \end{aligned} \tag{0.1}$$

Applying of [9, Lemma 2], we can construct a sequence of perfect sets P_n, $n \in \mathbb{N}$, such that

$$F_1 \subset_{\mathfrak{c}} P_1 \subset E$$

and

$$(F_{n+1} \cup P_n) \subset_{\mathfrak{c}} P_{n+1} \subset E \qquad \text{for every} \quad n \in \mathbb{N}.$$

It is evident that

$$P_1 \subset_{\mathfrak{c}} P_2 \subset_{\mathfrak{c}} P_3 \subset_{\mathfrak{c}} \cdots, \qquad E = \bigcup_{i=1}^{\infty} P_i.$$

Let for each $n \in \mathbb{N}$ the triple of functions (l_n, u_n, f_n) be associated with the perfect set P_n. The set E is $\mathfrak{c}$-dense in D^* and therefore F_n can be required to satisfy condition

$$\forall x \in D_n \quad \text{there is} \quad a, b \in F_n \subset P_n \quad \text{such that} \quad a < x < b \wedge b - a < \delta_n. \tag{0.2}$$

From the existence of a system of perfect sets P_n, $n = 1, 2, \ldots$, it follows that there exists a system of closed sets $P_\alpha, \alpha \geq 1$ (proof in [13, Theorem 8]) such that if

$$\alpha_1 < \alpha_2 \Rightarrow P_{\alpha_1} \subset_{\mathfrak{c}} P_{\alpha_2}.$$

Now, for each $i \leq \alpha < i+1$, we define a triple of functions $(l_\alpha, u_\alpha, f_\alpha)$,

$$\begin{aligned} l_\alpha &= l_i + (\alpha - i)\,(l_{i+1} - l_i)\,, \\ u_\alpha &= u_i + (\alpha - i)\,(u_{i+1} - u_i)\,, \\ f_\alpha &= f_i + (\alpha - i)\,(f_{i+1} - f_i)\,, \end{aligned}$$

associated with the set P_α.

It is easy to see that if

$$\alpha_1 < \alpha_2 \Rightarrow \begin{array}{rcl} l_{\alpha_1} & < & l_{\alpha_2}, \\ u_{\alpha_1} & > & u_{\alpha_2} \end{array}$$

and

$$f_\alpha = l_\alpha + u_\alpha \qquad \text{for each} \quad \alpha \geq 1.$$

Let $\alpha(x)$ have the same sense as in the proof of [10, Theorem 3] if:

$$x \in E \Rightarrow \alpha(x) = \inf\{\alpha \colon x \in P_\alpha\}.$$

Now, we define the functions l^*, u^*, f^* in the way as in the proof in [10, Theorem 3]:

$$l^*(x) = \begin{cases} l_{\alpha(x)}(x), & \text{if } x \in E, \\ l(x), & \text{if } x \notin E, \end{cases}$$

$$u^*(x) = \begin{cases} u_{\alpha(x)}(x), & \text{if } x \in E, \\ u(x), & \text{if } x \notin E, \end{cases}$$

$$f^*(x) = \begin{cases} f_{\alpha(x)}(x), & \text{if } x \in E, \\ f(x), & \text{if } x \notin E. \end{cases}$$

The function l^* fulfills identical conditions as the function g in the proof of [10, Theorem 3], that is

$$l^* \in Dlsc \quad \text{and} \quad \begin{array}{l} l^* < l \ \text{ on } E \\ l^* = l \ \text{ on } [0,1] \setminus E. \end{array}$$

A theorem analogous with Theorem 1 can be formulated for the function $u \in usc$, too. Therefore, the function

$$u^* \in Dusc \quad \text{and} \quad \begin{array}{l} u^* > u \ \text{ on } E, \\ u^* = u \ \text{ on } [0,1] \setminus E. \end{array}$$

Moreover, the function f^* is constructed in the same way as the function g in [13, Theorem 8]. According to the assertion of this theorem,

$$f^* \in DB_1 \quad \text{and} \quad \big\{x \colon f^*(x) \neq f(x)\big\} \subset E.$$

Because $f_\alpha = l_\alpha + u_\alpha$ for every $\alpha \geq 1$, from the construction of functions l^*, u^*, f^* it follows that

$$f^* = l^* + u^* \in S_1 \subset B_1.$$

Using the last two results, we show that the function

$$f^* \in DB_1 \cap S_1 = DS_1 \quad \text{and for} \quad \big\{x \colon f^*(x) \neq f(x)\big\} \subset E,$$

that is, the function f^* satisfies the assertion of theorem. □

The next theorem proves that every Sierpiński one function can be represented as a sum of two Sierpiński one functions with Darboux property.

Theorem 3. *Let f be a Sierpínski one function on an interval I, $f = l + u$, where $l \in lsc$ and $u \in usc$. If a Borel set $E \subset C_l \cap C_u$ is bilaterally $\mathfrak{c}$-dense in the set of points of discontinuity of the functions l and u, then there exists a function $g \in DS_1$ such that for $\{x\colon f(x) \neq g(x)\} \subset E$ and the function $f - g \in DS_1$.*

Proof. Again, by [13, Lemma 7], there exists a set $E^* \subset E \subset C_f$ of type F_σ of the first category, bilaterally $\mathfrak{c}$-dense in itself such that $D_l \cup D_u \subset_{\mathfrak{c}} E^*$. Theorem 2 implies the existence of a function $g \in DS_1$, $g = l^* + u^*$, where $l^* \in lsc$ and $u^* \in usc$ such that the set for $\{x\colon f(x) \neq g(x)\} \subset E^*$. Because the function

$$f - g = (l + u) - (l^* + u^*) = (l - u^*) + (u - l^*) \in S_1,$$

it suffices to prove that the function $f-g$ has Darboux property. Let us consider an arbitrary point $x_0 \in I$.

If $f(x_0) = g(x_0)$, then since the set for $\{x : f(x) = g(x)\}$ is residual in I, there exist sequences $x_n \nearrow x_0$, $y_n \searrow x_0$, $n \in \mathbb{N}$, $f(x_n) = g(x_n)$, $f(y_n) = g(y_n)$. Thus,

$$\lim_{n\to\infty} (f-g)(x_n) = \lim_{n\to\infty} (f-g)(y_n) = (f-g)(x_0) = 0.$$

If $f(x_0) \neq g(x_0)$, then $x_0 \in C_f$. The function $g \in DS_1$, hence there exist sequences $x_n \nearrow x_0$, $y_n \searrow x_0$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(y_n) = g(x_0).$$

The function f is continuous at the point x_0. Then,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(x_0).$$

From the foregoing, it follows that

$$\lim_{n\to\infty} (f-g)(x_n) = \lim_{n\to\infty} (f-g)(y_n) = (f-g)(x_0).$$

Following Young in [1], $f - g \in DB_1$.

From the above, it follows:

$$f - g \in DB_1 \cap S_1 = DS_1 \quad \text{and for} \quad \{x; g(x) \neq f(x)\} \subset E.$$

□

Of course, if we choose a first Baire category set E of Lebesque measure zero from Theorem 3, then there exists a Darboux Sierpiński one function g such that

$$f = g + (f - g), \quad \text{where} \quad f - g \in DS_1$$

and $f - g = 0$ except for on the first category measure zero set.

On the other hand, if we do not require validity of $f - g = 0$ except for on the measure zero set, then we can show that every Sierpiński one function can be expressed as a sum of two Sierpiński one functions with a stronger property than the Darboux property. Based on results of A. Maliszewski [7],

R. M e n k y n a [14] characterized the class of differences of lower semicontinuous strong Świątkowski functions by equations

$$(lsc - lsc) \cap DQ = lscDQ - lscDQ = \acute{S}_s lsc - \acute{S}_s lsc.$$

Additionally, in [12], it is proved that

$$lsc = \acute{S}_s lsc + \acute{S}_s lsc \text{ or } usc = \acute{S}_s usc + \acute{S}_s usc, \quad \text{respectively.}$$

If we use the validity of the above relations, we can prove the following statement:

Theorem 4. *Let f be a Sierpiński one function on an interval I. Then, there exist Darboux quasi-continuous Sierpiński one functions f_1, f_2 defined on the interval I such that $f = f_1 + f_2$.*

P r o o f. Because f is the Sierpiński one function, there exists a lower semicontinuous function l and a upper semicontinuous function u such that $f = l + u$. In accord with [12, Theorem 8], there exist lower semicontinuous strong Świątkowski functions l_1, l_2 and upper semicontinuous strong Świątkowski functions u_1, u_2 such that

$$l = l_1 + l_2 \quad \text{and} \quad u = u_1 + u_2.$$

Therefore,

$$f = (l_1 + l_2) + (u_1 + u_2) = \big(l_1 - (-u_1)\big) + \big(l_2 - (-u_2)\big).$$

The function

$$f_1 = l_1 - (-u_1) \in \acute{S}_s lsc - \acute{S}_s lsc = lscDQ - lscDQ = (lsc - lsc) \cap DQ = DQS_1$$

and for the same reason, the function

$$f_2 = l_2 - (-u_2) \in DQS_1.$$

We have shown that any Sierpiński one function can be written as a sum of two Darboux quasi-continuous Sierpiński one functions. □

In addition, our goal is to prove that every Sierpiński one function can be expressed as a sum of two strong Świątkowski functions of Sierpiński one class. By small modifications to the proof of Lemma A.3.2 and to the proof of the implication $ii \Rightarrow iii$ in the proof of [5, Theorem A.3.3], we can prove the folowing lemmas. Naturally, let symbols $\underline{\lim}\,(f,x)$, $\overline{\lim}\,(f,x)$, $\|g\|$, $\omega\,(f,x)$ have the same meaning as in [5].

Lemma 5. *Assume that K is nowhere dense and closed, the functions $f_1, f_2, \dots$ $\dots, f_m \in DQB_1$ are locally bounded on $\mathbb{R} \setminus K$, and $\tau > 0$. There is a nowhere dense closed set $F \supset K$ such that $F \setminus K \subset \cap_{j=1}^{m} C_{f_j}$, and continuous function g, $g = 0$ on K, $\|g\| \le \tau$ such that*

$$(f_j + g)\,(I \cap F \setminus K) \supset \big[\underline{\lim}\,(f_j, x)\,, \overline{\lim}\,(f_j, x)\big]$$

for every $x \in K$, $j = 1, 2, \dots, m$ and every open interval $I \ni x$.

P r o o f. Write $\mathbb{R}\setminus K$ as the union of a sequence of compact intervals $\{I_n,\, n \in \mathbb{N}\}$ with boundary points from $\cap_{j=1}^{m} C_{f_j}$ such that $int(I_n) \cap int(I_m) = \emptyset$, for every $n \neq m$. For each n, put $c_n^j = \inf\, f_j\,(I_n)$ and $L_n^j = \sup\, f_j\,(I_n)$. Let $\{J_p,\, p \in \mathbb{N}\}$ be an enumeration of all open intervals with rational endpoints which intersect K. (The lemma is trivial in case $K = \emptyset$.) Fix $p \in \mathbb{N}$. First, observe that for each $k_0 \in \mathbb{N}$, $j = 1, 2, \ldots, m$

$$\bigcup_{n>k_0, I_n \subset J_p} \left(c_n^j - \frac{\tau}{2^p}, L_n^j + \frac{\tau}{2^p}\right) \supset cl \bigcup_{x \in K \cap J_p} \left[\underline{\lim}\,(f_j, x)\,, \overline{\lim}\,(f_j, x)\right]. \qquad (0.3)$$

Indeed, let y belong to the set on the right-hand side of (0.3). There are $x \in K \cap J_p$ and $y' \in \left[\underline{\lim}\,(f_j, x)\,, \overline{\lim}\,(f_j, x)\right]$ such that $|y - y'| < \frac{\tau}{2^{p+1}}$. Choose $n > k_0$ and $x' \in I_n$ such that $I_n \subset J_p$ and $|f_j\,(x') - y'| < \frac{\tau}{2^{p+1}}$. Then clearly, $y \in \left(c_n^j - \frac{\tau}{2^p}, L_n^j + \frac{\tau}{2^p}\right)$.

Set $n_0^m = 0$. By (0.3), there is $n_p^m > \cdots > n_p^2 > n_p^1 > n_{p-1}^m$ such that

$$\bigcup_{I_n \subset J_p,\, n_{p-1}^m < n \le n_p^1} \left(c_n^1 - \frac{\tau}{2^p}, L_n^1 + \frac{\tau}{2^p}\right) \supset \bigcup_{x \in K \cap J_p} \left[\underline{\lim}\,(f_1, x)\,, \overline{\lim}\,(f_1, x)\right] \cap [-p, p]$$

and

$$\bigcup_{I_n \subset J_p,\, n_p^{j-1} < n \le n_p^j} \left(c_n^j - \frac{\tau}{2^p}, L_n^j + \frac{\tau}{2^p}\right) \supset \bigcup_{x \in K \cap J_p} \left[\underline{\lim}\,(f_j, x)\,, \overline{\lim}\,(f_j, x)\right] \cap [-p, p]$$

for $j = 2, \ldots, m$. Define $\varepsilon_n = \frac{\tau}{2^p}$ for $n \in \left\{n_p^1, \ldots, n_p^m\right\}$.

For each $n = n_p^j$, apply [5, Lemma A.3.1] to find a nowhere dense closed set $F_n \subset I_n \cap \cap_{j=1}^{m} C_{f_j}$ and a continuous function g_n such that $g_n = 0$ outside of I_n, $\|g_n\| \le 2\varepsilon_n$, and

$$(f_j + g_n)\,(F_n) \supset \left[c_n^j - \varepsilon_n, L_n^j + \varepsilon_n\right]. \qquad (0.4)$$

Define $F = K \cup \bigcup_{n \in \mathbb{N}} F_n$ and $g = \sum_{n \in \mathbb{N}} g_n$. Note that F is nowhere dense and closed, and since $\varepsilon_n \to 0$, so g is continuous.

Let $x \in K$, $I \ni x$ be an open interval, $j \in \{1, 2, \ldots, m\}$ and $y \in \left[\underline{\lim}\,(f_j, x), \overline{\lim}\,(f_j, x)\right]$. There exists $p > |y|$ such that $x \in J_p \subset I$. For $n = n_p^j$, the interval $I_n \subset J_p$ and $y \in \left(c_n^j - \frac{\tau}{2^p}, L_n^j + \frac{\tau}{2^p}\right)$. Then, $y \in \left(c_n^j - \varepsilon_n, L_n^j + \varepsilon_n\right)$, and by (0.4), there is $t \in F_n \subset F$ with $(f_j + g)\,(t) = y$. The other requirements are easy to prove. □

Lemma 6. *Let functions $f_1, f_2, \ldots, f_m \in DQB_1$, and let $\varepsilon > 0$ be arbitrary real number. Then, there is a continuous function g, $\|g\| \le \varepsilon$ such that $f_j + g \in \acute{S}_s B_1$ for every $j = 1, 2, \ldots, m$.*

P r o o f. Set $g_0 = 0$ and $F_0 = \emptyset$. We will procced by induction. Fix $n \in \mathbb{N}$ and assume that we have defined a closed and nowhere dense set F_{n-1}. Define $K_n = F_{n-1} \cup \cup_{j=1}^{m} \left\{x \in \mathbb{R} \colon \omega\,(f_j, x) \ge n^{-1}\right\}$. Then, K_n is closed and nowhere dense. Use Lemma 5 to find a nowhere dense closed set $F_n \supset K_n$ and a continuous function g_n such that

$$F_n \setminus K_n \subset \cap_{j=1}^{m} C_{f_j},\ g_n = g_{n-1} \quad \text{on} \quad K_n,\ \|g_n - g_{n-1}\| \leq \frac{\varepsilon}{2^n},$$

and

$$(f_j + g_n)(I \cap F_n \setminus K_n) \supset \left[\underline{\lim}\,(f_j + g_{n-1}, x)\,, \overline{\lim}\,(f_j + g_{n-1}, x)\right] \tag{0.5}$$

for every $x \in K_n$, $j = 1, 2, \ldots, m$ and every open interval $I \ni x$.

Define $g = \lim_{n\to\infty} g_n$ and $h_j = f_j + g$. Then clearly, g is continuous and $\|g\| \leq \varepsilon$. To prove that h_j is strong Świątkowski, we will use [5, Proposition 1.3.18]. Take $x \in D_{h_j} = D_{f_j}$ and an open interval $I \ni x$. There is $n \in \mathbb{N}$ with $x \in K_n$. By (0.5), we obtain

$$\begin{aligned} h_j\left(I \cap C_{h_j}\right) &= h_j\left(I \cap C_{f_j}\right) \supset h_j\left(I \cap F_n \setminus K_n\right) \\ &= (f_j + g_{n+1})(I \cap F_n \setminus K_n) \supset \left[\underline{\lim}\,(f_j + g_n, x)\,, \overline{\lim}\,(f_j + g_n, x)\right] \\ &= \left[\underline{\lim}\,(h_j, x)\,, \overline{\lim}\,(h_j, x)\right]. \end{aligned}$$

(We used that $g_p = g_{n+1}$ on F_n for $p > n$ and $g_p(x) = g_n(x)$ for $p \geq n$.) So, by [5, Proposition I.3.18], h_j is strong Światkowski. □

THEOREM 7. *Let f be a Sierpiński one function. Then, there are strong Świątkowski Sierpiński one functions g and h such that $f = g + h$.*

P r o o f. If f is a Sierpiński one function on an interval I, according to Theorem 4, there exist Darboux quasi-continuous Sierpiński one functions f_1, f_2 defined on the interval I such that $f = f_1 + f_2$. Since $S_1 \subset B_1$, by Lemma 6, there exists a continuous function g such that $f_1 + g$ and $-f_2 + g$ are strong Świątkowski functions. Moreover, because $S_1 + C = (lsc - lsc) + C = S_1$, the functions $f_1 + g$ and $-f_2 + g$ are strong Świątkowski Sierpiński one functions. Therefore, the function $f_2 - g$ is again strong Świątkowski Sierpiński one function, the function f can be represented as a sum of two strong Świątkowski Sierpiński one functions $f = (f_1 + g) + (f_2 - g)$. □

REFERENCES

[1] BRUCKNER, A. M.: *Differentiation of Real Functions*, in: Lecture Notes in Math., Vol. 659, Springer, Berlin, 1978.

[2] BRUCKNER, A. M.—CEDER, J. G.—KESTON, R.: *Representations and approximations by Darboux functions in the first class of Baire*, Rev. Roum. Math. Pures Appl. **9** (1968), 1247–1254.

[3] GUREVIČ, A. B.: *D-continuous Sierpiński components,* Dokl. Akad. Nauk BSSR **10** (1966), 539–571. (In Russian)

[4] KEMPISTY, S.: *Sur les functions quasicontinues*, Fund. Math. **19** (1932), 184–197.

[5] MALISZEWSKI, A.: *Darboux Property and Quasi-Continuity: A Uniform Approach.* Wydaw. Uczelniane WSP, 1996.

[6] MALISZEWSKI, A.: *On the sums of Darboux upper semicontinuous quasi-continuous functions,* Real Anal. Exchange **20** (1994–1995), 244–249.

[7] MALISZEWSKI, A.: *On the differences of upper semicontinuous quasi-continuous functions*, Math. Slovaca **48** (1998), 245–252.

[8] MALISZEWSKI, A.: *On the limits of strong Świątkowski functions*, Zeszyty Nauk. Politech. Łódz. Mat. **27** (1995), 87–93.

[9] MENKYNA, R.: *On approximations of semicontinuous functions by Darboux semicontinuous functions*, Real Anal. Exchange **35** (2009–2010), 423–430.

[10] MENKYNA, R.: *On the differences of lower semicontinuous functions,* Real Anal. Exchange **41** (2015–2016), 123–136.

[11] MENKYNA, R.: *On representations of Baire one functions as the sum of lower and upper semicontinuous functions*, Real Anal. Exchange **38** (2012–2013), 169–176.

[12] MENKYNA, R.: *On the sums of lower semicontinuous strong Swiątkowski functions*, Real. Anal. Exchange **39** (2013–2014), 15–32.

[13] MENKYNA, R.—MYDIELKA, Ľ.: *Approximations by Darboux functions in the Baire one class,* Tatra Mt. Math. Publ. **55** (2013), 57–66.

[14] MENKYNA, R.: *Difference of two strong Świątkowski lower semicontinuous functions*, Math. Slovaca **67** (2017), 99–108.

[15] MORAYNE, M.: *Sierpiński's hierarchy and locally Lipschitz functions,* Fund. Math. **147** (1995), 73–82.

[16] SIERPIŃSKI, W.: *Sur les fonctions développables en séries absolument convergentes de fonctions continues*, Fund. Math. **2** (1921), 15–27.

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