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COMPARISON OF SOME FAMILIES OF REAL FUNCTIONS IN ALGEBRAIC TERMS

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ABSTRACT. We compare families of functions related to the Darboux property (functions having the $\mathcal{A}$-Darboux property) with family of strong Świątkowski functions using the notions of strong $\mathfrak{c}$-algebrability. We also compare families of functions associated with density topologies.

1. Introduction

We will work with real functions defined on the real line, so if we write "function", then we mean $f\colon \mathbb{R} \to \mathbb{R}$. Throught the whole paper, when open or closed sets are mentioned, we mean the Euclidean topology in $\mathbb{R}$. We denote by $\overline{A}$ ($Int\,(A)$) the closure (interior) of the set A.

Recall that a function f is quasi-continuous at a point x if for every neighbourhood U of x and for every neighbourhood V of $f(x)$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$. A function f is *quasi-continuous* (briefly $f \in \mathcal{Q}$) if it is quasi-continuous at each point. The notion of quasi-continuity was introduced by S. Kempisty in 1932 [12].

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called a *strong Świątkowski function* if for each interval $(a,b) \subset \mathbb{R}$ and for each λ between $f(a)$ and $f(b)$ there exists a point $x \in (a,b)$ such that $f(x) = \lambda$ and f is continuous at x [17]. Hence, the family $\mathcal{D}_s$ of strong Świątkowski functions is situated somewhere in the road between the family $\mathcal{C}$ of all continuous functions and the family $\mathcal{DQ}$ of all Darboux quasi-continuous functions.

The notion of strong Świątkowski property was introduced by A. Maliszewski in [17]. He showed that $\mathcal{D}_s$ is dense in $\mathcal{DQ}$, so $\mathcal{D}_s$ is "not so far" from $\mathcal{DQ}$ (for more information about this class, see [16]–[19]).

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On the other hand, J. Wódka [27] proved that the set $(\mathcal{DQ} \setminus \mathcal{D}_s) \cup \{\Theta\}$ contains a $\mathfrak{c}$-generated algebra, i.e., $\mathcal{DQ}$ is much larger than $\mathcal{D}_s$ if we compare these families in algebraic terms.

In this paper, we will generalize the result obtained by J. Wódka. We will compare, in algebraic terms, $\mathcal{D}_s$ and $\mathcal{DQ}$ with the family $\mathcal{D}_{ap}$ of approximately Darboux functions and the family $\mathcal{D}_{\mathcal{I}-ap}$ of $\mathcal{I}$-approximately Darboux functions. We will use the exponential-like method introduced in [2].

Let $\mathcal{B}r$ denotes a family of all sets having the Baire property, and $n \cdot A$ stands for a set $\{n \cdot a \colon a \in A\}$. We will write $< a, b >$ instead of $\big(\min\{a,b\}, \max\{a,b\}\big)$.

2. $\mathcal{A}$-continuity and $\mathcal{A}$-Darboux property

We start with the definition which helps us describe approximate continuity and $\mathcal{I}$-approximate continuity as well as quasi-continuity in the same way.

Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is a power set of $\mathbb{R}$.

Definition 1 ([10]). We say that f is $\mathcal{A}$-continuous at the point $x \in \mathbb{R}$ if for each open set $V \subset \mathbb{R}$ with $f(x) \in V$ there exists a set $A \in \mathcal{A}$ such that $x \in A$ and $f(A) \subset V$. We say that f is $\mathcal{A}$-continuous ($f \in \mathcal{C}_{\mathcal{A}}$) if f is $\mathcal{A}$-continuous at each point $x \in \mathbb{R}$.

Remind that x is a density point of a measurable set $A \subset \mathbb{R}$ when

$$\lim_{h \to 0^+} \frac{m\left(A \cap [x-h, x+h]\right)}{2h} = 1$$

and the family of all measurable sets which any point is its density point is a topology called the density topology.

Now, let $\mathcal{I}$ be a σ-ideal of sets of the first category. We will say that a property holds $\mathcal{I}$-almost everywhere (briefly $\mathcal{I}$-a.e.) if the set of all points which do not have this property belongs to $\mathcal{I}$.

We will say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions with the Baire property converges with respect to $\mathcal{I}$ to some real function f with the Baire property ($f_n \xrightarrow[n\to\infty]{\mathcal{I}} f$) if every subsequence $\{f_{m_n}\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ contains a subsubsequence $\{f_{m_{p_n}}\}_{n\in\mathbb{N}}$ which converges to f $\mathcal{I}$-a.e. (see [21]).

The point 0 is an $\mathcal{I}$-density point of A if

$$\chi_{(n\cdot A)\cap(-1,1)} \xrightarrow[n\to\infty]{\mathcal{I}} \chi_{(-1,1)}.$$

We say that x is an $\mathcal{I}$-density point of A if 0 is an $\mathcal{I}$-density point of $A - x = \{a - x \colon a \in A\}$. Put $\Phi_{\mathcal{I}}(A) = \{x \in \mathbb{R} \colon x \text{ is an } \mathcal{I}\text{-density point of } A\}$.

The family $\tau_{\mathcal{I}} = \big\{A \subset \mathbb{R} \colon A \in \mathcal{B}r \wedge A \subset \Phi_{\mathcal{I}}(A)\big\}$, called the $\mathcal{I}$-density topology, was studied in [21], [22] and [25].

A set $A \subset \mathbb{R}$ is said to be *semi-open* if there is an open set U such that $U \subset A \subset \overline{U}$ (see [15]). It is not difficult to see that A is semi-open if and only if $A \subset \overline{Int\,(A)}$. The family of all semi-open sets will be denoted by $\mathcal{S}$. A function f is semi-continuous if for each open set V the set $f^{-1}(V)$ is semi-open [15].

In [20], A. Neubrunnová proved that f is semi-continuous if and only if it is quasi-continuous.

Now, we are in the position to describe all the mentioned classes of functions using the notion of $\mathcal{A}$-continuity.

Remark 2. $\mathcal{A}$-continuity coincides with:

(1) classical continuity whenever $\mathcal{A} = \tau_e$;

(2) approximate continuity whenever $\mathcal{A} = \tau_d$;

(3) $\mathcal{I}$-approximate continuity whenever $\mathcal{A} = \tau_{d_{\mathcal{I}}}$;

(4) quasi-continuity whenever $\mathcal{A} = \mathcal{S}$.

Definition 3 ([10])**.** We will say that f has the $\mathcal{A}$-Darboux property ($f \in \mathcal{D}_{\mathcal{A}}$) if for each interval $(a, b) \subset \mathbb{R}$ and each $\lambda \in< f(a), f(b) >$ there exists a point $x \in (a, b)$ such that $f(x) = \lambda$ and f is $\mathcal{A}$-continuous at x.

If $\mathcal{A} = \mathcal{P}(\mathbb{R})$ then $\mathcal{D}_{\mathcal{A}}$ is a family $\mathcal{D}$ of all Darboux functions. If $\mathcal{A}$ is the Euclidean topology τ_e, then $\mathcal{D}_{\mathcal{A}}$ is a family $\mathcal{D}_s$ of functions having strong Świątkowski property. If $\mathcal{A}$ is the density topology τ_d, then $\mathcal{D}_{\mathcal{A}}$ is a family $\mathcal{D}_{ap}$ of functions with the so called ap-Darboux property introduced by Z. Grande in [6]. If $\mathcal{A}$ is the $\mathcal{I}$-density topology, then $\mathcal{D}_{\mathcal{A}}$ is a family of $\mathcal{I}$-ap Darboux functions investigated by G. Ivanova and E. Wagner-Bojakowska in [7] and [9]. In [10], it is shown that if $\mathcal{A} = \mathcal{S}$, then $\mathcal{D}_{\mathcal{A}} = \mathcal{DQ}$.

It is easy to see that

$$\mathcal{D}_s \subset \mathcal{D}_{ap} \cap \mathcal{D}_{\mathcal{I}-ap} \subset \mathcal{D}_{ap} \cup \mathcal{D}_{\mathcal{I}-ap} \subset \mathcal{D}.$$

In [7], it is proved that all these inclusions are proper.

In [10], it is proved that for some families $\mathcal{A}$ we also have $\mathcal{D}_{\mathcal{A}} \subset \mathcal{DQ}$. Let us briefly describe these results. We say that the set A is of the *first category at the point* x if there exists an open neighbourhood G of x such that $A \cap G$ is of the first category (see [13]). $D(A)$ will denote the set of all points x such that A is not of the first category at x.

Definition 4 ([10])**.** We will say that a family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ has the (*)-property, if

(1) $\tau_e \subset \mathcal{A} \subset \mathcal{B}r$;

(2) $A \subset D(A)$ for each $A \in \mathcal{A}$.

It is not difficult to see that a wide class of topologies has the (*)-property. For example, the Euclidean topology, $\mathcal{I}$-density topology, topologies constructed in [14] by E. Łazarow, R. A. Johnson and W. Wilczyński or the topology constructed by R. Wiertelak in [23]. Certain families of sets, which are not topologies, have the (*)-property: the family of semi-open sets is a good example. On the other hand, the density topology does not have this property.

3. Algebrability

To compare families $\mathcal{D}_s$, $\mathcal{D}\mathcal{Q}$, $\mathcal{D}_{ap}$ and $\mathcal{D}_{\mathcal{I}-ap}$ in algebraic terms, we need some definitions.

Definition 5 ([3])**.** Let $\mathcal{L}$ be a linear commutative algebra. We say that $A \subset \mathcal{L}$ is strongly $\mathfrak{c}$-algebrable if $A \cup \{\Theta\}$ contains a $\mathfrak{c}$-generated algebra B that is isomorphic with a free algebra. We denote by $X = \{x_\alpha \colon \alpha < \mathfrak{c}\}$ the set of generators of this free algebra.

Let us remark that the set $X = \{x_\alpha \colon \alpha < \mathfrak{c}\}$ is the set of generators of some free algebra contained in $A \cup \{\Theta\}$ if and only if the set $\widetilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \dots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from $\widetilde{X}$ are in $A \cup \{\Theta\}$.

In [2] there was presented a useful method of proving that a fixed family is strongly $\mathfrak{c}$-algebrable. We say that a function f is *exponential-like* (briefly $f \in \mathcal{E}$) whenever f is given by the formula

$$f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x},$$

for some distinct nonzero real numbers $\beta_1, \dots, \beta_m$ and some nonzero real numbers $a_1, \dots, a_m$ (see [1]).

It is not difficult to check that:

Lemma 6 ([1])**.** *For every positive integer m, any exponential-like function f, and each $c \in \mathbb{R}$, the preimage $f^{-1}[\{c\}]$ is finite. Consequently, f is not constant in any subinterval of $\mathbb{R}$. In particular, there exists a decomposition of $\mathbb{R}$ to a finite number of intervals such that the function f is strictly monotone on each of them.*

Exponential-like functions may be used to prove $\mathfrak{c}$-algebrability of a fixed family of functions in the following way:

Theorem 7 ([1])**.** *Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ and assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{\Theta\}$ for every exponential-like function f [1]. Then, $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable.*

4. Comparison of classes related to Darboux property

In [8], it is shown that the family $\mathcal{DQ}$ is strongly porous (so, it is very small from the topological point of view) in the family $\mathcal{DB}a$ of Darboux functions having the Baire property. Therefore, $\mathcal{DB}a \setminus \mathcal{DQ}$ is "topologically large". Let us show that it is large also in algebraic terms. For this purpose, we need the following lemma:

Lemma 8 ([11]). *There exists a Darboux Baire 1 function* $F\colon \mathbb{R} \xrightarrow{onto} [0,1]$ *such that*

(1) F *vanishes* $\mathcal{I}$*-a.e. on* $(0,1)$*;*

(2) F *vanishes on* $\mathbb{R}\setminus(0,1)$*;*

(3) $[0,1] \setminus F^{-1}(\{0\})$ *is a first category set dense in* $[0,1]$.

Indeed, let $\{C_n\}$ be a sequence of pairwise disjoint closed and nowhere dense subsets of $[0,1]$ of cardinality continuum such that for each interval $(a,b) \subset [0,1]$ there exists $n \in \mathbb{N}$ with $C_n \subset (a,b)$. Obviously, the set $C = \bigcup_{n\in\mathbb{N}} C_n$ is of type F_σ and is bilaterally $\mathfrak{c}$-dense-in-itself. Therefore, (see [4]) there exists a function $F \in \mathcal{DB}_1$ such that $F(x) = 0$ if $x \notin C$ and $0 < F(x) \leq 1$ for $x \in C$ (for details, see [11]).

Using this function, we can prove strong $\mathfrak{c}$-algebrability of the family $\mathcal{DB}a \setminus \mathcal{DQ}$:

Theorem 9. *The family* $\mathcal{DB}a \setminus \mathcal{DQ}$ *is strongly* $\mathfrak{c}$*-algebrable.*

P r o o f. Let F be the function defined above. Fix an exponential-like function f. By Theorem 7, to prove that $\mathcal{DB}a \setminus \mathcal{DQ}$ is strongly $\mathfrak{c}$-algebrable, it is suffices to show that $f \circ F \in \mathcal{DB}a \setminus \mathcal{DQ}$.

As f is continuous and $F \in \mathcal{DB}_1$, $f \circ F \in \mathcal{DB}a$. To show that $f \circ F$ is not quasi--continuous, fix an open set V such that $0 \notin f^{-1}(V)$ and $V \cap (f \circ F)(\mathbb{R}) \neq \emptyset$. The set $F^{-1}\left(f^{-1}(V)\right)$ is a nonempty set of the first category, so it is not semi--open, and due to A. N e u b r u n n o v á's result [20], the function $f \circ F$ is not quasi-continuous. □

Put $\mathcal{P} := \{\mathcal{D}, \mathcal{DB}a\}$, and $\mathcal{P}_\mathcal{A} := \{\mathcal{D}_\mathcal{A} : \mathcal{A} \text{ has the (*)-property}\}$. As for each $\mathcal{A}$ with the (*)-property, the family $\mathcal{D}_\mathcal{A}$ is contained in $\mathcal{DQ}$ (see [10]), it is easy to see that for any $\mathcal{F}_1 \in \mathcal{P}$ and $\mathcal{F}_2 \in \mathcal{P}_\mathcal{A}$ we have $\mathcal{DB}a \subset \mathcal{F}_1$ and $\mathcal{F}_2 \subset \mathcal{DQ}$. Consequently, by the latter theorem, we obtain

Corollary 10. *If* $\mathcal{F}_1 \in \mathcal{P}$ *and* $\mathcal{F}_2 \in \mathcal{P}_\mathcal{A}$ *then the family* $\mathcal{F}_1 \setminus \mathcal{F}_2$ *is strongly* $\mathfrak{c}$*-algebrable.*

Remind that any family from $\mathcal{P}_{\mathcal{A}}$ is strongly porous in any family from $\mathcal{P}$ [8]. Therefore, if $\mathcal{F}_1 \in \mathcal{P}$ and $\mathcal{F}_2 \in \mathcal{P}_{\mathcal{A}}$, then $\mathcal{F}_1 \setminus \mathcal{F}_2$ is residual (so, "topologically large") in $\mathcal{F}_1$.

Let us now consider some subfamilies of the family $\mathcal{B}_1$ of all Baire 1 functions. In [11], it is shown that a strong Świątkowski function need not be Baire 1, so no family from $\mathcal{P}_{\mathcal{A}}$ is contained in $\mathcal{B}_1$. Put $\mathcal{P}'_{\mathcal{A}} := \{\mathcal{D}_{\mathcal{A}} \cap \mathcal{B}_1 : \mathcal{A} \text{ has the (*)-property}\}$ and $\mathcal{DB}_1$—the family of all Darboux Baire 1 functions. In [11], it is proved that each family from $\mathcal{P}'_{\mathcal{A}}$ is strongly porous, so "topologically small", in $\mathcal{DB}_1$. Using the function from Lemma 8, we can show, in the same way as in Theorem 9, that:

Theorem 11. *If $\mathcal{F} \in \mathcal{P}'_{\mathcal{A}}$ then the family $\mathcal{DB}_1 \setminus \mathcal{F}$ is strongly $\mathfrak{c}$-algebrable.*

J. Wódka proved in [27] that the family $\mathcal{DQ} \setminus \mathcal{D}_s$ is strongly $\mathfrak{c}$-algebrable. We will show that both $(\mathcal{C}_{\tau_d} \cap \mathcal{C}_{\tau_\mathcal{I}}) \setminus \mathcal{D}_s$ and $\mathcal{D}_s \setminus (\mathcal{C}_{\tau_d} \cup \mathcal{C}_{\tau_\mathcal{I}})$ are strongly $\mathfrak{c}$-algebrable, too.

Recall that a set A is called a right-interval set if A is a union of intervals (a_n, b_n) with $\lim_{n\to\infty} a_n = 0$ and $0 < b_{n+1} < a_n < b_n$ for each $n \in \mathbb{N}$. Suppose that $A = \bigcup_{n=1}^{\infty} (a_n, b_n)$ is a right-hand interval set and $b_1 = 1$.

Put

$$t_A(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 1 - \frac{1}{n} & \text{for } x \in [a_n, b_n], \quad n \in \mathbb{N}, \\ 0 & \text{for } x = \frac{a_n + b_{n+1}}{2},\ n \in \mathbb{N} \text{ and for } x \in [b_1, \infty), \\ \text{linear} & \text{on the intervals } \left[b_{n+1}, \frac{a_n+b_{n+1}}{2}\right],\ \left[\frac{a_n+b_{n+1}}{2}, a_n\right],\ n \in \mathbb{N} \end{cases}$$

and

$$F_A(x) = \begin{cases} t_A(x-m) - m & \text{if } m \text{ is even and } x \in [m, m+1], \\ 1 - x & \text{if } m \text{ is odd and } x \in (m, m+1). \end{cases}$$

Note that F_A is not continuous at any even m, and F_A is continuous at any other point. Moreover, $F_A \notin \mathcal{D}_s$.

Lemma 12. *Let f be an exponential-like function. If A is a right-hand interval set, then $f \circ F_A \notin \mathcal{D}_s$.*

Proof. Fix an exponential-like function f and a right-hand interval set A. We will show that $f \circ F_A$ is not continuous and does not belong to $\mathcal{D}_s$. By Lemma 6, there exists an even number $m \in \mathbb{Z}$ such that f is strictly monotone on $(-m, -m+2)$. Without loss of generality, we can assume that f is strictly increasing on this interval.

Observe that $f \circ F_A$ is not continuous at m. Indeed, let (a, b) be an arbitrary interval such that $m - 1 < a < m < b < m + 1$. Then

$$f \circ F_A((a, b)) \supset f([-m, -m+1]).$$

Since f is strictly monotone and continuous on $[-m,-m+1]$, $f\big([-m,-m+1]\big)$ is a nondegenerate interval. Let $\epsilon < \operatorname{diam}\big(f([-m,-m+1])\big)/3$. Then,

$$\Big(f\big(F_A(m)\big)-\epsilon, f\big(F_A(m)\big)+\epsilon\Big) \not\supseteq f\big(F_A(a,b)\big),$$

for each (a,b) such that $m-1<a<m<b<m+1$.

Let us show that $f\circ F_A \notin \mathcal{D}_s$. For this purpose, fix numbers a_0 and b_0 such that $m-1<a_0<m<b_0<m+1$. Then, we have $F_A(a_0)>F_A(m)>F_A(b_0)$ and $F_A(m)=-m+1$. As f is strictly increasing on $(-m,-m+2)$, we obtain $f\big(F_A(a_0)\big)>f\big(F_A(x)\big)>f\big(F_A(m)\big)>f\big(F_A(y)\big)>f\big(F_A(b_0)\big)$ for all $x\in(a_0,m)$ and $y\in(m,b_0)$, so on the interval (a_0,b_0) the function $f\circ F_A$ has value $f(-m+1)$ only at the point m. Note that $f\circ F_A$ is not continuous at m. Therefore, $f\circ F_A$ is not a strong Świątkowski function. □

Fix $f\in\mathcal{C}_{\tau_d}$. It is well-known [26] that f has the Darboux property. Since f is approximately continuous at any point, $f\in\mathcal{D}_{ap}$. Analogously, from the fact that $\mathcal{C}_{\tau_d}\subset\mathcal{D}$ [21], we obtain that $\mathcal{C}_{\tau_\mathcal{J}}\subset\mathcal{D}_{\mathcal{J}-ap}$.

As $\tau_\mathcal{J}$ has the (*)-property, $\mathcal{D}_{\mathcal{J}-ap}\subset\mathcal{DQ}$. Thus,

$$\mathcal{C}_{\tau_d}\cap\mathcal{C}_{\tau_\mathcal{J}}\subset\mathcal{D}_{ap}\cap\mathcal{D}_{\mathcal{J}-ap}\subset\mathcal{DQ}.$$

Observe that families $\mathcal{C}_{\tau_d}\cap\mathcal{C}_{\tau_\mathcal{J}}$ and $\mathcal{D}_s$ are incomparable.

EXAMPLE 13. If 0 is a right-hand density point and a right-hand $\mathcal{J}$-density point of a right-hand interval set A, then $t_A\in(\mathcal{C}_{\tau_d}\cap\mathcal{C}_{\tau_\mathcal{J}})\setminus\mathcal{D}_s$.

P r o o f. Indeed, t_A is continuous at any point $x\neq 0$. Moreover, t_A is approximately continuous and $\mathcal{J}$-approximately continuous at zero. On the other hand, $1\in t_A((-1/2,1/2))$ and the only point x_0 such that $t_A(x_0)=1$ is equal to 0, and t_A is not continuous at zero. □

EXAMPLE 14. There exists a function belonging to $\mathcal{D}_s\setminus(\mathcal{C}_{\tau_d}\cup\mathcal{C}_{\tau_\mathcal{J}})$.

P r o o f. Suppose that A is a right-hand interval set at 0 such that 0 is a right--hand density point and $\mathcal{J}$-density point of A. Put

$$\hat{t}_A(x)=\begin{cases}1 & \text{for } x=\frac{a_n+b_{n+1}}{2},\ n\in\mathbb{N}, \text{ and for } x\le 0,\\ 0 & \text{for } x\in[a_n,b_n],\quad n\in\mathbb{N}, \text{ and for } x\in(b_1,\infty),\\ \text{linear} & \text{on intervals } \Big[b_{n+1},\frac{a_n+b_{n+1}}{2}\Big],\ \Big[\frac{a_n+b_{n+1}}{2},a_n\Big],\ n\in\mathbb{N}\end{cases}$$

and

$$\hat{F}_A(x)=\begin{cases}\hat{t}_A(x-m)-m & \text{if } m \text{ is even and } x\in[m,m+1],\\ 1-x & \text{if } m \text{ is odd and } x\in(m,m+1).\end{cases}$$

Then, $\hat{F}_A$ is neither approximately nor $\mathcal{J}$-approximately continuous at each $m\in\mathbb{Z}$ and has the strong Świątkowski property, so $\hat{F}_A\in\mathcal{D}_s\setminus(\mathcal{C}_{\tau_d}\cup\mathcal{C}_{\tau_\mathcal{J}})$. □

Using functions F_A and $\hat{F}_A$ considered in Example 13 and Example 14, we obtain that families $(\mathcal{C}_{\tau_d} \cap \mathcal{C}_{\tau_\mathcal{I}}) \setminus \mathcal{D}_s$ and $\mathcal{D}_s \setminus (\mathcal{C}_{\tau_d} \cup \mathcal{C}_{\tau_\mathcal{I}})$ are strongly $\mathfrak{c}$-algebrable. Indeed, using the exponential method, we obtain the following facts:

Theorem 15. *The family* $(\mathcal{C}_{\tau_d} \cap \mathcal{C}_{\tau_\mathcal{I}}) \setminus \mathcal{D}_s$ *is strongly* $\mathfrak{c}$*-algebrable.*

Proof. Suppose that A is a right-hand interval set such that $b_1 = 1$ and 0 is right-hand density and $\mathcal{I}$-density point of A. Let f be an exponential-like function. It is easy to check that $f \circ F_A$ is approximately and $\mathcal{I}$-approximately continuous. By Lemma 12, we obtain $f \circ F_A \in (\mathcal{C}_{\tau_d} \cap \mathcal{C}_{\tau_\mathcal{I}}) \setminus \mathcal{D}_s$. □

Corollary 16. *Let*

$$\mathcal{P} = \big\{\mathcal{DQ}, \mathcal{D}_{\mathcal{I}-ap}, \mathcal{D}_{ap}, \mathcal{D}_{ap} \cap \mathcal{D}_{\mathcal{I}-ap}, \mathcal{C}_{\tau_d}, \mathcal{C}_{\tau_d} \cup \mathcal{D}_s, \mathcal{C}_{\tau_\mathcal{I}}, \mathcal{C}_{\tau_\mathcal{I}} \cup \mathcal{D}_s, \mathcal{C}_{\tau_d} \cap \mathcal{C}_{\tau_\mathcal{I}}, (\mathcal{C}_{\tau_d} \cap \mathcal{C}_{\tau_\mathcal{I}}) \cup \mathcal{D}_s\big\}.$$

Each $\mathcal{F} \in \mathcal{P}$ *contains* $\mathcal{C}_{\tau_d} \cap \mathcal{C}_{\tau_\mathcal{I}}$. *Therefore, by the latter theorem, the family* $\mathcal{F} \setminus \mathcal{D}_s$ *is strongly* $\mathfrak{c}$*-algebrable for any* $\mathcal{F}$ *belonging to* $\mathcal{P}$.

Theorem 17. *The family* $\mathcal{D}_s \setminus (\mathcal{C}_{\tau_d} \cup \mathcal{C}_{\tau_\mathcal{I}})$ *is strongly* $\mathfrak{c}$*-algebrable.*

Proof. Fix an exponential-like function f, and let $\hat{F}_A$ be the function from Example 14. Let us show that $f \circ \hat{F}_A \in \mathcal{D}_s$. Indeed, fix $(a, b) \subset \mathbb{R}$ such that $f\big(\hat{F}_A(a)\big) \neq f\big(\hat{F}_A(b)\big)$, and $\lambda \in < f\big(\hat{F}_A(a)\big), f\big(\hat{F}_A(b)\big) >$. Then, as $f \circ \hat{F}_A$ has the Darboux property, there exists a point $x' \in (a, b)$ such that $f \circ \hat{F}_A(x') = \lambda$.

If $x' \notin \mathbb{Z}$ or x' is odd, then put $x = x'$. It is easy to see that $f \circ \hat{F}_A$ is continuous at x.

Assume that there exists even m such that $x' = m$. Let us show that there exists a point $x \in (x', b)$ such that $f \circ \hat{F}_A(x) = \lambda$. Indeed, we can find a point $x \in (x', b)$ with $\hat{F}_A(x) = 1 - m$, so $f\big(\hat{F}_A(x)\big) = f\big(\hat{F}_A(m)\big) = \lambda$.

It is easy to see that $f \circ \hat{F}_A$ is continuous at x, so it has the strong Świątkowski property.

We will show that $f \circ \hat{F}_A$ is neither approximately nor $\mathcal{I}$-approximately continuous. By Lemma 6, there exists an even number $m \in \mathbb{Z}$ such that f is strictly monotone on $[-m, -m + 1]$. Without loss of generality, we can assume that f is strictly increasing function on this interval, so $f(-m) < f(1 - m)$. Let $\epsilon \in \big(0, f(1 - m) - f(-m)\big)$. Then,

$$A + m \subset \mathbb{R} \setminus \left[\hat{F}_A^{-1}\left(f^{-1}\left(\left(f\big(\hat{F}_A(m)\big) - \epsilon, f\big(\hat{F}_A(m)\big) + \epsilon\right)\right)\right)\right],$$

so m is a right-hand dispersion and a right-hand $\mathcal{I}$-dispersion point of $A + m$, and $f \circ \hat{F}_A$ is neither approximately nor $\mathcal{I}$-approximately continuous at m. □

Theorem 18. *The family* $\mathcal{DQ} \setminus (\mathcal{D}_{ap} \cup \mathcal{D}_{\mathcal{I}-ap})$ *is strongly* $\mathfrak{c}$*-algebrable.*

P r o o f. Let $B = \bigcup_{n=1}^{\infty} (a_n, b_n)$ be a right-hand interval set such that 0 is a density and an $\mathcal{I}$-density point of the set $(-1, 0] \cup B$.

We will use a function f_B which is a modification of t_B.

Let

$$f_B(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 0 & \text{for } x \in [a_n, b_n], \quad n \in \mathbb{N}, \text{ and for } x \in (b_1, \infty), \\ 1 - \frac{1}{n} & \text{for } x = \frac{a_n + b_{n+1}}{2}, \ n \in \mathbb{N}, \\ \text{linear} & \text{on intervals } \left[b_{n+1}, \frac{a_n + b_{n+1}}{2}\right], \ \left[\frac{a_n + b_{n+1}}{2}, a_n\right], \ n \in \mathbb{N}. \end{cases}$$

Now, we use the function

$$F_B(x) = \begin{cases} f_B(x - m) - m, & \text{if } m \text{ is even and } x \in [m, m+1], \\ 1 - x & \text{if } m \text{ is odd and } x \in (m, m+1). \end{cases}$$

It is not difficult to see that $F_B \in \mathcal{DQ}$.

Fix an exponential-like function f. Then, as f is continuous, $f \circ F_B \in \mathcal{DQ}$. There exists an even number $m \in \mathbb{Z}$ such that f is strictly monotone on $(-m, -m+2)$. Again, we can assume that f is strictly increasing on this interval.

Let $y_0 = f(-m)$ and $y_1 = f \circ F_B(m) = f(-m+1)$. Observe that $F_B(B) = \{0\}$, so $f\big(F_B(B + m)\big) = \{y_0\}$ and $y_0 < y_1$.

Fix $\epsilon \in (0, y_1 - y_0)$ and put $W = (y_1 - \epsilon, y_1 + \epsilon)$. The complement of the set $F_B^{-1}\left(f^{-1}(W)\right)$ contains the set $B + m$. As $f\big(F_B(m)\big) = y_1$ and m is a right-hand density and a right-hand $\mathcal{I}$-density point of $B + m$, $f \circ F_B$ is neither approximately nor $\mathcal{I}$-approximately continuous at m. Therefore, as $f \circ F_B$ assumes value $-m$ on the interval $(m-1, m+1)$ only at the point m, we obtain $f \circ F_B \in \mathcal{DQ} \setminus (\mathcal{D}_{ap} \cup \mathcal{D}_{\mathcal{J}-ap})$. □

Corollary 19. *Put* $\mathcal{P} = \{\mathcal{D}_{\mathcal{I}-ap}, \mathcal{D}_{ap}, \mathcal{D}_{ap} \cup \mathcal{D}_{\mathcal{I}-ap}, \mathcal{C}_{\tau_d}, \mathcal{C}_{\tau_\mathcal{I}}, \mathcal{C}_{\tau_d} \cup \mathcal{C}_{\tau_\mathcal{I}}\}$. *If* $\mathcal{F} \in \mathcal{P}$, *then the family* $\mathcal{DQ} \setminus \mathcal{F}$ *is strongly* $\mathfrak{c}$*-algebrable.*

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