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ON SOME DENSITY TOPOLOGY WITH RESPECT TO AN EXTENSION OF LEBESGUE MEASURE

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ABSTRACT. This paper presents a density type topology with respect to an extension of Lebesgue measure involving sequence of intervals tending to zero. Some properties of such topologies are investigated.

Let $\mathbb{R}$ denote a set of real numbers, $\mathbb{N}$ a set of natural numbers, and λ a Lebesgue measure on $\mathbb{R}$. By $\mathcal{L}$ we understand a family of Lebesgue measurable sets, by $\mathbb{L}$ a family of Lebesgue measurable null sets, and by $|I|$ a length of an interval I. By μ we denote any complete extension of Lebesgue measure λ, by S_μ a domain of function μ, and by $\mathcal{I}_\mu$ a family of μ-null sets. If $\mathcal{A}$, $\mathcal{B}$ are families of subsets of the space X, then we use notation $\mathcal{A} \ominus \mathcal{B} = \{C \subset X : C = A \setminus B, A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A}\Delta\mathcal{B} = \{C \subset X \colon C = A\Delta B, A \in \mathcal{A}, B \in \mathcal{B}\}$, where Δ is an operation of the symmetric difference. It is well-known that if $\mathcal{A}$ is σ-algebra of sets in X and $\mathcal{B}$ is σ-ideal of sets in X, then the family $\mathcal{A}\Delta\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{A} \cup \mathcal{B}$.

It is clear that $x_0 \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ if

$$\lim_{h\to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

It is equivalent to

$$\lim_{\substack{h_1\to 0^+, h_2\to 0^+,\\ h_1+h_2>0}} \frac{\lambda(A \cap [x_0 - h_1, x_0 + h_2])}{h_1 + h_2} = 1.$$

The above condition can be written as in [8]:

$$\forall_{\{J_n\}_{n\in\mathbb{N}}} \left(0 \in \bigcap_{n\in\mathbb{N}} J_n \wedge |J_n| \underset{n\to\infty}{\longrightarrow} 0\right) \Longrightarrow \lim_{n\to\infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 1,$$

where $\{J_n\}_{n\in\mathbb{N}}$ is a sequence of closed intervals.

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Let $A \in \mathcal{L}$ and $\Phi_d(A) = \{x \in \mathbb{R}\colon x \text{ is a density point of } A\}$. Then the family $\mathcal{T}_d = \{A \in \mathcal{L}\colon A \subset \Phi_d(A)\}$ is a topology called density topology (see [8]). We say that a sequence of closed intervals $J = \{J_n\}_{n\in\mathbb{N}}$ is convergent to 0 if $\operatorname{diam}\{\{0\} \cup J_n\} \underset{n\to\infty}{\longrightarrow} 0$. We will consider sequences of closed intervals.

Let $J = \{J_n\}_{n\in\mathbb{N}}$ be a sequence of closed intervals convergent to zero. We say that the point $x_0 \in \mathbb{R}$ is a J-density point of a set $A \in \mathcal{L}$, if

$$\lim_{n\to\infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 1.$$

Let $\Phi_J(A) = \{x \in \mathbb{R} : x \text{ is a } J\text{-density point of a set } A\}$.

Having regard to the results of the paper [4], we obtain the following properties:

Property 1. *If $A \in \mathcal{L}$, then $\Phi_J(A) \in \mathcal{F}_{\sigma\delta}$, so $\Phi_J(A) \in \mathcal{L}$.*

Property 2. *For arbitrary sets $A, B \in \mathcal{L}$ and arbitrary sequence J of intervals convergent to zero, we obtain that*

a) $\Phi_J(\mathbb{R}) = \mathbb{R}$, $\Phi_J(\emptyset) = \emptyset$,

b) $\lambda(A\Delta B) = 0 \Rightarrow \Phi_J(A) = \Phi_J(B)$,

c) $\Phi_J(A \cap B) = \Phi_J(A) \cap \Phi_J(B)$,

d) $\lambda(\Phi_J(A) \setminus A) = 0$.

From the above property we conclude that for an arbitrary sequence of intervals tending to zero, operator Φ_J is an almost lower density operator on $(\mathbb{R}, \mathcal{L}, \mathbb{L})$ (see [3]).

The paper [4] also contains the proofs of the following two theorems:

Theorem 1 (cf. [4])**.** *If $J = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero, then the family*

$$\mathcal{T}_J = \{A \in \mathcal{L}\colon A \subset \Phi_J(A)\}$$

is a topology such that $\mathcal{T}_{nat} \subsetneq \mathcal{T}_J$, where $\mathcal{T}_{nat}$ denotes a natural topology on $\mathbb{R}$. Topology $\mathcal{T}_J$ described above will be called the topology generated by the operator Φ_J on the space $(\mathbb{R}, \mathcal{L}, \mathbb{L})$.

For an arbitrary sequence $J = \{J_n\}_{n\in\mathbb{N}}$ tending to zero we define

$$\alpha(J) = \limsup_{n\to\infty} \frac{\operatorname{diam}\{\{0\} \cup J_n\}}{|J_n|}.$$

Theorem 2 (cf. [4])**.** *If $J = \{J_n\}_{n\in\mathbb{N}}$ is a sequence of intervals tending to zero such that $\alpha(J) < \infty$, then for an arbitrary $A \in \mathcal{L}$,*

$$\lambda(\Phi_J(A)\Delta A) = 0.$$

Taking into account Property 2 and Theorem 2, we conclude that if $\alpha(J) < \infty$, then operator Φ_J is the lower density operator on space $(\mathbb{R}, \mathcal{L}, \mathbb{L})$ (see [4]).

Let $J = \{J_n\}_{n\in\mathbb{N}}$ be a sequence of intervals tending to zero. Let $A \in S_\mu$. We say that a point $x_0 \in \mathbb{R}$ is a J-density point of the set A if

$$\lim_{n\to\infty} \frac{\mu(A \cap (J_n + x_0))}{|J_n|} = 1.$$

Let $\Phi_J^\mu(A) = \{x \in \mathbb{R}\colon x \text{ is a } J\text{-density point of the set } A\}$.

Property 3 (cf. [4]). *For an arbitrary sequence of intervals $J = \{J_n\}_{n\in\mathbb{N}}$ tending to zero and for an arbitrary set $A \in S_\mu$ we obtain that $\Phi_J^\mu(A) \in \mathcal{L}$.*

Proof. Let $A \in S_\mu$. Then, $x \in \Phi_J^\mu(A)$ if and only if

$$\forall_{k\in\mathbb{N}} \exists_{m\in\mathbb{N}} \forall_{n>m} \frac{\mu(A \cap (J_n + x))}{|J_n|} \geq 1 - \frac{1}{k}.$$

Hence,

$$\Phi_J^\mu(A) = \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n>m} \left\{x \in \mathbb{R}\colon \frac{\mu(A \cap (J_n + x))}{|J_n|} \geq 1 - \frac{1}{k}\right\}.$$

A function $f(x) = \mu\big(A \cap (J_n + x)\big)$ is continuous for a fixed $n \in \mathbb{N}$, furthermore, it satisfies Lipschitz's condition. In fact, for every x_1, x_2 we have

$$\begin{aligned} \big|f(x_1) - f(x_2)\big| &= \big|\mu\big(A \cap (J_n + x_1)\big) - \mu\big(A \cap (J_n + x_2)\big)\big| \\ &\leq \big|\mu\big(A \cap ((J_n + x_1))\Delta(J_n + x_2))\big)\big| \\ &\leq \big|\mu\big((J_n + x_1)\big)\Delta\big(J_n + x_2)\big)\big| \\ &\leq 2|x_1 - x_2|. \end{aligned} \tag{1}$$

Hence, $\Phi_J^\mu(A) \in F_{\delta\sigma}$, so in particular, $\Phi_J^\mu(A) \in \mathcal{L}$. □

Directly from the definition of the operator Φ_J^μ, we can deduce the following property.

Property 4. *For any sequence of intervals $J = \{J_n\}_{n\in\mathbb{N}}$ tending to zero and any sets $A, B \in S_\mu$ we obtain:*

a) $\Phi_J^\mu(\mathbb{R}) = \mathbb{R}$, $\Phi_J^\mu(\emptyset) = \emptyset$,

b) $\Phi_J^\mu(A \cap B) = \Phi_J^\mu(A) \cap \Phi_J^\mu(B)$,

c) $\mu(A\Delta B) = 0 \Rightarrow \Phi_J^\mu(A) = \Phi_J^\mu(B)$.

Now, we prove another property of the operator Φ_J^μ for a sequence of intervals $J = \{J_n\}_{n\in\mathbb{N}}$ tending to zero and such that $0 \in J_n$ for any $n \in \mathbb{N}$. Let $\mathfrak{J}^0$ denote a family of all sequences of intervals tending to zero and containing zero.

Theorem 3. *If $J \in \mathfrak{I}^0$, then for any $A \in S_\mu$ such that $A \subset \Phi^\mu_J(A)$ we obtain*

$$\mu\big(\Phi^\mu_J(A) \setminus A\big) = 0.$$

P r o o f. Let $A \subset \mathbb{R}$ be bounded. So, there exists an interval K such that $A \subset K$. Now, we show that for any $0 < \varepsilon < \lambda(K)$ the family

$$\mathcal{K}(\varepsilon) = \left\{ H \subset K \colon \mu(H \cap A) > \left(1 - \frac{\varepsilon}{\lambda(K)}\right) \lambda(H) \right\},$$

where H is a closed interval, is a Vitali cover of the set A.

This is a consequence of the fact that if $x \in A$, then $x \in \Phi^\mu_J(A)$, namely,

$$\forall_{\eta>0} \exists_{\delta>0} \forall_{|J_n|<\delta} \ \frac{\mu(A \cap (J_n + x))}{\lambda(J_n)} > 1 - \eta.$$

Putting $\eta = \frac{\varepsilon}{\lambda(\mathcal{K})}$, we find arbitrary short intervals such that $(J_n + x) \in \mathcal{K}(\varepsilon)$, and x belongs to the interval $(J_n + x)$ for any $n \in \mathbb{N}$.

Hence, from the Vitali theorem there exists a sequence of closed, pairwise disjoint intervals $\{P_n\}_{n\in\mathbb{N}}$ from the family $\mathcal{K}(\varepsilon)$ such that

$$\lambda\left(A \setminus \bigcup_{n=1}^{\infty} P_n\right) = 0.$$

Then,

$$\mu\left(\bigcup_{n=1}^{\infty} (P_n \setminus A)\right) \le \sum_{n=1}^{\infty} \mu\big(P_n \setminus (A \cap P_n)\big) \le \frac{\varepsilon}{\lambda(\mathcal{K})} \sum_{n=1}^{\infty} \lambda(P_n) \le \varepsilon \frac{\lambda(\mathcal{K})}{\lambda(\mathcal{K})} = \varepsilon.$$

Let

$$C = \left(A \setminus \bigcup_{n=1}^{\infty} P_n\right) \cup \bigcup_{n=1}^{\infty} P_n.$$

Then, $A \subset C$, C is Lebesgue measurable set and $\mu(C \setminus A) < \varepsilon$. Therefore,

$$\forall_{n\in\mathbb{N}} \exists_{\substack{C_n \in \mathcal{L}, \\ A \subset C_n}} \ \mu(C_n \setminus A) < \frac{1}{n},$$

so,

$$A \subset \bigcap_{n=1}^{\infty} C_n \quad \text{and} \quad \mu\left(\bigcap_{n\in\mathbb{N}} C_n \setminus A\right) = 0.$$

Then,

$$A = \bigcap_{n\in\mathbb{N}} C_n \setminus \left(\bigcap_{n\in\mathbb{N}} C_n \setminus A\right).$$

Putting

$$B = \bigcap_{n=1}^{\infty} C_n, \qquad D = \bigcap_{n=1}^{\infty} C_n \setminus A,$$

we obtain

$$A = B \setminus D, \qquad \text{where} \quad B \in \mathcal{L} \quad \text{and} \quad \mu(D) = 0.$$

At the same time, $\mu\big(\Phi_J^\mu(A)\setminus A\big)=0$, because $\mu\big(\Phi_J^\mu(A)\setminus A\big)=\mu\big(\Phi_J^\mu(B\setminus D)\setminus (B\setminus D)\big)=\mu\big(\Phi_J^\mu(B)\setminus B\big)=\lambda\big(\Phi_J(B)\setminus B\big)$. If $J\in\mathfrak{J}^0$, then $\alpha(J)=1$, so from Theorem 2, we obtain $\lambda\big(\Phi_J(B)\setminus B\big)=0$. □

Theorem 4. *If $J\in\mathfrak{J}^0$, then the family*

$$\mathcal{T}_J^\mu=\big\{A\in S_\mu\colon A\subset\Phi_J^\mu(A)\big\}$$

is a topology on $\mathbb{R}$ containing topology $\mathcal{T}_d$.

Proof. By Property 4, we obtain that $\emptyset,\mathbb{R}\in\mathcal{T}_J^\mu$, and $\mathcal{T}_J^\mu$ is closed under intersection. Let $\{A_t\}_{t\in T}\subset\mathcal{T}_J^\mu$. We show that $\bigcup_{t\in T}A_t\in\mathcal{T}_J^\mu$. The pair $(S_\mu,\mathcal{I}_\mu)$ satisfies ccc, because the measure μ is σ-finite, so it has the hull property. Let $B\subset\bigcup_{t\in T}A_t$ be a measurable kernel. Then, $\mu\big((B\cap A_t)\Delta A_t\big)=0$. Hence,

$$B\subset\bigcup_{t\in T}A_t\subset\bigcup_{t\in T}\Phi_J^\mu(A_t)=\bigcup_{t\in T}\Phi_J^\mu(B\cap A_t)\in\Phi_J^\mu(B),$$

but $\mu\big(\Phi_J^\mu(B)\setminus B\big)=0$, then $\bigcup_{t\in T}A_t\in S_\mu$, and consequently, $\bigcup_{t\in T}A_t\subset\Phi_J^\mu(\bigcup_{t\in T}A_t)$, which means that $\bigcup_{t\in T}A_t\in\mathcal{T}_J^\mu$. Obviously, $\mathcal{T}_d\subset\mathcal{T}_J^\mu$. □

Theorem 5. *For any sequence $J\in\mathfrak{J}^0$ we have*

$$\mathcal{T}_J^\mu=\mathcal{T}_J\ominus\mathcal{I}_\mu,$$

where $\mathcal{T}_J$ is a topology generated by the operator Φ_J on the space $(\mathbb{R},\mathcal{L},\mathbb{L})$.

Proof. Let $A=B\setminus C$, $B\in\mathcal{T}_J$, $C\in\mathcal{I}_\mu$. Then, $A\in S_\mu$ and $\Phi_J^\mu(A)=\Phi_J^\mu(B\setminus C)=\Phi_J^\mu(B)\supset B\supset B\setminus C=A$, so $A\in\mathcal{T}_J^\mu$.

Let $A\in\mathcal{T}_J^\mu$, then $A\in S_\mu$ and $A\subset\Phi_J^\mu(A)$. From the proof of Theorem 3, we have

$$A=D\setminus E,\qquad\text{where}\quad D\in\mathcal{L}\quad\text{and}\quad E\in\mathcal{I}_\mu.$$

At the same time, $A=\Phi_J^\mu(A)\setminus\big(\Phi_J^\mu(A)\setminus A\big)=\Phi_J^\mu(D\setminus E)\setminus F=\Phi_J^\mu(D)\setminus F=\Phi_J(D)\setminus F$, where $F=\Phi_J^\mu(A)\setminus A$. Since $J\in\mathfrak{J}^0$, then $\alpha(J)=1$, so Φ_J is the lower density operator on $(\mathbb{R},\mathcal{L},\mathbb{L})$. It implies that $\Phi_J\big(\Phi_J(D)\big)=\Phi_J(D)$. Hence, $\Phi_J(D)\in\mathcal{T}_J$. From Theorem 3 we have $F\in\mathcal{I}_\mu$, so $A\in\mathcal{T}_J\ominus\mathcal{I}_\mu$. □

Now, we quote a theorem on extension of measure.

Theorem 6 (see [6]). *Let (X,S,μ) be a measurable space with a σ-finite measure μ. Let $J\subset 2^X$ be a σ-ideal such that $\mu_*(B)=0$ for $B\in J$, where μ_* is an inner measure induced by μ. Then, the function μ' defined on σ-field $S\Delta J$ by $\mu'(A\Delta B)=\mu(A)$, where $A\in S$, $B\in J$, is an extension of the measure μ, and if μ is a complete measure, then μ' is a complete measure as well.*

Corollary 1. *If μ is a complete extension of Lebesgue measure, then the function μ' defined on σ-field $\mathcal{L}\Delta\mathcal{I}_\mu$, where $\mathcal{I}_\mu$ is σ ideal of μ-zero sets by $\mu'(A\Delta B)=\lambda(A)$, where $A\in\mathcal{L}$, $B\in\mathcal{I}_\mu$, is a complete extension of Lebesgue measure.*

Let μ be a complete extension of Lebesgue measure. Let us recall the definition of the density operator on the space $(\mathbb{R}, S_\mu, \mathcal{I}_\mu)$.

Definition 1. We shall say that operator $\Phi\colon S_\mu \to 2^X$ is the lower density operator on the space $(\mathbb{R}, S_\mu, \mathcal{I}_\mu)$ if the following conditions are satisfied:

a) $\Phi(\mathbb{R}) = \mathbb{R}$, $\Phi(\emptyset) = \emptyset$,

b) $\forall_{A,B\in S_\mu}\ \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

c) $\forall_{A,B\in S_\mu}\ \mu(A\Delta B) = 0 \Rightarrow \Phi(A) = \Phi(B)$,

d) $\forall_{A\in S_\mu}\ \mu\big(A\Delta\Phi(A)\big) = 0$.

Similarly to Theorem 4, we can prove the following theorem (see [4]).

Theorem 7. *If $\Phi\colon S_\mu \to 2^{\mathbb{R}}$ is the lower density operator on $(\mathbb{R}, S_\mu, \mathcal{I}_\mu)$, then the family $\mathcal{T}_\Phi = \big\{A \in S_\mu\colon A \subset \Phi(A)\big\}$ is a topology on $\mathbb{R}$.*

We now justify that topology $\mathcal{T}_J^\mu$ generated by operator Φ_J^μ can also be generated by some lower density operator on the space $(\mathbb{R}, \mathcal{L}\Delta\mathcal{I}_\mu, \mathcal{I}_\mu)$.

Theorem 8. *If $J \in \mathfrak{I}^0$ and $\mathcal{T}_J^\mu$ is a topology generated by the operator Φ_J^μ, then there exists a complete extension μ' of Lebesgue measure such that $S_{\mu'} \subset S_\mu$, $\mathcal{I}_{\mu'} = \mathcal{I}_\mu$, and $\Phi_{\mu'} = \Phi_J^\mu|_{S_{\mu'}}$, where $\Phi_J^\mu|_{S_{\mu'}}$ denotes the restriction of Φ_J^μ to the family $S_{\mu'}$, is the lower density operator on $(\mathbb{R}, S_{\mu'}, \mathcal{I}_{\mu'})$ and topology $\mathcal{T}_{\mu'} = \big\{A \in S_{\mu'}\colon A \subset \Phi_{\mu'}(A)\big\}$ is identical to the topology $\mathcal{T}_J^\mu$.*

P r o o f. Let μ' be a measure defined on σ-field $S_{\mu'} = \mathcal{L}\Delta\mathcal{I}_\mu$ as in Corollary 1. Then $S_{\mu'} \subset S_\mu$, and clearly $\mathcal{I}_{\mu'} = \mathcal{I}_\mu$. To prove that $\Phi_{\mu'} = \Phi_J^\mu|_{S_{\mu'}}$ is the lower density operator on $(\mathbb{R}, S_{\mu'}, \mathcal{I}_{\mu'})$ by Property 4, it is sufficient to show condition d) from Definition 1. Let $A \in S_{\mu'}$, then $A = B\Delta C$, where $B \in \mathcal{L}$, $C \in \mathcal{I}_\mu$. At the same time, $\Phi_{\mu'}(A)\Delta A = \Phi_{\mu'}(B\Delta C)\Delta(B\Delta C) = \Phi_J^\mu(B)\Delta B\Delta C$. By Theorem 2 we get $\Phi_J^\mu(B)\Delta B \in \mathcal{I}_\mu$, so $\Phi_\mu(A)\Delta A \in \mathcal{I}_\mu$. Hence, $\Phi_\mu(A)\Delta A \in \mathcal{I}_{\mu'}$. We show that $\mathcal{T}_J^\mu = \mathcal{T}_{\mu'}$. Let $A \in \mathcal{T}_J^\mu$. Then, by Theorem 5 we have $A = B \setminus C$, where $B \in \mathcal{L}$ and $C \in \mathcal{I}_\mu$. Since $\mathcal{I}_\mu = \mathcal{I}_{\mu'}$, we get that $A \in S_{\mu'}$ and $\Phi_{\mu'}(A) = \Phi_{\mu'}(B \setminus C) = \Phi_{\mu'}(B) = \Phi_J^\mu(B) = \Phi_J^\mu(B \setminus C) = \Phi_J^\mu(A) \supset A$. Hence, $A \in \mathcal{T}_{\mu'}$. Let $A \in \mathcal{T}_{\mu'}$. Then, $A \in S_{\mu'}$ and $A \subset \Phi_{\mu'}(A)$. Clearly, $A \in S_\mu$, because $S_{\mu'} \subset S_\mu$ and $\Phi_{\mu'}(A) = \Phi_J^\mu(A)$. Thus, $A \in \mathcal{T}_J^\mu$. □

If $J \in \mathfrak{I}^0$, then, by the previous theorem, it follows that the topology $\mathcal{T}_J^\mu$ coincides with the topology generated by the operator $\Phi_J^\mu|_{\mathcal{L}\Delta\mathcal{I}_\mu}$ which is the lower density operator on $(\mathbb{R}, \mathcal{L}\Delta\mathcal{I}_\mu, \mathcal{I}_\mu)$. Hence in the light of the properties of an abstract density topology generated by the lower density operator which are presented in Theorem 25.3 and Theorem 25.9 in [3], we obtain the following theorem.

Theorem 9. *Let $J \in \mathfrak{I}^0$. Let $\mathcal{T}_J^\mu$ be a topology generated by an operator Φ_J^μ on $(\mathbb{R}, S_\mu, \mathcal{I}_\mu)$. Then:*

a) $A \in \mathcal{I}_\mu$ if and only if A is $\mathcal{T}_J^\mu$-closed and $\mathcal{T}_J^\mu$-nowhere dense;

b) If $A \in \mathcal{I}_\mu$, then A is $\mathcal{T}_J^\mu$-closed and $\mathcal{T}_J^\mu$-discrete;

c) $\mathcal{I}_\mu = \mathcal{K}(\mathcal{T}_J^\mu)$, where $\mathcal{K}(\mathcal{T}_J^\mu)$ is a family of the first category sets with respect to topology $\mathcal{T}_J^\mu$;

d) $Bor(\mathcal{T}_J^\mu) = B(\mathcal{T}_J^\mu) = \mathcal{L}\Delta\mathcal{I}_\mu$, where $Bor(\mathcal{T}_J^\mu)$ is a family of Borel sets, $B(\mathcal{T}_J^\mu)$ is a family of Baire sets with respect to the topology $\mathcal{T}_J^\mu$;

e) $(\mathbb{R}, \mathcal{T}_J^\mu)$ is a Baire space;

f) A is $\mathcal{T}_J^\mu$-compact if and only if A is finite;

g) $(\mathbb{R}, \mathcal{T}_J^\mu)$ is not a first countable, not a second countable, and not a separable space;

h) $(\mathbb{R}, \mathcal{T}_J^\mu)$ is not a Lindelöf space;

i) $int_{\mathcal{T}_{\Phi_J^\mu}}(A) = A \cap \Phi_J^\mu(\mathcal{K}A)$, $A \subset \mathbb{R}$, $\mathcal{K}A$-μ-measurable kernel of the set A;

j) $int_{\mathcal{T}_{\Phi_J^\mu}}(A) = A \cap \Phi_J^\mu(A)$, $A \in S_\mu$.

As a consequence of the previous theorem, we obtain that in the case of a measure μ such that $S_\mu = \mathcal{L}\Delta\mathcal{I}_\mu$ and a sequence $J \in \mathfrak{I}^0$, topology $\mathcal{T}_J^\mu$ generated by the operator Φ_J^μ is such that $B(\mathcal{T}_J^\mu) = \mathcal{L}\Delta\mathcal{I}_\mu$, $\mathcal{K}(\mathcal{T}_J^\mu) = \mathcal{I}_\mu$, $(\mathbb{R}, \mathcal{T}_J^\mu)$ is a Baire space. It is easy to see that the family of nonempty $\mathcal{T}_J^\mu$-open and pairwise disjoint sets is at most countable. Hence, in the case $S_\mu = \mathcal{L}\Delta\mathcal{I}_\mu$, topology $\mathcal{T}_J^\mu$ is a von Neumann topology associated with the measure μ (see [8]). We can prove the following theorem.

Theorem 10. *Let $J \in \mathfrak{I}^0$ and let and $\mathcal{T}_J^\mu$ be a topology generated by the operator Φ_J^μ. Then, the next conditions are equivalent:*

a) *Φ_J^μ is a lower density operator on $(\mathbb{R}, S_\mu, \mathcal{I}_\mu)$;*

b) *$S_\mu = \mathcal{L}\Delta\mathcal{I}_\mu$;*

c) *$\mathcal{T}_J^\mu$ is a von Neumann topology with respect to the measure μ.*

P r o o f. *a)* $\Rightarrow$ *b)* Of course, $\mathcal{L}\Delta\mathcal{I}_\mu \subset S_\mu$. Let $A \in S_\mu$. Then, $A\Delta\Phi_J^\mu(A) \in \mathcal{I}_\mu$. From Property 3 we have $\Phi_J^\mu(A) \in \mathcal{L}$, so $A \in \mathcal{L}\Delta\mathcal{I}_\mu$.

The implication *b)* $\Rightarrow$ *c)* is a consequence of the previous theorem.

We prove the implication *c)* $\Rightarrow$ *a)*. By Property 4, it suffices to prove that $A\Delta\Phi_J^\mu(A) \in \mathcal{I}_\mu$ for $A \in S_\mu$.

If $\mathcal{T}_J^\mu$ is a von Neumann topology with respect to the measure μ, then $S_\mu = B(\mathcal{T}_J^\mu) = \mathcal{L}\Delta\mathcal{I}_\mu$ so $A \in \mathcal{L}\Delta\mathcal{I}_\mu$. Hence $A = B\Delta C$, where $B \in \mathcal{L}$ and $C \in \mathcal{I}_\mu$. Therefore $B\Delta C\Delta\Phi_J^\mu(B\Delta C) = B\Delta\Phi_J^\mu(B)\Delta C \in \mathcal{I}_\mu$, because $B\Delta\Phi_J^\mu(B) \in \mathbb{L}$ by Theorem 2. □

Now, we discuss some results related separation axioms.

For any sequence $J \in \mathfrak{I}^0$ we have that $\mathcal{T}_{nat} \subset \mathcal{T}_J^\mu$. Hence, the space $(\mathbb{R}, \mathcal{T}_J^\mu)$ is Hausdorff. Paper [2] demonstrates that the space $(\mathbb{R}, \mathcal{T}_J)$ is regular, and in paper [5] it was shown that $(\mathbb{R}, \mathcal{T}_J)$ is completely regular for any sequence of intervals J tending to zero such that $\alpha(J) < \infty$. So, for any sequence $J \in \mathfrak{I}^0$, the space $(\mathbb{R}, \mathcal{T}_J)$ is regular.

Investigating paper [7], we get that for $J \in \mathfrak{I}^0$ the family of $\mathcal{T}_J$-continuous functions and $\mathcal{T}_J^\mu$-continuous functions with value in a topological regular space are equal. It implies that if $(\mathbb{R}, \mathcal{T}_J)$ is a regular space, then the space $(\mathbb{R}, \mathcal{T}_J^\mu)$ is regular if and only if $\mathcal{T}_J^\mu = \mathcal{T}_J$. Finally, we get

Property 5. *For any sequence $J \in \mathfrak{I}^0$, the following conditions are equivalent:*

a) $(\mathbb{R}, \mathcal{T}_J^\mu)$ *is completely regular;*

b) $(\mathbb{R}, \mathcal{T}_J^\mu)$ *is regular;*

c) $\mathcal{T}_J^\mu = \mathcal{T}_J$*;*

d) $\mathcal{I}_\mu = \mathbb{L}$.

P r o o f. $a \Rightarrow b$ is obvious. Let $(\mathbb{R}, \mathcal{T}_J^\mu)$ be regular. Let $A \in \mathcal{T}_J^\mu$. Then $A = V \setminus B$, if $V \in \mathcal{T}_J$ and $B \in \mathcal{I}_\mu$. Let us assume that $A \notin \mathcal{T}_J$. Then, $B \in \mathcal{I}_\mu \setminus \mathcal{L}$. Let C be a measurable hull of B. We see that $\Phi_J(C) \setminus B \neq \emptyset$ because otherwise $\Phi_J(C) \subset B \subset C$ would be measurable. Let $x \in \Phi_J(C) \setminus B$. Since the space $(\mathbb{R}, \mathcal{T}_\mu)$ is regular and B is $\mathcal{T}_\mu$-closed, then there exist $V_1, V_2 \in \mathcal{T}_\mu$, $V_1 \cap V_2 = \emptyset$ and $x \in V_1$, $B \subset V_2$. Since $V_1 = W_1 \setminus D_1$, $V_2 = W_2 \setminus D_2$, where $W_1, W_2 \in \mathcal{T}_J$, $D_1, D_2 \in \mathcal{I}_\mu$ we get $W_1 \cap W_2 = \emptyset$. Then, $W_1 \subset \mathbb{R} \setminus W_2$ and $C \cap W_1 \subset C \setminus W_2 \subset C \setminus B$. From the definition of a hull, $\lambda(C \setminus W_2) = 0$. This implies that $\Phi_J(C \cap W_1) = \Phi_J(C) \cap \Phi_J(W_2) = \emptyset$. At the same time, $x \in \Phi_J(C) \cap W_1 \subset \Phi_J(C) \cap \Phi_J(W_2) = \emptyset$, which is a contradiction. Since $\mathcal{T}_J \subset \mathcal{T}_J^\mu$, implication $b \Rightarrow c$ has been proved. Let $\mathcal{T}_J^\mu = \mathcal{T}_J$ and $A \in \mathcal{I}_\mu \setminus \mathbb{L}$. Then, $X \setminus A \in \mathcal{T}_J^\mu$. Hence $X \setminus A \in \mathcal{T}_J$ and finally, $A \in \mathcal{L}$. So, $A \in \mathbb{L}$. Since $\mathbb{L} \subset \mathcal{I}_\mu$, implication $c \Rightarrow d$ has been proved. If $\mathcal{I}_\mu = \mathbb{L}$, then $\mathcal{T}_J^\mu = \mathcal{T}_J$. Since $J \in \mathcal{I}^0$, then by Theorem 13 in [5], $(\mathbb{R}, \mathcal{T}_J^\mu)$ is completely regular. □

It is worth mentioning that there exist extensions of the Lebesgue measure maintaining Lebesgue null sets.

Theorem 11 (cf. [1])**.** *Under the Continuum Hypothesis there exists a nonseparable extension of the Lebesgue measure on $\mathbb{R}$ whose null sets coincide with the null sets of the Lebesgue measure.*

Theorem 12. *Let $J \in \mathfrak{I}^0$. Then the space $(\mathbb{R}, \mathcal{T}_J^\mu)$ is not normal.*

P r o o f. By Theorem 3.15 in [2], we conclude that the space $(\mathbb{R}, \mathcal{T}_J)$ is not normal. Hence there are $\mathcal{T}_J$-closed sets $F_1, F_2 \neq \emptyset$ and $F_1 \cap F_2 = \emptyset$ such that for any open set $V_1, V_2 \in \mathcal{T}_J$ such that $F_1 \subset V_1$, $F_2 \subset V_2$, we obtain $V_1 \cap V_2 \neq \emptyset$. The sets F_1, F_2 are also $\mathcal{T}_J^\mu$-closed. In case $(\mathbb{R}, \mathcal{T}_J^\mu)$ is a normal space, then there are sets $W_1, W_2 \in \mathcal{T}_J^\mu$ such that $F_1 \subset W_1$, $F_2 \subset W_2$, $W_1 \cap W_2 = \emptyset$. Since by the form of the topology $\mathcal{T}_J^\mu$, $W_1 = V_1 \setminus Z_1$, $W_2 = V_2 \setminus Z_2$, where $V_1, V_2 \in \mathcal{T}_J$ and $Z_1, Z_2 \in \mathcal{I}_\mu$. We conclude that V_1, V_2 are disjoint and also $F_1 \subset V_1$, $F_2 \subset V_2$. This contradiction finishes the proof. $\square$

REFERENCES

[1] KHARAZISHVILI, A. B.: *A nonseparable extension of the Lebesgue measure without new nullsets*, Real Anal. Exchange **33** (2007/2008), 259–268.

[2] HEJDUK, J.—LORANTY, A.—WIERTELAK, R.: *J-approximately continuous functions*, Tatra Mt. Math. Publ. **62** (2015), 45–55.

[3] HEJDUK, J.—WIERTELAK, R.: *On the abstract density topologies generated by lower and almost lower density operators*, in: Traditional and present-day topics in real analysis (M. Filipczak et al., eds.), Łódź University Press, Łódź 2013, pp. 431–447.

[4] HEJDUK, J.—WIERTELAK, R.: *On the generalization of the density topologies on the real line*, Math. Slovaca **64** (2014), 1267–1276.

[5] HEJDUK, J.—WIERTELAK, R.: *On some properties of J-approximately continuous functions*, Math. Slovaca (to appear).

[6] MARCZEWSKI, E.: *Sur l'extension de la measure lebesguienne*, Fund. Math. **25** (1935), 551–558.

[7] MARTIN, N. F. G.: *Generalized condensation points*, Duke Math. J. **28** (1961), 507–5014.

[8] OXTOBY, J. C.: *Measure and Category*, 2nd edition. Graduate Texts in Math. Vol. 2, Springer-Verlag, Berlin, 1980.

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