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# ON SOME GENERALIZATIONS OF POROSITY AND POROUSCONTINUITY

Stanisław Kowalczyk — Małgorzata Turowska

ABSTRACT. In the present paper, we introduce notions of v-porosity and v-porouscontinuity. We investigate some properties of v-poroucontinuity and its connections with porouscontinuity introduced by Borsík and Holos. Moreover, we show that v-porouscontinuous functions may not belong to Baire class one.

## 1. v-porosity

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The symbol $|I|$ stands for the length of an open interval $I \subset \mathbb{R}$. For a set $A \subset \mathbb{R}$ and an open interval $I \subset \mathbb{R}$ let $\Lambda(A, I)$ denote the length of the largest open subinterval of $I$ having an empty intersection with $A$. According to [1], [4], the right porosity of the set $A$ at $x \in \mathbb{R}$ is defined as

$$p^+(A,x) = \limsup_{h\to 0^+} \frac{\Lambda(A,(x,x+h))}{h},$$

the left porosity of $A$ at $x$ is defined as

$$p^-(A,x) = \limsup_{h\to 0^+} \frac{\Lambda(A,(x-h,x))}{h},$$

and the porosity of $A$ at $x$ is defined as

$$p(A,x) = \max\left\{p^-(A,x), p^+(A,x)\right\}.$$

The set $A$ is called right porous at $x$ if $p^+(A,x) > 0$, left porous at $x$ if $p^-(A,x) > 0$, and porous at $x$ if $p(A,x) > 0$. Some properties of porosity can be found in [2], [4].

We give a definition of similar property. The right v-porosity of the set $A$ at $x$ is defined as

$$vp^+(A,x) = \liminf_{h\to 0^+} \frac{\Lambda(A,(x,x+h))}{h},$$


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the left v-porosity of $A$ at $x$ is defined as

$$vp^-(A,x) = \liminf_{h\to 0^+} \frac{\Lambda(A,(x-h,x))}{h},$$

and the v-porosity of $A$ at $x$ is defined as

$$vp(A,x) = \max\{vp^-(A,x), vp^+(A,x)\}.$$

The set $A \subset \mathbb{R}$ is called right v-porous at $x \in \mathbb{R}$ if $vp^+(A,x) > 0$, left v-porous at $x$ if $vp^-(A,x) > 0$, and v-porous at $x$ if $vp(A,x) > 0$. It is clear that $A$ is v-porous at $x$ if and only if it is v-porous from the right **or** from the left at $x$.

The set $A$ is called v-porous if $A$ is v-porous at each point $x \in A$.

In [4], Zajíček investigates similar notion, which we remind. Let $X$ be a metric space. The open ball with the center $x \in X$ and with the radius $R$ will be denoted by $B(x,R)$. Let $M \subset X$, $x \in X$ and $R > 0$. Then, according to [4], by $\gamma(x,R,M)$, we denote the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$. The number $2\liminf_{R\to 0^+} \frac{\gamma(x,R,M)}{R}$ is called the very porosity of $M$ at $x$. We say that the set $M$ is very porous at $x$ if $\liminf_{R\to 0^+} \frac{2\gamma(x,R,M)}{R} > 0$.

The definition of v-porosity of the set is very similar to the definition of very porosity in the space $\mathbb{R}$ with the euclidian metric, however, they are not equivalent. Directly from the definitions, we obtain

**Corollary 1.1.** *Let $A \subset \mathbb{R}$, $x \in \mathbb{R}$. If the set $A$ is v-porous at $x$, then $A$ is very porous at $x$.*

The next example presents a subset of $\mathbb{R}$ which is very porous at 0 but is not v-porous at 0. Moreover, the porosity of this set at 0 is equal to 1.

Example 1.2. Let

$$A = \bigcup_{n\in\mathbb{N}} \left[\frac{1}{(2n)!}, \frac{1}{(2n-1)!}\right] \cup \left[-\frac{1}{(2n)!}, -\frac{1}{(2n+1)!}\right].$$

Then,

$$vp^+(A,0) = \liminf_{h\to 0^+} \frac{\Lambda(A,(0,h))}{h} \le \liminf_{n\to\infty} \frac{\Lambda(A,(0,\frac{1}{(2n-1)!}))}{\frac{1}{(2n-1)!}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{(2n)!} - \frac{1}{(2n+1)!}}{\frac{1}{(2n-1)!}} = \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

and
$$vp^-(A,0) = \liminf_{h\to 0^+} \frac{\Lambda(A,(-h,0))}{h} \le \liminf_{n\to\infty} \frac{\Lambda(A,(-\frac{1}{(2n)!},0))}{\frac{1}{(2n)!}}$$
$$= \lim_{n\to\infty} \frac{-\frac{1}{(2n+2)!}+\frac{1}{(2n+1)!}}{\frac{1}{(2n)!}} = \lim_{n\to\infty} \frac{1}{2n+2} = 0.$$
Hence, $vp(A,0)=0$.

Denote
$$t_n = \frac{1}{n!} + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = \frac{2n+1}{(n+1)!} \qquad \text{for each} \quad n \in \mathbb{N}.$$
Let $h \in (0,1)$. There exists $n \in \mathbb{N}$ such that $h \in \left[\frac{1}{(2n+1)!}, \frac{1}{(2n-1)!}\right)$.

If $h \in \left[\frac{1}{(2n+1)!}, t_{2n+1}\right)$, then $\left(-\frac{1}{(2n+1)!}, -\frac{1}{(2n+2)!}\right) \cap A = \emptyset$ and
$$\frac{2\gamma(0,h,A)}{h} \ge \frac{\frac{1}{(2n+1)!} - \frac{1}{(2n+2)!}}{t_{2n+1}} = \frac{\frac{2n+1}{(2n+2)!}}{\frac{4n+3}{(2n+2)!}} = \frac{2n+1}{4n+3}.$$
If $h \in \left[t_{2n+1}, \frac{1}{(2n)!}\right)$, then $\left(\frac{1}{(2n+1)!}, h\right) \cap A = \emptyset$ and
$$\frac{2\gamma(0,h,A)}{h} \ge \frac{h - \frac{1}{(2n+1)!}}{h} \ge \frac{t_{2n+1} - \frac{1}{(2n+1)!}}{t_{2n+1}} = \frac{\frac{2n+1}{(2n+2)!}}{\frac{4n+3}{(2n+2)!}} = \frac{2n+1}{4n+3}.$$
If $h \in \left[\frac{1}{(2n)!}, t_{2n}\right)$, then $\left(\frac{1}{(2n+1)!}, \frac{1}{(2n)!}\right) \cap A = \emptyset$ and
$$\frac{2\gamma(0,h,A)}{h} \ge \frac{\frac{1}{(2n)!} - \frac{1}{(2n+1)!}}{h} \ge \frac{\frac{2n}{(2n+1)!}}{t_{2n}} = \frac{\frac{2n}{(2n+1)!}}{\frac{4n+1}{(2n+1)!}} = \frac{2n}{4n+1}.$$
Finally, if $h \in \left[t_{2n}, \frac{1}{(2n-1)!}\right)$, then $\left(-h, -\frac{1}{(2n)!}\right) \cap A = \emptyset$ and
$$\frac{2\gamma(0,h,A)}{h} \ge \frac{h - \frac{1}{(2n)!}}{h} \ge \frac{t_{2n} - \frac{1}{(2n)!}}{t_{2n}} = \frac{\frac{2n}{(2n+1)!}}{\frac{4n+1}{(2n+1)!}} = \frac{2n}{4n+1}.$$
Therefore, $\liminf_{h\to 0^+} \frac{2\gamma(0,h,A)}{h} = \frac{1}{2}$ and $A$ is very porous at $0$.

Moreover,
$$\begin{aligned} p(A,0) &\ge p^+(A,0) \\ &\ge \lim_{n\to\infty} \frac{\Lambda(A,(0,\frac{1}{(2n)!}))}{\frac{1}{(2n)!}} \\ &= \lim_{n\to\infty} \frac{\frac{1}{(2n)!} - \frac{1}{(2n+1)!}}{\frac{1}{(2n)!}} \\ &= \lim_{n\to\infty} \frac{2n}{2n+1} = 1. \end{aligned}$$

**Theorem 1.3.** *Let $A \subset \mathbb{R}$, $x \in A$. If there exists a decreasing sequence $(x_n)_{n\in\mathbb{N}} \subset A$ converging to $x$, then $vp^+(A,x) \le \frac{1}{2}$.*

P r o o f. It is sufficient to observe that

$$vp^+(A,x) \le \liminf_{n\to\infty} \frac{\Lambda(A,(x,2x_n-x))}{2(x_n-x)} \le \liminf_{n\to\infty} \frac{x_n-x}{2(x_n-x)} = \frac{1}{2},$$

because if $I \subset [x, 2x_n - x]$ and $I \cap A = \emptyset$, then either $I \subset [x, x_n]$ or $I \subset \big[x_n, x_n + (x_n - x)\big]$. $\square$

**Definition 1.4** ([1])**.** A point $x \in \mathbb{R}$ will be called a point of $\pi_r$-density of a set $A \subset \mathbb{R}$ for $0 \le r < 1$ ($\mu_r$-density of a set $A$ for $0 < r \le 1$) if $p(\mathbb{R}\setminus A, x) > r$ ($p(\mathbb{R}\setminus A, x) \ge r$).

We give similar definitions.

**Definition 1.5.** Let $0 \le r < 1$ ($0 < r \le 1$). A point $x \in \mathbb{R}$ will be called a point of $v\pi_r$-density of a set $A \subset \mathbb{R}$ ($v\mu_r$-density of a set $A$ ) if $vp(\mathbb{R}\setminus A, x) > r$ ($vp(\mathbb{R}\setminus A, x) \ge r$).

In an obvious way, we may define one-sided $v\pi_r$-densities and $v\mu_r$-densities.

Example 1.6. Let $A = \bigcup_{n\in\mathbb{N}} [x_{n+1}, y_n]$, where $x_n = \frac{1}{2^n}$, $y_n = \frac{3}{2^{n+2}} \in (x_{n+1}, x_n)$ for $n \in \mathbb{N}$. Denote $t_n = y_n + x_{n+1} - y_{n+1} = \frac{7}{2^{n+3}}$ for each $n \in \mathbb{N}$. If $h \in [x_{n+1}, t_n)$, then

$$\frac{\Lambda(A,(0,h))}{h} = \frac{x_{n+1}-y_{n+1}}{h} \ge \frac{\frac{1}{2^{n+3}}}{t_n} = \frac{\frac{1}{2^{n+3}}}{\frac{7}{2^{n+3}}} = \frac{1}{7}.$$

If $h \in [t_n, x_n)$, then

$$\frac{\Lambda(A,(0,h))}{h} = \frac{h-y_n}{h} \ge 1 - \frac{y_n}{t_n} = 1 - \frac{\frac{3}{2^{n+2}}}{\frac{7}{2^{n+3}}} = \frac{1}{7}.$$

Therefore, $vp^+(A,0) = \liminf_{h\to 0^+} \frac{\Lambda(A,(0,h))}{h} \ge \frac{1}{7}$ (it is clear that $vp^+(A,0) = \frac{1}{7}$).

Example 1.7. Let

$$B = \left\{\left(\frac{7}{8}\right)^n : n \in \mathbb{N}\right\} \quad \text{and} \quad h_n = \left(\frac{7}{8}\right)^{n+1} + \left(\left(\frac{7}{8}\right)^{n+1} - \left(\frac{7}{8}\right)^{n+2}\right) = \frac{9\cdot 7^{n+1}}{8^{n+2}}$$

for $n \in \mathbb{N}$. Then,

$$\begin{aligned} vp^+(B,0) &= \liminf_{h\to 0^+} \frac{\Lambda(B,(0,h))}{h} = \lim_{n\to\infty} \frac{\Lambda(B,(0,h_n))}{h_n} \\ &= \lim_{n\to\infty} \frac{\left(\frac{7}{8}\right)^{n+1} - \left(\frac{7}{8}\right)^{n+2}}{\frac{9\cdot 7^{n+1}}{8^{n+2}}} = \lim_{n\to\infty} \frac{\frac{7^{n+1}}{8^{n+2}}}{\frac{9\cdot 7^{n+1}}{8^{n+2}}} = \frac{1}{9}. \end{aligned}$$

Thus, $vp^+(B,0) = \frac{1}{9}$.

**Theorem 1.8.** *Let $A \subset \mathbb{R}$, $x \in \mathbb{R}$, $r \in [0,1)$. If $x$ is a point of $v\pi_r$-density of the set $A$, then there exists an open set $B$ such that $x$ is a point of $v\pi_r$-density of the set $B$.*

P r o o f. (The proof is similar to the proof of [1, Theorem 2]). We may assume that $vp(\mathbb{R} \setminus A, x) = vp^+(\mathbb{R} \setminus A, x)$. Then, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that

$$vp^+(\mathbb{R} \setminus A, x) = \lim_{n\to\infty} \frac{\Lambda(\mathbb{R} \setminus A, (x, a_n))}{a_n - x} > r.$$

It means that for each $n$ we can find an open interval $I_n$ such that $I_n \subset A$ and $|I_n| = \Lambda\big(\mathbb{R} \setminus A, (x, a_n)\big)$. Denote $B = \bigcup_{n\in\mathbb{N}} I_n$. It is obvious that $vp^+(\mathbb{R} \setminus B, x) = vp^+(\mathbb{R} \setminus A, x)$. □

**Lemma 1.9.** *Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence convergent to $x \in \mathbb{R}$ satisfying conditions $\lim_{n\to\infty} \frac{x_n - x_{n+1}}{x_{n+1} - x} = \infty$ and $x_n - x_{n+1} \geq 2(x_{n+1} - x_{n+2})$ for $n \geq 1$. Take any $\alpha \in (0,1]$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence such that $y_n = x_{n+1} + \alpha(x_n - x_{n+1})$ if $\alpha < 1$ or $y_n = x_{n+1} + \frac{n}{n+1}(x_n - x_{n+1})$ if $\alpha = 1$. Put $A = \bigcup_{n=1}^{\infty} [x_{n+1}, y_n]$. Then, $vp^+(\mathbb{R} \setminus A, x) = \frac{\alpha}{1+\alpha}$.*

P r o o f. Clearly, $\lim_{n\to\infty} \frac{x_n - x}{x_{n+1} - x} = \infty$, because

$$\lim_{n\to\infty} \frac{x_n - x_{n+1}}{x_{n+1} - x} = \lim_{n\to\infty} \left( \frac{x_n - x}{x_{n+1} - x} - 1 \right) = \infty.$$

Denote

$$t_n = x_{n+1} + y_{n+1} - x_{n+2} \qquad \text{for} \quad n \in \mathbb{N}.$$

Then, $x_{n+1} < t_n$. Moreover,

$$t_n = x_{n+1} + \alpha(x_{n+1} - x_{n+2}) \leq x_{n+1} + \alpha(x_n - x_{n+1}) = y_n$$

if $\alpha < 1$ and

$$\begin{aligned} t_n &= x_{n+1} + \frac{n+1}{n+2}(x_{n+1} - x_{n+2}) \\ &< x_{n+1} + \frac{n+1}{2(n+2)}(x_n - x_{n+1}) \\ &< x_{n+1} + \frac{n}{n+1}(x_n - x_{n+1}) = y_n \end{aligned}$$

if $\alpha = 1$. Hence, $t_n < y_n$ for every $n \in \mathbb{N}$ and $\alpha \in (0,1]$. Take any $z \in (x, x_1)$. There exists $n \in \mathbb{N}$ such that $z \in [y_{n+1}, y_n)$. First, assume that $z \in [y_{n+1}, t_n)$. Then,

$$\frac{\Lambda(\mathbb{R} \setminus A, (x, z))}{z - x} = \frac{y_{n+1} - x_{n+2}}{z - x} \geq \frac{y_{n+1} - x_{n+2}}{t_n - x} = \frac{\Lambda(\mathbb{R} \setminus A, (x, t_n))}{t_n - x}.$$

If $z \in [t_n, y_n)$, then

$$\frac{\Lambda(\mathbb{R}\setminus A,(x,z))}{z-x} = \frac{z-x_{n+1}}{z-x} \geq \frac{t_n-x_{n+1}}{t_n-x} = \frac{\Lambda(\mathbb{R}\setminus A,(x,t_n))}{t_n-x}.$$

Moreover, for each $n \in \mathbb{N}$,

$$\begin{aligned}\frac{\Lambda(\mathbb{R}\setminus A,(x,t_n))}{t_n-x} &= \frac{\alpha(x_{n+1}-x_{n+2})}{x_{n+1}-x+\alpha(x_{n+1}-x_{n+2})}\\ &= \frac{\alpha}{\frac{x_{n+1}-x}{x_{n+1}-x_{n+2}}+\alpha}\\ &= \frac{\alpha}{\frac{1}{1-\frac{x_{n+2}-x}{x_{n+1}-x}}+\alpha}\end{aligned}$$

if $\alpha < 1$ and

$$\begin{aligned}\frac{\Lambda(\mathbb{R}\setminus A,(x,t_n))}{t_n-x} &= \frac{\frac{n+1}{n+2}(x_{n+1}-x_{n+2})}{x_{n+1}-x+\frac{n+1}{n+2}(x_{n+1}-x_{n+2})}\\ &= \frac{\frac{n+1}{n+2}}{\frac{x_{n+1}-x}{x_{n+1}-x_{n+2}}+\frac{n+1}{n+2}}\\ &= \frac{\frac{n+1}{n+2}}{\frac{1}{1-\frac{x_{n+2}-x}{x_{n+1}-x}}+\frac{n+1}{n+2}}\end{aligned}$$

if $\alpha = 1$. Therefore,

$$\liminf_{z\to x^+}\frac{\Lambda(\mathbb{R}\setminus A,(x,z))}{z-x} = \lim_{n\to\infty}\frac{\Lambda(\mathbb{R}\setminus A,(x,t_n))}{t_n-x} = \frac{\alpha}{1+\alpha}$$

for every $\alpha \in (0,1]$. Hence, $vp^+(\mathbb{R}\setminus A,x) = \frac{\alpha}{1+\alpha}$. □

**Corollary 1.10.** *For each $x \in \mathbb{R}$ and $r \in \left(0,\frac{1}{2}\right]$, there is a set $A$ such that $vp^+(\mathbb{R}\setminus A,x) = r$. Moreover, we can assume that the set $A$ is of the form $A = \{x\} \cup \bigcup_{n=1}^{\infty}[x_{n+1},y_n]$, where $x < x_{n+1} < y_n < x_n$ for every $n \in \mathbb{N}$.*

P r o o f. It is sufficient to take $x_n = x + \frac{1}{(n+1)!}$ and $\alpha = \frac{r}{1-r}$ in Lemma 1.9. □

## 2. v-porouscontinuity

**Definition 2.1** ([1]). Let $r \in [0,1)$. The function $f\colon \mathbb{R}\to\mathbb{R}$ will be called:

- $\mathcal{P}_r$-continuous at the point $x$ if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $\pi_r$-density of $A$, and $f|_A$ is continuous at the point $x$;
- $\mathcal{S}_r$-continuous at the point $x$ if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $\pi_r$-density of $A$, and $f(A) \subset \big(f(x)-\varepsilon, f(x)+\varepsilon\big)$.

Let $r \in (0,1]$. The function $f\colon \mathbb{R} \to \mathbb{R}$ will be called:

- $\mathcal{M}_r$-continuous at the point $x$ if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $\mu_r$-density of $A$, and $f|_A$ is continuous at the point $x$;
- $\mathcal{N}_r$-continuous at the point $x$ if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $\mu_r$-density of $A$, and $f(A) \subset \big(f(x)-\varepsilon, f(x)+\varepsilon\big)$.

All of these functions are called porouscontinuous functions. Porouscontinuity is an example of the so-called path continuity, which was exhaustively studied.

Now, we give similar definition applying the notion of v-porosity instead of porosity.

**Definition 2.2.** Let $r \in [0,1)$ and $x \in \mathbb{R}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ will be called:

- $v\mathcal{P}_r$-continuous at the point $x$ if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $v\pi_r$-density of $A$, and $f|_A$ is continuous at the point $x$;
- $v\mathcal{S}_r$-continuous at the point $x$ if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $v\pi_r$-density of $A$, and $f(A) \subset \big(f(x)-\varepsilon, f(x)+\varepsilon\big)$.

Let $r \in (0,1]$ and $x \in \mathbb{R}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ will be called:

- $v\mathcal{M}_r$-continuous at the point $x$ if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $v\mu_r$-density of $A$, and $f|_A$ is continuous at the point $x$;
- $v\mathcal{N}_r$-continuous at the point $x$ if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, $x$ is a point of $v\mu_r$-density of $A$, and $f(A) \subset \big(f(x)-\varepsilon, f(x)+\varepsilon\big)$.

The symbols $\mathcal{P}_r(f)$, $\mathcal{S}_r(f)$, $\mathcal{M}_r(f)$ and $\mathcal{N}_r(f)$ denote the sets of all points at which the function $f$ is $\mathcal{P}_r$-continuous, $\mathcal{S}_r$-continuous, $\mathcal{M}_r$-continuous and $\mathcal{N}_r$-continuous, respectively. These sets are called the sets of porouscontinuity points of the function $f$. And analogously, symbols $v\mathcal{P}_r(f)$, $v\mathcal{S}_r(f)$, $v\mathcal{M}_r(f)$ and $v\mathcal{N}_r(f)$ will denote the sets of all points at which the function $f$ is $v\mathcal{P}_r$-continuous, $v\mathcal{S}_r$-continuous, $v\mathcal{M}_r$-continuous and $v\mathcal{N}_r$-continuous, respectively. These sets will be called the sets of v-porouscontinuity points of the function $f$.

Analogously, symbols $v\mathcal{P}_r^+(f)$, $v\mathcal{S}_r^+(f)$, $v\mathcal{M}_r^+(f)$ and $v\mathcal{N}_r^+(f)$ will denote the sets of all points at which the function $f$ is $v\mathcal{P}_r^+$-continuous, $v\mathcal{S}_r^+$-continuous, $v\mathcal{M}_r^+$-continuous and $v\mathcal{N}_r^+$-continuous, respectively. And similarly, for the left v-porouscontinuity. Moreover, let $\mathcal{C}(f)$ denote the set of all points at which $f$ is continuous and, similarly, $\mathcal{C}^{\pm}(f)$ denotes the set of all points at which $f$ is continuous from the left or from the right.

**Corollary 2.3.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$. If $f$ is continuous from the right or from the left at $x$ then $x \in v\mathcal{P}_r(f) \cap v\mathcal{S}_r(f) \cap v\mathcal{M}_s(f) \cap v\mathcal{N}_s(f)$ for every $r \in [0,1)$ and every $s \in (0,1]$.*

Applying Theorem 1.3, we get

**Corollary 2.4.** *$v\mathcal{P}_r(f) = v\mathcal{S}_r(f) = \mathcal{C}^{\pm}(f)$ for each $r \geq \frac{1}{2}$. Similarly, $v\mathcal{M}_r(f) = v\mathcal{N}_r(f) = \mathcal{C}^{\pm}(f)$ for each $r > \frac{1}{2}$.*

**Theorem 2.5.** *Let $f\colon \mathbb{R} \to \mathbb{R}$, $r, q, s \in [0, 1]$, $q < s$. Then:*

(1) $v\mathcal{P}_s(f) \subset v\mathcal{P}_q(f)$, $v\mathcal{S}_s(f) \subset v\mathcal{S}_q(f)$, $v\mathcal{M}_s(f) \subset v\mathcal{M}_q(f)$, $v\mathcal{N}_s(f) \subset v\mathcal{N}_q(f)$, $v\mathcal{P}_r(f) \subset v\mathcal{M}_r(f)$, $v\mathcal{S}_r(f) \subset v\mathcal{N}_r(f)$;
(2) $v\mathcal{M}_s(f) \subset v\mathcal{P}_q(f)$, $v\mathcal{N}_s(f) \subset v\mathcal{S}_q(f)$;
(3) $v\mathcal{P}_r(f) \subset v\mathcal{S}_r(f)$, $v\mathcal{M}_r(f) \subset v\mathcal{N}_r(f)$;
(4) $v\mathcal{P}_r(f) \subset \mathcal{P}_r(f)$, $v\mathcal{S}_r(f) \subset \mathcal{S}_r(f)$, $v\mathcal{M}_r(f) \subset \mathcal{M}_r(f)$, $v\mathcal{N}_r(f) \subset \mathcal{N}_r(f)$;

The corresponding subscripts should belong to intervals specified in Definitions 2.1 and 2.2.

Proof. (1), (2) and (4) are obvious. The proof of (3) is the same as in [1] for porouscontinuity and we omit it. □

**Theorem 2.6.** *Let $r \in \left(0, \frac{1}{2}\right]$, $x \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$. Then, $x \in v\mathcal{M}_r(f)$ if and only if $x \in v\mathcal{N}_r(f)$.*

Proof. The proof is very similar to the proof of Theorem 3 from [1] and we omit it. □

**Corollary 2.7.** *$v\mathcal{M}_r(f) = v\mathcal{N}_r(f)$ for every $f\colon \mathbb{R} \to \mathbb{R}$ and for every $r \in \left(0, \frac{1}{2}\right]$.*

**Lemma 2.8.** *Let $r \in \left(0, \frac{1}{2}\right]$, $x \in \mathbb{R}$ and let $(E_n)_{n \geq 1}$ be a decreasing family of subsets of $\mathbb{R}$ such that $vp^+(\mathbb{R} \setminus E_n, x) \geq r$ for $n \geq 1$. There exists $E \subset (x, \infty)$ such that $vp^+(\mathbb{R} \setminus E, x) \geq r$ and for every $n \geq 1$ we have $E \cap (x, y_n) \subset E_n$ for some $y_n > x$.*

Proof. By assumptions, for every $n \geq 1$ there exists $x_n > x$ such that

$$\frac{\Lambda(\mathbb{R} \setminus E_n, (x, t))}{t - x} > r - \frac{1}{n} \qquad \text{for all} \quad t \in (x, x_n].$$

We may assume that $x_{n+1} < x_n$ for $n \geq 1$ and $\lim_{n\to\infty} x_n = x$. Fix $n \geq 1$ and define $f_n\colon [x_{n+1}, x_n] \to \mathbb{R}$ by $f_n(t) = \Lambda\big(\mathbb{R} \setminus E_n, (x, t)\big) - (t - x)\big(r - \frac{1}{n}\big)$. It is easily seen that $f_n$ is continuous and positive. Hence, $\gamma_n = \inf_{t \in [x_{n+1}, x_n]} f_n(t) > 0$. Define $E = \bigcup_{n=1}^{\infty} E_n \cap [x + \gamma_n, x_n]$. We claim that $E$ has all the required conditions. Clearly, $E \cap \big(x, x + \min\{\gamma_i \colon i \in \{1, 2, \ldots, n-1\}\}\big) \subset E_n$,

$$\begin{aligned}\Lambda\big(\mathbb{R} \setminus E, (x, t)\big) &\geq \Lambda\big(\mathbb{R} \setminus E_n, (x + \gamma_n, t)\big) \\ &\geq \Lambda\big(\mathbb{R} \setminus E_n, (x, t)\big) - \gamma_n \\ &\geq (t - x)\left(r - \frac{1}{n}\right)\end{aligned}$$

and
$$\frac{\Lambda(\mathbb{R}\setminus E,(x,t))}{t-x} \ge r-\frac{1}{n} \qquad \text{for each} \quad t\in[x_{n+1},x_n].$$
It follows that $vp^{+}(\mathbb{R}\setminus E,x)\ge r$. □

**Theorem 2.9.** *Let $r\in\left[0,\frac{1}{2}\right)$, $x\in\mathbb{R}$ and $f\colon\mathbb{R}\to\mathbb{R}$. Then, $x\in v\mathcal{P}_r(f)$ if and only if $\lim_{\varepsilon\to0^+} vp\big(\mathbb{R}\setminus\{t\colon |f(x)-f(t)|<\varepsilon\},x\big)>r$.*

P r o o f. First, assume that $x\in v\mathcal{P}_r(f)$ and let $E$ be a set witnessing this fact. For every $\varepsilon>0$, $E\cap(x-h,x+h)\subset\big\{t\colon |f(x)-f(t)|<\varepsilon\big\}$ for some $h>0$, and therefore,
$$vp\Big(\mathbb{R}\setminus\big\{t\colon |f(x)-f(t)|<\varepsilon\big\},x\Big)\ge vp(\mathbb{R}\setminus E,x).$$
Thus,
$$\lim_{\varepsilon\to0^+} vp\Big(\mathbb{R}\setminus\big\{t\colon |f(x)-f(t)|<\varepsilon\big\},x\Big)\ge vp(\mathbb{R}\setminus E,x)>r.$$

Now, assume that $\lim_{\varepsilon\to0^+} vp\big(\mathbb{R}\setminus\{t\colon |f(x)-f(t)|<\varepsilon\},x\big)=s>r$, and let $E_n=\big\{t\colon |f(x)-f(t)|<\frac{1}{n}\big\}$ for $n\ge1$. Since $(E_n)_{n\ge1}$ is decreasing, either $\lim_{n\to\infty} vp^{+}(\mathbb{R}\setminus E_n,x)=s>r$ or $\lim_{n\to\infty} vp^{-}(\mathbb{R}\setminus E_n,x)=s>r$. Without loss of generality, we may assume that the first possibility occurs. By Lemma 2.8, there exists $E\subset(x,\infty)$ such that $vp^{+}(\mathbb{R}\setminus E,x)\ge r$, and for every $n\ge1$ we have $E\cap(x,y_n)\subset\big\{t\colon |f(x)-f(t)|<\frac{1}{n}\big\}$ for some $y_n>x$. The last statement proves that $f|_{E\cup\{x\}}$ is continuous at $x$. It follows that $x\in v\mathcal{P}_r(f)$. □

**Theorem 2.10.** *Let $r\in[0,1)$, $f\colon\mathbb{R}\to\mathbb{R}$, $x\in\mathbb{R}$.*

- *$x\in v\mathcal{P}_r(f)$ if and only if there is an open nonempty set $B$ such that $x$ is a point of $v\pi_r$-density of $B$ and $f|_{B\cup\{x\}}$ is continuous at the point $x$,*
- *$x\in v\mathcal{S}_r(f)$ if and only if for each $\varepsilon>0$ there is an open set $B$ such that $x$ is a point of $v\pi_r$-density of $B$ and $f(B)\subset\big(f(x)-\varepsilon,f(x)+\varepsilon\big)$.*

*Let $r\in(0,1]$, $f\colon\mathbb{R}\to\mathbb{R}$, $x\in\mathbb{R}$.*

- *$x\in v\mathcal{M}_r(f)$ if and only if there is an open nonempty set $B$ such that $x$ is a point of $v\mu_r$-density of $B$ and $f|_{B\cup\{x\}}$ is continuous at the point $x$,*
- *$x\in v\mathcal{N}_r(f)$ if and only if for each $\varepsilon>0$ there is an open set $B$ such that $x$ is a point of $v\mu_r$-density of $B$ and $f(B)\subset\big(f(x)-\varepsilon,f(x)+\varepsilon\big)$.*

P r o o f. All the assertions follow from Theorem 1.8. □

**Theorem 2.11.** *Let $f\colon\mathbb{R}\to\mathbb{R}$, $x\in\mathbb{R}$. Then,*
$$x\in v\mathcal{P}_r^{+}(f)\iff$$
$$\exists_{s>r}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{\eta\in(0,\delta)}\ \exists_{I\subset(x,x+\eta)}\big(|I|\ge\eta s, I\subset\{t\colon |f(t)-f(x)|<\varepsilon\}\big),$$

$$x \in v\mathcal{S}_r^+(f) \iff$$
$$\forall_{\varepsilon>0}\ \exists_{s>r}\ \exists_{\delta>0}\ \forall_{\eta\in(0,\delta)}\ \exists_{I\subset(x,x+\eta)}\big(|I| \geq \eta s, I \subset \{t\colon |f(t)-f(x)|<\varepsilon\}\big) \quad \textit{for } r \in \left[0, \frac{1}{2}\right)$$

*and*

$$x \in v\mathcal{M}_r^+(f) \iff$$
$$\forall_{\varepsilon>0}\ \forall_{s>r}\ \exists_{\delta>0}\ \forall_{\eta\in(0,\delta)}\ \exists_{I\subset(x,x+\eta)}\big(|I| \geq \eta s, I \subset \{t\colon |f(t)-f(x)|<\varepsilon\}\big) \quad \textit{for } r \in \left(0, \frac{1}{2}\right].$$

P r o o f. All the statements follow from these two implications:

$$\exists_{\delta>0}\ \forall_{\eta\in(0,\delta)}\ \exists_{I\subset(x,x+\eta)}\Big(|I| \geq \eta s, I \subset \big\{t\colon |f(t)-f(x)|<\varepsilon\big\}\Big) \implies vp^+\Big(\mathbb{R}\setminus\big\{t\colon |f(t)-f(x)|<\varepsilon\big\}, x\Big) \geq s$$

and

$$vp^+(\mathbb{R}\setminus\{t\colon |f(t)-f(x)|<\varepsilon\}, x) > s \implies \exists_{\delta>0}\ \forall_{\eta\in(0,\delta)}\ \exists_{I\subset(x,x+\eta)}\Big(|I| \geq \eta s, I \subset \{t\colon |f(t)-f(x)|<\varepsilon\}\Big),$$

which hold for every $\varepsilon, s > 0$. □

## 3. Main results

**Theorem 3.1.** *Let* $0 < r < s < \frac{1}{2}$ *and* $f\colon \mathbb{R} \to \mathbb{R}$. *Then,*

$$\mathcal{C}^{\pm}(f) \subset v\mathcal{M}_{\frac{1}{2}}(f) \subset v\mathcal{P}_s(f) \subset v\mathcal{S}_s(f) \subset v\mathcal{M}_s(f) \subset v\mathcal{P}_r(f) \subset v\mathcal{P}_0(f) \subset v\mathcal{S}_0(f).$$

P r o o f.
The first inclusion follows from Corollary 2.3. The others, from Theorem 2.5. □

We introduce the following denotations:

$$\mathcal{C}^{\pm} = \big\{f\colon \mathcal{C}^{\pm}(f) = \mathbb{R}\big\}, \qquad \mathcal{S}_0 = \big\{f\colon \mathcal{S}_0(f) = \mathbb{R}\big\},$$
$$v\mathcal{P}_r = \big\{f\colon v\mathcal{P}_r(f) = \mathbb{R}\big\} \text{ for } r \in \big[0, \tfrac{1}{2}\big), \quad v\mathcal{S}_r = \big\{f\colon v\mathcal{S}_r(f) = \mathbb{R}\big\} \text{ for } r \in \big[0, \tfrac{1}{2}\big),$$
$$v\mathcal{M}_r = \big\{f\colon v\mathcal{M}_r(f) = \mathbb{R}\big\} \text{ for } r \in \big(0, \tfrac{1}{2}\big].$$

The following example presents, for a given $r \in \big[0, \frac{1}{2}\big)$, a function belonging to $v\mathcal{S}_r \setminus v\mathcal{P}_r$.

Example 3.2. Fix $r \in \left[0, \frac{1}{2}\right)$. Let $x_n = \frac{1}{(n+1)!}$ and $y_n = x_{n+1} + \frac{r}{1-r}(x_n - x_{n+1})$ for $n \geq 1$. Define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in (-\infty, 0) \cup [x_1, \infty), \\ 0, & x \in \bigcup_{n=1}^{\infty} [x_{n+1}, y_n] \cup \{0\}, \\ \frac{x-y_n}{x_n-y_n}, & x \in (y_n, x_n), n \in \mathbb{N}. \end{cases}$$

The function $f$ is continuous from the right at each point except 0. Let $\varepsilon \in (0,1)$. Then

$$vp^+\Big(\mathbb{R} \setminus \big\{x\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) = vp^+\left(\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} \big[x_{n+1}, y_n + \varepsilon(x_n - y_n)\big), 0\right).$$

Moreover,

$$\begin{aligned} y_n + \varepsilon(x_n - y_n) - x_{n+1} &= \frac{r}{1-r}(x_n - x_{n+1}) + \varepsilon\left(1 - \frac{r}{1-r}\right)(x_n - x_{n+1}) \\ &= \frac{r + \varepsilon(1-2r)}{1-r}(x_n - x_{n+1}). \end{aligned}$$

By Lemma 1.9,

$$vp^+\left(\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} \big[x_{n+1}, y_n + \varepsilon(x_n - y_n)\big), 0\right) = \frac{\frac{r+\varepsilon(1-2r)}{1-r}}{\frac{r+\varepsilon(1-2r)}{1-r} + 1} = \frac{r + \varepsilon(1-2r)}{1 + \varepsilon(1-2r)} > r.$$

It follows that $f \in v\mathcal{S}_r$. Since

$$vp^+\Big(\mathbb{R}\setminus \big\{x\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) = \frac{r + \varepsilon(1-2r)}{1 + \varepsilon(1-2r)}$$

and

$$vp^-\Big(\mathbb{R}\setminus \big\{x\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) = 0,$$

we obtain

$$vp\Big(\mathbb{R} \setminus \big\{x\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) = \frac{r + \varepsilon(1-2r)}{1 + \varepsilon(1-2r)}.$$

Hence,

$$\lim_{\varepsilon\to 0^+} vp\Big(\mathbb{R} \setminus \big\{x\colon |f(x) - f(0)| < \varepsilon\big\}, 0\Big) = \lim_{\varepsilon\to 0^+} \frac{r + \varepsilon(1-2r)}{1 + \varepsilon(1-2r)} = r,$$

and by Theorem 2.9, $f \notin v\mathcal{P}_r$.

**Theorem 3.3.** *Let $0 < r < s < \frac{1}{2}$. Then*

$$\mathcal{C}^{\pm} \subset v\mathcal{M}_{\frac{1}{2}} \subset v\mathcal{P}_s \subset v\mathcal{S}_s \subset v\mathcal{M}_s \subset v\mathcal{P}_r \subset v\mathcal{P}_0 \subset v\mathcal{S}_0 \subset \mathcal{S}_0.$$

*Moreover, all inclusions are proper.*

P r o o f. All inclusions follow from Theorem 3.1. We have only to show that all of them are proper.

Let $A_t$, for $t \in \left(0, \frac{1}{2}\right]$, be the set of the form $A_t = \{0\} \cup \bigcup_{n=1}^{\infty} \left[x_n^t, y_n^t\right]$ such that $0 < y_{n+1}^t < x_n^t < y_n^t$ for $n \geq 1$ and $vp^+(\mathbb{R} \setminus A_t, 0) = t$. Such a set exists by Corollary 1.10. Let $f_t$ be the characteristic function of $A_t$. Then, $\mathcal{C}^{\pm}(f_t) = \mathbb{R} \setminus \{0\}$ and $\left\{x \colon |f_t(x) - f_t(0)| < \varepsilon\right\} = A_t$ for $\varepsilon \in (0,1)$. Since $vp^-(\mathbb{R} \setminus A_t, 0) = 0$ and $vp^+(\mathbb{R} \setminus A_t, 0) = t$, we obtain $vp(\mathbb{R} \setminus A_t, 0) = t$.

Then, $f_{\frac{1}{2}} \in v\mathcal{M}_{\frac{1}{2}} \setminus \mathcal{C}^{\pm}$.

If $s \in \left(0, \frac{1}{2}\right)$, then $f_s \in v\mathcal{M}_s \setminus v\mathcal{S}_s$.

If $0 \leq r < s \leq \frac{1}{2}$, then for $t \in (r, s)$ we have $f_t \in v\mathcal{P}_r \setminus v\mathcal{M}_s$.

The functions from $v\mathcal{S}_s \setminus v\mathcal{P}_s$ for $s \in \left[0, \frac{1}{2}\right)$ were constructed in Example 3.2.

As a member of $\mathcal{S}_0 \setminus v\mathcal{S}_0$, we can take a characteristic function of the complement of the set $A$ constructed in Example 1.2. □

**Lemma 3.4.** *For every $E \subset \mathbb{R}$ and $x \in \mathbb{R}$ such that $x$ is a bilateral point of accumulation of $E$, the following inequality $p(E,x) \geq \frac{vp(E,x)}{1-vp(E,x)}$ holds.*

P r o o f. Let $(x_n)_{n \geq 1}$ be a decreasing sequence from $E$ converging to $x$. For every $n \in \mathbb{N}$, let $(a_n, b_n) \subset (x, x_n)$ satisfy condition $|(a_n, b_n)| = \Lambda\big(E, (x, x_n)\big)$. Since $x_n \in E$, we obtain

$$\Lambda\big(E, (x, b_n + b_n - a_n)\big) = |(a_n, b_n)|.$$

Put

$$\alpha_n = \frac{b_n - a_n}{2b_n - a_n - x}.$$

Then

$$\frac{(b_n - x) + (b_n - a_n)}{b_n - a_n} = \frac{1}{\alpha_n}, \qquad \frac{b_n - x}{b_n - a_n} = \frac{1}{\alpha_n} - 1,$$

and finally,

$$\frac{b_n - a_n}{b_n - x} = \frac{\alpha_n}{1 - \alpha_n}.$$

Hence,

$$p^+(E,x) \geq \limsup_{n \to \infty} \frac{\Lambda(E,(x,b_n))}{b_n - x} \geq \limsup_{n \to \infty} \frac{\alpha_n}{1 - \alpha_n} \geq \frac{vp^+(E,x)}{1 - vp^+(E,x)}.$$

Similarly, we can show that $p^-(E,x) \geq \frac{vp^-(E,x)}{1-vp^-(E,x)}$. □

**Theorem 3.5.**

$$v\mathcal{P}_r \subsetneq \mathcal{P}_{\frac{r}{1-r}} \quad \textit{and} \quad v\mathcal{S}_r \subsetneq \mathcal{S}_{\frac{r}{1-r}} \quad \textit{for } r \in \left[0, \frac{1}{2}\right).$$

*Similarly,*

$$v\mathcal{M}_r \subsetneq \mathcal{M}_{\frac{r}{1-r}} \quad \textit{for } r \in \left(0, \frac{1}{2}\right].$$

Proof. All inclusions follow directly from Lemma 3.4.

Let $A$ be the set from Example 1.2. Then, the characteristic function of the set $\mathbb{R} \setminus A$ belongs to $\mathcal{M}_1$ and does not belong to $v\mathcal{S}_0$. This proves that all inclusions are proper, by Theorem 3.1 and Theorem 3.3. □

**Question.** To find the best $s$ with $v\mathcal{P}_r \subset \mathcal{P}_s$ (and similarly for $\mathcal{S}_s$ and $\mathcal{M}_s$). Is it true that $s = \frac{r}{1-r}$?

**Theorem 3.6.** *The families $v\mathcal{S}_r$ for $r \in \left[0, \frac{1}{2}\right)$ and $v\mathcal{M}_r$ for $r \in \left(0, \frac{1}{2}\right]$ are closed under the operation of uniform convergence.*

Proof. Assume that a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n\colon \mathbb{R} \to \mathbb{R}$, is uniformly convergent to $f\colon \mathbb{R} \to \mathbb{R}$. Let $x \in \mathbb{R}$, $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ and for each $t \in \mathbb{R}$ the inequality

$$|f_n(t) - f(t)| < \frac{\varepsilon}{3}$$

holds. Fix $n \geq n_0$. Notice that

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq |f(x_1) - f_n(x_1)| + |f_n(x_1) - f_n(x_2)| + |f_n(x_2) - f(x_2)| \\ &< \frac{2}{3}\varepsilon + |f_n(x_1) - f_n(x_2)| \quad \text{for every } \ x_1, x_2 \in \mathbb{R}. \end{aligned}$$

Therefore,

$$\left\{t \in \mathbb{R}\colon |f_n(x) - f_n(t)| < \frac{\varepsilon}{3}\right\} \subset \left\{t \in \mathbb{R}\colon |f(x) - f(t)| < \varepsilon\right\}.$$

Hence,

$$vp\Big(\mathbb{R} \setminus \big\{t\colon |f(x) - f(t)| < \varepsilon\big\}, x\Big) \geq vp\left(\mathbb{R} \setminus \left\{t\colon |f_n(x) - f_n(t)| < \frac{\varepsilon}{3}\right\}, x\right).$$

It means that if $x \in v\mathcal{S}_r(f_n)$ for every $n \in \mathbb{N}$, then $x \in v\mathcal{S}_r(f)$. Similarly, if $x \in v\mathcal{M}_r(f_n)$ for every $n \in \mathbb{N}$, then $x \in v\mathcal{M}_r(f)$. □

We will show that the family of $v\mathcal{P}_r$-continuous functions is not closed under uniform convergence.

Example 3.7. Let $r \in \left[0, \frac{1}{2}\right)$. Consider the function $f$ described in Example 3.2. Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $f_n = \max\left\{f, \frac{1}{n}\right\}$ for each $n \in \mathbb{N}$. Then, clearly, the sequence $(f_n)_{n\in\mathbb{N}}$ uniformly converges to $f$ and $f \notin v\mathcal{P}_r$. Since

$$\left\{x\colon f_n(x) = f_n(0)\right\} = \left\{x\colon |f(x) - f(0)| < \frac{1}{n}\right\}$$

and

$$vp^+\left(\mathbb{R}\setminus\left\{x\colon |f(x)-f(0)|<\frac{1}{n}\right\},0\right)=\frac{r+\frac{1}{n}(1-2r)}{1+\frac{1}{n}(1-2r)}>r,$$

we conclude that $f_n \in v\mathcal{P}_r$ for each $n \in \mathbb{N}$.

**Theorem 3.8.** *Let $r \in \left[0, \frac{1}{2}\right)$ and $f_n \in v\mathcal{P}_r$ for each $n \in \mathbb{N}$. If the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to $f$, then $f \in v\mathcal{S}_r$.*

Proof. Since $v\mathcal{P}_r \subset v\mathcal{S}_r$, $f_n \in v\mathcal{S}_r$. By Theorem 3.6, we obtain $f \in v\mathcal{S}_r$. $\square$

**Question.** Let $r \in \left[0, \frac{1}{2}\right)$. Is the closure of the set $v\mathcal{P}_r$ in the topology of uniform convergence equal to $v\mathcal{S}_r$?

Using standard arguments, one can easily prove that all v-porouscontinuous functions are measurable (see, for example, proof of [3, Theorem 2.1], and observe that $\overline{d}\big(\{x\colon |f(x)-f(x_0)|<\varepsilon\},x_0\big) \ge p\big(\mathbb{R}\setminus\{x\colon |f(x)-f(x_0)|<\varepsilon\},x_0\big)$, where $\overline{d}(A,x)$ is the upper density of a measurable set $A \subset \mathbb{R}$ at $x$). We will show that they may not belong to Baire class one.

**Theorem 3.9.** *There exists $f\colon \mathbb{R}\to\mathbb{R}$ such that $f \in v\mathcal{M}_{\frac{1}{2}}$ and $f$ does not belong to Baire class one.*

Proof. Denote $D_1^0 = A_0 = [0,1]$ and $|D_1^0| = 1 = \beta_0$. Let $I_1^0$ be an open interval included in $D_1^0$ with the same center as $D_1^0$ and of the length $\frac{1}{2}|D_1^0| = \alpha_0$. Denote $A_1 = A_0 \setminus I_1^0$. Then, $A_1 = D_1^1 \cup D_2^1$, where $D_1^1$, $D_2^1$ are closed intervals and $|D_1^1| = |D_2^1| = \beta_1$. Let $I_1^1$, $I_2^1$ be open intervals included in $D_1^1$, $D_2^1$ with the same center as $D_1^1$, $D_2^1$, respectively, and of the length $\frac{2}{3}|D_1^1| = \alpha_1$.

In this way, we can construct inductively two families of intervals: closed

$$\big\{D_i^n : i \in \{1,\dots,2^n\}, n \in \{0,1,2,\dots\}\big\}$$

and open

$$\big\{I_i^n : i \in \{1,\dots,2^n\}, n \in \{0,1,2,\dots\}\big\}$$

with the following properties

- $|D_i^n| = |D_j^n| = \beta_n$; if $i \neq j$ then $D_i^n \cap D_j^n = \emptyset$; $\beta_0 = 1$;
- $|I_i^n| = \frac{n+1}{n+2}|D_i^n| = \alpha_n$;
- $I_i^n$, $D_i^n$ have the same center;
- $D_i^n = D_{2i-1}^{n+1} \cup I_i^n \cup D_{2i}^{n+1}$; $D_{2i-1}^{n+1} \cap I_i^n = \emptyset = D_{2i}^{n+1} \cap I_i^n$;
- $D_1^0 = [0,1]$,

for each $i \in \{1,\dots,2^n\}$, $j \in \{1,\dots,2^n\}$ and $n \ge 0$. In particular, the sequence $(\alpha_n)_{n\ge 0}$ is decreasing and convergent to 0. Since $\beta_n = 2\beta_{n+1}+\alpha_n$ and $\alpha_n = \frac{n+1}{n+2}\beta_n$, we conclude $\beta_{n+1} = \frac{1}{2}\left(\frac{n+2}{n+1}-1\right)\alpha_n = \frac{1}{2(n+1)}\alpha_n < \frac{\alpha_n}{n}$. Denote $A = \bigcap_{n=0}^{\infty}\bigcup_{i=1}^{2^n} D_i^n$

and $I_i^n = (a_i^n, b_i^n)$ for each $i \in \{1, \dots, 2^n\}$ and $n \geq 0$. Let

$$B = \{b_i^n : i \in \{1, \dots, 2^n\}, n \geq 0\}.$$

Obviously, $B \subset A$ and $A$ is a perfect set. For each $n \geq 0$ and $i \in \{1, \dots, 2^n\}$, let $J_i^n = (a_i^n, c_i^n]$, $c_i^n < b_i^n$, be an interval such that $\alpha_n - |J_i^n| = \frac{1}{n+1}\alpha_{n+1}$.

Then, for each $n \geq 0$ and $i \in \{1, \dots, 2^n\}$, define a mapping $f_n^i \colon (a_i^n, b_i^n] \to \mathbb{R}$ by

$$f_i^n(x) = \begin{cases} 0, & x \in J_i^n, \\ \frac{x - c_i^n}{b_i^n - c_i^n}, & x \in (c_i^n, b_i^n]. \end{cases}$$

Finally, let $f \colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} f_i^n(x), & x \in (a_i^n, b_i^n], \quad i \in \{1, \dots, 2^n\}, \quad n \geq 0, \\ 0, & x \in (-\infty, 0) \cup (1, \infty) \cup (A \setminus B). \end{cases}$$

The sets $A \setminus B$ and $B$ are dense in $A$ and $f|_{A \setminus B} = 0$, $f|_B = 1$. Therefore, $f|_A$ has no point of continuity. Thus, $f$ does not belong to Baire class one.

If $x \in \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{2^n} I_i^n \cup (-\infty, 0) \cup (1, \infty)$, then $f$ is continuous at $x$. If $x \in B$, then $f$ is continuous from the left at $x$. Take any $x \in A \setminus B$. Let $h \in (0, \alpha_1)$. There is $m$ such that $h \in [\alpha_m, \alpha_{m-1})$. Then we can find $i \in \{1, \dots, 2^m\}$ such that $x \in D_i^m = D_{2i-1}^{m+1} \cup I_i^m \cup D_{2i}^{m+1}$. Therefore, $x \in D_{2i-1}^{m+1}$ or $x \in D_{2i}^{m+1}$. By construction, there exists an open interval $K$ from the family $\{I_i^n : i \in \{1, \dots, 2^n\}, n \geq 0\}$ such that its left endpoint is the same as the right endpoint of $D_{2i}^{m+1}$ or its right endpoint is the same as the left endpoint of $D_{2i-1}^{m+1}$. Without loss of generality, we may assume that the first case occurs. Then, $|K| \geq \alpha_{m-1}$. Put $E = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{2^n} J_i^n$. We will estimate $\frac{\Lambda(\mathbb{R} \setminus E, (x, x+h))}{h}$.

First, consider the case, where $x \in D_{2i-1}^{m+1}$. If $h < 2\alpha_m$, then

$$\begin{aligned} \frac{\Lambda(\mathbb{R} \setminus E, (x, x+h))}{h} &\geq \frac{\alpha_m - |D_{2i}^{m+1}|}{h} \\ &= \frac{\alpha_m}{h} - \frac{|D_{2i}^{m+1}|}{h} > \frac{\alpha_m}{h} - \frac{\frac{1}{m}\alpha_m}{h} \geq \frac{1}{2} - \frac{1}{m}. \end{aligned}$$

If $h \geq 2\alpha_m$, then $x + h \in K$ and

$$\begin{aligned} \frac{\Lambda(\mathbb{R} \setminus E, (x, x+h))}{h} &\geq \frac{h - \alpha_m - 2|D_{2i}^{m+1}|}{h} \\ &> 1 - \frac{\alpha_m}{h} - \frac{2\frac{1}{m}\alpha_m}{h} \\ &\geq 1 - \frac{1}{2} - \frac{1}{m} \\ &= \frac{1}{2} - \frac{1}{m}. \end{aligned}$$

Now, consider the case where $x \in D_{2i}^{m+1}$. Then, $x+h \in K$. Hence,

$$\frac{\Lambda(\mathbb{R}\setminus E,(x,x+h))}{h} \geq \frac{h-|D_{2i}^{m+1}|}{h} > \frac{h-\frac{1}{m}\alpha_m}{h} = 1-\frac{\alpha_m}{hm} \geq 1-\frac{1}{m}.$$

Thus we have showed that for each $h \in (0,\alpha_1)$ there exists $m(h) \in \mathbb{N}$ such that $h \in \left[\alpha_{m(h)},\alpha_{m(h)-1}\right)$ and

$$\frac{\Lambda(\mathbb{R}\setminus E,(x,x+h))}{h} \geq \frac{1}{2}-\frac{1}{m(h)}.$$

Certainly, if $h \to 0^+$, then $m(h) \to \infty$. Therefore,

$$vp^+(\mathbb{R}\setminus E,x) = \liminf_{h\to 0^+}\frac{\Lambda(\mathbb{R}\setminus E,(x,x+h))}{h} \geq \frac{1}{2}.$$

Thus $f|_{E\cup\{x\}}$ is continuous at $x$ and $x$ is a point of $v\mu_{\frac{1}{2}}$-density of the set $E\cup\{x\}$. It follows that $f \in v\mathcal{M}_{\frac{1}{2}}$. □

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*Institute of Mathematics*
*Pomeranian University in Słupsk*
*ul. Kozietulskiego 6–7*
*PL–76-200 Słupsk*
*POLAND*

*E-mail*: stanislaw.kowalczyk@apsl.edu.pl
malgorzata.turowska@apsl.edu.pl