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# ON GENERALIZATION OF THE $\mathcal{T}_{A_I}$-DENSITY TOPOLOGY

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ABSTRACT. We present a further generalization of the $\mathcal{T}_{A_I}$-density topology introduced in [W. Wojdowski, *A category analogue of the generalization of Lebesgue density topology*, Tatra Mt. Math. Publ. **42** (2009), 11–25] as a generalization of the $I$-density topology. We construct an ascending sequence $\{\mathcal{T}_{A_{I(n)}}\}_{n\in\mathbb{N}}$ of density topologies which leads to the $\mathcal{T}_{A_{I(\omega)}}$-density topology including all previous topologies. We examine several basic properties of the topologies.

The aim of the paper is to present a category analogue of results presented for the case of measure in [WO4].

Let $S$ be a $\sigma$-algebra of subsets of the real line $\mathbb{R}$, and $I \subset S$ a proper $\sigma$-ideal. We shall say that the sets $A, B \in S$ are equivalent ($A \sim B$), if and only if $A \triangle B \in I$. We will denote by $\lambda$ the Lebesgue measure on the real line.

Recall that the point $x \in \mathbb{R}$ is a Lebesgue density point of a measurable set $A$, if

$$\lim_{h\to 0} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1. \tag{$*$}$$

In [W1], W. Wilczyński gave his reformulation of the notion of a density point of a measurable set $A$ in terms of convergence almost everywhere of the sequence of characteristic functions of dilations of a set $A$:

A point $x \in \mathbb{R}$ is Lebesgue density point of a measurable set $A$ if and only if every subsequence

$$\left\{\chi_{(n_m\cdot(A-x))\cap[-1,1]}\right\}_{m\in\mathbb{N}} \quad \text{of} \quad \left\{\chi_{(n\cdot(A-x))\cap[-1,1]}\right\}_{n\in\mathbb{N}}$$

contains a subsequence

$$\left\{\chi_{\left(n_{m_p}\cdot(A-x)\right)\cap[-1,1]}\right\}_{p\in\mathbb{N}}$$

which converges to $\chi_{[-1,1]}$ almost everywhere on $[-1,1]$ (which means, except for on a null set).


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Wilczynski's approach relieved the definition of the notion of a measure. His definition requires null sets only. Instead of the notion of convergence in measure of a sequence of measurable function, he uses a convergence almost everywhere. This has opened a new space for study of more subtle properties of the notion of the Lebesgue density point and density topology, their various modifications, and, most of all, category analogues (see [PWW1], [PWW2], [CLO]).

The reformulated definition could be considered in more general settings as follows:

A point $x \in \mathbb{R}$ is an $I$-density point of a set $A \in S$ if every subsequence

$$\left\{\chi_{(n_m\cdot(A-x))\cap[-1,1]}\right\}_{m\in\mathbb{N}} \quad \text{of} \quad \left\{\chi_{(n\cdot(A-x))\cap[-1,1]}\right\}_{n\in\mathbb{N}}$$

contains a subsequence

$$\left\{\chi_{\left(n_{m_p}\cdot(A-x)\right)\cap[-1,1]}\right\}_{p\in\mathbb{N}}$$

which converges to $\chi_{[-1,1]}$ $I$-almost everywhere on $[-1,1]$ (which means, except for on a set belonging to $I$).

In [PWW2, Corollary 1, p. 556] in the category case, and in [W2] in measure case, it is proved that the following conditions are equivalent:

1. $x$ is an $I$-density point of a set $A \in S$,
2. for any decreasing to zero sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, there is its subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges $I$-almost everywhere on $[-1,1]$ to $\chi_{[-1,1]}$.

Following Wilczyński's approach in [WO1], we have introduced a notion of $\mathcal{A}_d$-density of a Lebesgue measurable set leading to a notion of $\mathcal{T}_{\mathcal{A}_d}$ topology on the real line stronger than the Lebesgue density topology. The generalization was related to a given appropriate family of subsets of $[-1,1]$, namely the family of measurable sets having density one at zero.

In [WO3] and [WO5], an $\mathcal{A}_I$-density as a category analogue of $\mathcal{A}_d$-density was introduced.

From now on, $(S, I)$ stands for $\sigma$-algebra of sets with the Baire property and $\sigma$-ideal of first category sets.

In [WO3], the author introduced $\mathcal{A}_I$, the family of subsets of interval $[-1,1]$ that are from $S$ and have 0 as their $I$-density point.

**Definition 1** ([WO3], [WO5]). We say that $x$ is an $\mathcal{A}_I$-density point of $A \in S$ if, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_I$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges $I$-almost everywhere on $B$ to 1.

**Remark 1.** Originally, in the definition of an $\mathcal{A}_I$-density point in [WO3], we required for the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ to be convergent $I$-almost everywhere on $[-1,1]$ to $\chi_B$. In [WO5], it was observed that the operator $\Phi_{\mathcal{A}_I}(A)$ corresponding to this condition is not monotonic, and thus, it is not a lower $I$-density.

The correct form is given in Definition 1 above. We shall show (after [WO5]) that the operator $\Phi_{\mathcal{A}_I}$ introduced according to the above definition is a lower density:

**Theorem 1.** *The mapping $\Phi_{\mathcal{A}_I} : S \to 2^{\mathbb{R}}$ has the following properties*:

(0) For each $A \in S$, $\Phi_{\mathcal{A}_I}(A) \in S$.

(1) For each $A \in S$, $A \sim \Phi_{\mathcal{A}_I}(A)$.

(2) For each $A, B \in S$, if $A \sim B$, then $\Phi_{\mathcal{A}_I}(A) = \Phi_{\mathcal{A}_I}(B)$.

(3) $\Phi_{\mathcal{A}_I}(\emptyset) = \emptyset$, $\Phi_{\mathcal{A}_I}(\mathbb{R}) = \mathbb{R}$.

(4) For each $A, B \in S$, $\Phi_{\mathcal{A}_I}(A \cap B) = \Phi_{\mathcal{A}_I}(A) \cap \Phi_{\mathcal{A}_I}(B)$.

Proof. It suffices to prove (4). Observe first that if $A \subset B$, $A$, $B \in S$, then $\Phi_{\mathcal{A}_I}(A) \subset \Phi_{\mathcal{A}_I}(B)$, so $\Phi_{\mathcal{A}_I}(A \cap B) \subset \Phi_{\mathcal{A}_I}(A) \cap \Phi_{\mathcal{A}_I}(B)$. To prove the opposite inclusion, assume $x \in \Phi_{\mathcal{A}_I}(A) \cap \Phi_{\mathcal{A}_I}(B)$. Let $\{t_n\}_{n\in N}$ be an arbitrary sequence of real numbers decreasing to zero. From $x \in \Phi_{\mathcal{A}_I}(A)$, by definition, there is its subsequence $\{t_{n_m}\}_{m\in N}$ and a set $A_1 \in \mathcal{A}_I$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$ of characteristic functions converges $I$-almost everywhere on $A_1$ to 1. Similarly, for $\{t_{n_m}\}_{m\in N}$ from $x \in \Phi_{\mathcal{A}_I}(B)$, by definition, there is a subsequence $\left\{t_{n_{m_k}}\right\}_{k\in N}$ and a set $B_1 \in \mathcal{A}_I$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot(A-x)\cap[-1,1]}\right\}_{k\in N}$ of characteristic functions converges $I$-almost everywhere on $B_1$ to 1. It is clear that the sequence $\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot((A\cap B)-x)\cap[-1,1]}\right\}_{k\in N}$ converges $I$-almost everywhere on $A_1 \cap B_1$ to 1, i.e., $x$ is a $\Phi_{\mathcal{A}_I}$-density point of $A \cap B$. $\square$

**Remark 2.** With Definition 1, all the results from [WO3] stay valid. Since, we do not require any convergence of the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$ on the set $[-1,1] \setminus B$, some proofs may be even shorter.

## Generalization

**Definition 2.** $\mathcal{A}_{II}$ will denote a family of subsets of $[-1,1]$ having the Baire property that have zero as an $\mathcal{A}_I$-density point. $\mathcal{A}_{II}^{+}$ (and $\mathcal{A}_{II}^{-}$) will denote a family of subsets of $[-1,1]$ having the Baire property that have zero as a right (left) $\mathcal{A}_I$-density point. If $A \in \mathcal{A}_{II}^{+}$ $\left(A \in \mathcal{A}_{II}^{-}\right)$, we say that 0 is an $\mathcal{A}_I$-density point of $A$ from the right (left).

**Remark 3.** Since $[-1,1] \in \mathcal{A}_I$, we clearly have $\mathcal{A}_I \subset \mathcal{A}_{II}$.

**Definition 3.** We shall say that $x$ is an $\mathcal{A}_{II}$-density point of $A \in S$ if, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_{II}$, such that the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges $I$-almost everywhere on $B$ to 1.

The set of all $\mathcal{A}_{II}$-density points of $A \in S$ will be denoted by $\Phi_{\mathcal{A}_{II}}(A)$.

**Proposition 1.** *For each $A \in S$,*

$$\Phi_{\mathcal{A}_I}(A) \subset \Phi_{\mathcal{A}_{II}}(A)\,.$$

Proof. This is a simple consequence of the inclusion $\mathcal{A}_I \subset \mathcal{A}_{II}$. □

**Lemma 1** ([WO3]). *Let $A \subset [0,1]$ be a set with the Baire property and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 1, $a_n < \frac{3}{2}$. Then, the sequence of characteristic functions $\{\chi_{a_n\cdot A}\}_{n\in\mathbb{N}}$ converges $I$-a.e. on $[-1,1]$ to $\chi_A$.*

To shorten the notation, given a sequence $\{a_n\}_{n\in\mathbb{N}}$, we will use $\left\{a_n^{(1)}\right\}_{n\in\mathbb{N}}$, $\left\{a_n^{(2)}\right\}_{n\in\mathbb{N}}, \ldots, \left\{a_n^{(k)}\right\}_{n\in\mathbb{N}}$ for its consecutive subsequences, and $\left\{a_n^{(k)-1}\right\}_{n\in\mathbb{N}}$ will denote the sequence of elements of the sequence $\{a_n\}_{n\in\mathbb{N}}$ directly preceding elements of $\left\{a_n^{(k)}\right\}_{n\in\mathbb{N}}$ (immediate predecessors of elements of $\left\{a_n^{(k)}\right\}_{n\in\mathbb{N}}$ in $\{a_n\}_{n\in\mathbb{N}}$). The sequence of immediate successors will be denoted by $\left\{a_n^{(k)+1}\right\}_{n\in\mathbb{N}}$.

**Proposition 2.** *There exists a set $A$ such that $\Phi(A) = \Phi_{\mathcal{A}_I}(A) \subsetneqq \Phi_{\mathcal{A}_{II}}(A)$.*

Proof. We start with the notion of the density from the right. We shall define a set $E$ such that:

1) 0 is not an $\mathcal{A}_I$-density point of $E$ from the right.
2) 0 is not an $\mathcal{A}_I$-density point of $\mathbb{R} \setminus E$ from the right.
3) 0 is an $\mathcal{A}_{II}$-density point of $E$ from the right.

The proof is analogous to the proof presented in [WO4] for the measure case with the appropriate changes. For the convenience of the reader, we present it in a complete form.

Let $D \in \mathcal{A}_I$ be a set such that $[0,1] \setminus D \in S \setminus I$, and $\{c_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, $c_1 < 1$, such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$.

The set $A \in S$ is defined as

$$A = \bigcup_{n=1}^{\infty} \big[(c_n \cdot D) \cap (c_{n+1}, c_n)\big].$$

From the definition of $A$, we have

$$\left(\frac{1}{c_n} \cdot A\right) \cap \left(\frac{c_{n+1}}{c_n}, 1\right) = D \cap \left(\frac{c_{n+1}}{c_n}, 1\right),$$

and by the proof in [WO3, Proposition 3 ], $A \in \mathcal{A}^+_{II}$, i.e., 0 is an $\mathcal{A}_I$-density point of $A$ from the right.

Let $\{d_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0 such that
$$\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = 0, \quad d_1 < 1.$$

We define $E \in S$ as follows
$$E = \bigcup_{n\in\mathbb{N}} \big[(d_n \cdot A) \cap (d_{n+1}, d_n)\big].$$

Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. We can find subsequences $\{t_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{d_n^{(1)}\}_{n\in\mathbb{N}}$ such that $d_n^{(1)} \le t_n^{(1)}$, and there are neither elements of $\{t_n\}_{n\in\mathbb{N}}$ nor of $\{d_n\}_{n\in\mathbb{N}}$ between $d_n^{(1)}$ and $t_n^{(1)}$.

Let us consider the sequence $\{\frac{d_n^{(1)}}{t_n^{(1)}}\}_{n\in\mathbb{N}} \subset (0,1]$. It is possible to find a convergent subsequence $\{\frac{d_n^{(2)}}{t_n^{(2)}}\}_{n\in\mathbb{N}}$. There are two possibilities:

(a) $\lim_{n\to\infty} \frac{d_n^{(2)}}{t_n^{(2)}} = c \neq 0$, i.e., $\lim_{n\to\infty} \frac{d_n^{(2)}}{ct_n^{(2)}} = 1$ (of course, $c \le 1$).

Since $\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = 0$, we have $\lim_{n\to\infty} \frac{d_n^{(2)+1}}{c\cdot t_n^{(2)}} = 0$. By the definition of $E$, for every $0 < \varepsilon < \frac{1}{2}$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have
$$\left(\frac{1}{c\cdot t_n^{(2)}} \cdot E\right) \cap (\varepsilon, 1-\varepsilon) = \left(\frac{d_n^{(2)}}{c\cdot t_n^{(2)}} \cdot A\right) \cap\, (\varepsilon, 1-\varepsilon)\,.$$

By Lemma 1, $\left\{\chi_{\left(\frac{d_n^{(2)}}{c\cdot t_n^{(2)}}\cdot A\cap[0,1]\right)}\right\}_{n\in\mathbb{N}}$ converges $I$-a.e. to $\chi_{A\cap[0,1]}$ on $(\varepsilon, 1-\varepsilon)$.

Since $\varepsilon$ was arbitrary, $\left\{\chi_{\left(\frac{d_n^{(2)}}{c\cdot t_n^{(2)}}\cdot A\cap[0,1]\right)}\right\}_{n\in\mathbb{N}}$ and thus also $\left\{\chi_{\left(\left(\frac{1}{c\cdot t_n^{(2)}}\cdot E\right)\cap[0,1]\right)}\right\}_{n\in\mathbb{N}}$ converge $I$-a.e. to $\chi_{A\cap[0,1]}$ on $[0,1]$.

Equivalently, $\left\{\chi_{\left(\left(\frac{1}{t_n^{(2)}}\cdot E\right)\cap[0,c]\right)}\right\}_{n\in\mathbb{N}}$ converges $I$-a.e. to $\chi_{(c\cdot A)\cap[0,c]}$ on $[0,c]$, and thus, $I$-a.e. on $(c\cdot A)\cap[0,c]$ to 1. Since 0 is an $\mathcal{A}_I$-density point of $(c\cdot A)\cap[0,c]$ from the right, we put
$$C = (c\cdot A)\cap[0,c] \quad \text{and} \quad C \in \mathcal{A}^+_{II}.$$

(b) $\lim_{n\to\infty} \frac{d_n^{(2)}}{t_n^{(2)}} = 0$.

Again, there are two possibilities:

(b1) The sequence $\{\frac{d_n^{(2)-1}}{t_n^{(2)}}\}_{n\in\mathbb{N}}$ has a subsequence $\{\frac{d_n^{(3)-1}}{t_n^{(3)}}\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{d_n^{(3)-1}}{t_n^{(3)}} = c < \infty$. Then, we proceed similarly as in $(a)$.

We clearly have $c > 1$, and, by an argument similar to that used in part (a), $\left\{\chi_{\left(\frac{1}{t_n^{(3)}}\cdot E\right)\cap[0,c]}\right\}_{n\in\mathbb{N}}$ is convergent $I$-a.e. on $[0,1]$ to $\chi_{(c\cdot A)\cap[0,1]}$, and thus, convergent $I$-a.e. on $(c\cdot A)\cap[0,1]$ to 1. Since 0 is an $\mathcal{A}_I$-density point of $(c\cdot A)\cap[0,1]$ from the right, we put

$$B = (c\cdot A)\cap[0,1] \quad \text{and} \quad B\in\mathcal{A}_{II}^{+}.$$

(b2) The sequence $\left\{\frac{d_n^{(2)-1}}{t_n^{(2)}}\right\}_{n\in\mathbb{N}}$ tends to infinity. By the definition of $E$, for every $0<\varepsilon<\frac{1}{2}$, there exists $n_0\in\mathbb{N}$ such that for all $n>n_0$

$$\left(\frac{1}{t_n^{(2)}}\cdot E\right)\cap\left(\varepsilon,\frac{1}{\varepsilon}\right) = \left(\frac{d_n^{(2)-1}}{t_n^{(2)}}\cdot A\right)\cap\left(\varepsilon,\frac{1}{\varepsilon}\right).$$

We can find a subsequence $\left\{\frac{d_n^{(3)-1}}{t_n^{(3)}}\right\}_{n\in\mathbb{N}}$ of $\left\{\frac{d_n^{(2)-1}}{t_n^{(2)}}\right\}_{n\in\mathbb{N}}$ and a subsequence $\left\{c_n^{(1)}\right\}_{n\in\mathbb{N}}$ of sequence $\{c_n\}_{n\in\mathbb{N}}$ such that

$$\frac{d_n^{(3)-1}}{t_n^{(3)}}c_n^{(1)}\le 1, \quad \frac{d_{n+1}^{(3)-1}}{t_{n+1}^{(3)}}c_n^{(1)}>1, \quad \frac{d_n^{(3)-1}}{t_n^{(3)}}c_n^{(1)-1}>1.$$

We shall consider the behavior of the sequence $\left\{\frac{d_n^{(3)-1}}{t_n^{(3)}}c_n^{(1)}\right\}_{n\in\mathbb{N}}$. There are two cases (b2a) and (b2b).

(b2a) There is a subsequence $\left\{\frac{d_n^{(4)-1}}{t_n^{(4)}}c_n^{(2)}\right\}_{n\in\mathbb{N}}$ convergent to some $0<c\le 1$.

Then, the sequence of characteristic functions $\left\{\chi_{\left(\frac{1}{t_n^{(4)}}\cdot E\right)\cap(0,c)}\right\}_{n\in\mathbb{N}}$ is convergent to $\chi_{(c\cdot D)\cap(0,c)}$ $I$-a.e. on $(0,c)$, and thus, it converges $I$-a.e. on $(c\cdot D)\cap(0,c)$ to 1. Since 0 is an $I$-density point of $(c\cdot D)\cap(0,c)$ from the right and as $b\le 1$, put

$$B=(c\cdot D)\cap[0,c] \quad \text{and} \quad B\in\mathcal{A}_I^{+}\subset\mathcal{A}_{II}^{+}.$$

(b2b) There is a subsequence $\left\{\frac{d_n^{(4)-1}}{t_n^{(4)}}c_n^{(2)}\right\}_{n\in\mathbb{N}}$ convergent to zero. We have the following two subcases.

(b2b1) The sequence $\left\{\frac{d_n^{(4)-1}}{t_n^{(4)}}c_n^{(2)-1}\right\}_{n\in\mathbb{N}}$ is bounded. Then, we can find its subsequence $\left\{\frac{d_n^{(5)-1}}{t_n^{(5)}}c_n^{(3)-1}\right\}_{n\in\mathbb{N}}$ convergent to some $c\ge 1$. Then, the sequence of characteristic functions $\left\{\chi_{\left(\frac{1}{t_n^{(5)}}\cdot E\right)\cap(0,c)}\right\}_{n\in\mathbb{N}}$ is convergent $I$-a.e. on $(0,c)$ to $\chi_{(c\cdot D)\cap(0,c)}$, and thus, it converges $I$-a.e. on $(c\cdot D)\cap(0,1)$ to 1. Since 0 is an $I$-density point of $(c\cdot D)\cap(0,c)$ from the right and as $b\ge 1$, put

$$B=(c\cdot D)\cap[0,1] \quad \text{and} \quad B\in\mathcal{A}_I^{+}\subset\mathcal{A}_{II}^{+}.$$

(b2b2) We can find an increasing subsequence

$$\left\{\frac{d_n^{(5)-1}}{t_n^{(5)}}c_n^{(3)-1}\right\}_{n\in\mathbb{N}} \quad \text{of} \quad \left\{\frac{d_n^{(4)-1}}{t_n^{(4)}}c_n^{(2)-1}\right\}_{n\in\mathbb{N}},$$

convergent to infinity. Then, we can take one more subsequence $\left\{t_n^{(6)}\right\}_{n\in\mathbb{N}}$ to obtain the sequence of characteristic functions $\left\{\chi_{\left(\frac{1}{t_n^{(6)}}\cdot E\right)\cap(c,1)}\right\}_{n\in\mathbb{N}}$ convergent to $\chi_{[0,1]}$ $I$-a.e. on $[0,1]$ and thus, it converges $I$-a.e. on $[0,1]$ to 1. Since $[0,1]$ has 0 as an $I$-density point from the right, we determine the set $B$ as $B=[0,1]$ and $B\in\mathcal{A}_{II}^{+}$.

Finally, 0 is an $\mathcal{A}_{II}$-density point of $E$ from the right and we define

$$H=-E\cup E.$$

Clearly, 0 is an $\mathcal{A}_{II}$-density point of $H$, however, not an $\mathcal{A}_I$-density point of $H$ nor of $\mathbb{R}\setminus H$. □

**Remark 4.** Observe that in the above proof, the set $B\in\mathcal{A}_{II}$ associated, as required in Definition 3, with the appropriate subsequence of $\{t_n\}_{n\in\mathbb{N}}$ either equals the interval $[0,1]$ or is a dilation either of the set $D$ or the set $A$, when restricted to the interval $(0,c)$ for some $c\in(0,1]$.

**Theorem 2.** *The mapping $\Phi_{\mathcal{A}_{II}}: S\to 2^{\mathbb{R}}$ has the following properties*:

(0) For each $A\in S$, $\Phi_{\mathcal{A}_{II}}(A)\in S$.

(1) For each $A\in S$, $A\sim\Phi_{\mathcal{A}_{II}}(A)$.

(2) For each $A,B\in S$, if $A\sim B$, then $\Phi_{\mathcal{A}_{II}}(A)=\Phi_{\mathcal{A}_{II}}(B)$.

(3) $\Phi_{\mathcal{A}_{II}}(\emptyset)=\emptyset$, $\Phi_{\mathcal{A}_{II}}(\mathbb{R})=\mathbb{R}$.

(4) For each $A,B\in S$, $\Phi_{\mathcal{A}_{II}}(A\cap B)=\Phi_{\mathcal{A}_{II}}(A)\cap\Phi_{\mathcal{A}_{II}}(B)$.

Proof. Conditions (2) and (3) are immediate consequences of the definition of $\Phi_{\mathcal{A}_{II}}$.

(4) First, observe that if $A\subset B$, $A,B\in S$, then $\Phi_{\mathcal{A}_{II}}(A)\subset\Phi_{\mathcal{A}_{II}}(B)$, so $\Phi_{\mathcal{A}_{II}}(A\cap B)\subset\Phi_{\mathcal{A}_{II}}(A)\cap\Phi_{\mathcal{A}_{II}}(B)$. To prove the opposite inclusion, assume $x\in\Phi_{\mathcal{A}_{II}}(A)\cap\Phi_{\mathcal{A}_{II}}(B)$. Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. From $x\in\Phi_{\mathcal{A}_{II}}(A)$, by definition, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $A_1\in\mathcal{A}_{II}$, such that the sequence $\left\{\chi_{\left(\frac{1}{t_{n_m}}\cdot(A-x)\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ of characteristic functions converges $I$-a.e. on $A_1$ to 1. Similarly, for $\{t_{n_m}\}_{m\in\mathbb{N}}$, there is a subsequence $\left\{t_{n_{m_k}}\right\}_{k\in\mathbb{N}}$ and a set $B_1\in\mathcal{A}_{II}$ such that the sequence $\left\{\chi_{\left(\frac{1}{t_{n_{m_k}}}\cdot(B-x)\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ of characteristic functions converges $I$-a.e. on $B_1$ to 1.

Since $A_1 \cap B_1 \in \mathcal{A}_{II}$, the sequence $\left\{\chi_{\left(\frac{1}{t_{n_{m_k}}}\cdot((A\cap B)-x)\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ converges $I$-a.e. on $A_1 \cap B_1$ to 1, i.e., $x$ is an $\mathcal{A}_{II}$-density point of $A \cap B$.

(1) Let $A \in S$. Since $\Phi(A) \subset \Phi_{\mathcal{A}_{II}}(A)$, we have $A \setminus \Phi_{\mathcal{A}_{II}}(A) \subset A \setminus \Phi(A)$, and thus, $A \setminus \Phi_{\mathcal{A}_{II}}(A) \in I$, since $A \setminus \Phi(A) \in I$.

From (3) and (4) we have

$$\Phi_{\mathcal{A}_{II}}(A) \cap \Phi_{\mathcal{A}_{II}}(\mathbb{R} \setminus A) = \Phi_{\mathcal{A}_{II}}\big(A \cap (\mathbb{R} \setminus A)\big) = \emptyset,$$

so

$$\Phi_{\mathcal{A}_{II}}(\mathbb{R} \setminus A) \subset \mathbb{R} \setminus \Phi_{\mathcal{A}_{II}}(A)$$

and consequently

$$\Phi_{\mathcal{A}_{II}}(A) \setminus A = (\mathbb{R} \setminus A) \setminus \big(\mathbb{R} \setminus \Phi_{\mathcal{A}_{II}}(A)\big) \subset (\mathbb{R} \setminus A) \setminus \Phi_{\mathcal{A}_{II}}(\mathbb{R} \setminus A).$$

From the first part of the proof, we obtain $\Phi_{\mathcal{A}_{II}}(A) \setminus A \in I$, so $A \sim \Phi_{\mathcal{A}_{II}}(A)$.

Condition (0) is a consequence of (1). □

**Remark 5.** It is an immediate consequence of (0), (1) and (2) of Theorem 2 that $\Phi_{\mathcal{A}_{II}}$ is idempotent, i.e., $\Phi_{\mathcal{A}_{II}}(A) = \Phi_{\mathcal{A}_{II}}\big(\Phi_{\mathcal{A}_{II}}(A)\big)$.

**Theorem 3.** *The family $\mathcal{T}_{\mathcal{A}_{II}} = \big\{A \in S \colon A \subset \Phi_{\mathcal{A}_{II}}(A)\big\}$ is a topology stronger than the $\mathcal{A}_I$-density topology.*

P r o o f. (compare [W2]) By Theorem 2 (3), we have $\emptyset, \mathbb{R} \in \mathcal{T}_{\mathcal{A}_{II}}$, and the family $\mathcal{T}_{\mathcal{A}_{II}}$ is closed under finite intersections by (4). To prove that $\mathcal{T}_{\mathcal{A}_{II}}$ is closed under arbitrary unions, observe that from (1) $\Phi_{\mathcal{A}_{II}}(A) \setminus A \in I$ for each $A \in S$. Take a family $\{A_t\}_{t\in T} \subset \mathcal{T}_{\mathcal{A}_{II}}$. We have $A_t \subset \Phi_{\mathcal{A}_{II}}(A)$ for each $t \in T$. Choose a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that for each $t \in T$ we have $A_t \setminus \bigcup_{n\in\mathbb{N}} A_{t_n} \in I$. It is possible by the CCC property of $(S, I)$. Then,

$$\Phi_{\mathcal{A}_{II}}(A_t) = \Phi_{\mathcal{A}_{II}}\left(\left(A_t \cap \bigcup_{n\in\mathbb{N}} A_{t_n}\right) \cup \left(A_t \setminus \bigcup_{n\in\mathbb{N}} A_{t_n}\right)\right) \subset \Phi_{\mathcal{A}_{II}}\left(\bigcup_{n\in\mathbb{N}} A_{t_n}\right)$$

for each $t \in T$. Hence,

$$\bigcup_{n\in\mathbb{N}} A_{t_n} \subset \bigcup_{t\in T} A_t \subset \bigcup_{t\in T} \Phi_{\mathcal{A}_{II}}(A_t) \subset \Phi_{\mathcal{A}_{II}}\left(\bigcup_{n\in\mathbb{N}} A_{t_n}\right).$$

The first and the last set in the above sequence of inclusions differ in a set of first category and both belong to $S$, so $\bigcup_{t\in T} A_t \in S$. Also

$$\bigcup_{t\in T} A_t \subset \Phi_{\mathcal{A}_{II}}\left(\bigcup_{t\in T} A_t\right),$$

by central inclusion and the monotonicity of $\Phi_{\mathcal{A}_{II}}$, so finally $\bigcup_{t\in T} A_t \in \mathcal{T}_{\mathcal{A}_{II}}$.

The set $(-E \cup E) \cup \{0\}$, where $E$ is defined in Proposition 2, belongs to $\mathcal{T}_{\mathcal{A}_{II}}$ but not to $\mathcal{T}_{\mathcal{A}_I}$. □

**Remark 6.** Like the density topology, the $\mathcal{A}_{II}$-density topology can be described in the form $\mathcal{T}_{\mathcal{A}_{II}} = \{\Phi_{\mathcal{A}_{II}}(A) \setminus P : A \in S \text{ and } P \in I\}$, for if $A \in \mathcal{T}_{\mathcal{A}_{II}}$, then $A \in \Phi_{\mathcal{A}_{II}}(A)$. Consequently, $A = \Phi_{\mathcal{A}_{II}}(A) \setminus (\Phi_{\mathcal{A}_{II}}(A) \setminus A)$, and we take $P = \Phi_{\mathcal{A}_{II}}(A) \setminus A \in I$. Now, if $B = \Phi_{\mathcal{A}_{II}}(A) \setminus P$, for some $A \in S$ and $P \in I$, then

$$\Phi_{\mathcal{A}_{II}}(B) = \Phi_{\mathcal{A}_{II}}\big(\Phi_{\mathcal{A}_{II}}(A) \setminus P\big) = \Phi_{\mathcal{A}_{II}}\big(\Phi_{\mathcal{A}_{II}}(A)\big) = \Phi_{\mathcal{A}_{II}}(A) \supset \Phi_{\mathcal{A}_{II}}(A) \setminus P = B,$$

from Theorem 2 (1),(2).

**Theorem 4.** *For an arbitrary set $A \subset \mathbb{R}$,*

$$\mathrm{Int}_{\mathcal{T}_{\mathcal{A}_{II}}}(\mathrm{A}) = \mathrm{A} \cap \Phi_{\mathcal{A}_{II}}(\mathrm{B}),$$

*where $B \in S$ is such that $B \subset A$ and $A \setminus B \in I$.*

**Theorem 5.** *A set $A \in \mathcal{T}_{\mathcal{A}_{II}}$ is $\mathcal{T}_{\mathcal{A}_{II}}$-regular open if and only if $A = \Phi_{\mathcal{A}_{II}}(A)$.*

**Theorem 6.** *The following equalities hold*:

$$\begin{aligned} I &= \{A \subset \mathbb{R} : A \text{ is a } \mathcal{T}_{\mathcal{A}_{II}} - \text{nowhere dense set}\} \\ &= \{A \subset \mathbb{R} : A \text{ is a } \mathcal{T}_{\mathcal{A}_{II}} - \text{first category set}\} \\ &= \{A \subset \mathbb{R} : A \text{ is a } \mathcal{T}_{\mathcal{A}_{II}} - \text{closed and } \mathcal{T}_{\mathcal{A}_{II}} - \text{discrete set}\}. \end{aligned}$$

**Theorem 7.**

(a) *The $\sigma$-algebra of $\mathcal{T}_{\mathcal{A}_{II}}$-Borel sets coincides with $S$.*

(b) *If $E \subset \mathbb{R}$ is a $\mathcal{T}_{\mathcal{A}_{II}}$-compact set, then $E$ is finite.*

(c) *The space $(\mathbb{R}, \mathcal{T}_{\mathcal{A}_{II}})$ is neither first countable, nor second countable, nor Lindelöf, nor separable. It is a Baire space.*

## Further generalization

We will continue the generalization of a density point by induction, defining an $\mathcal{A}_{III}$-density point of a set $A \in S$, later $\mathcal{A}_{IIII}$-density point of the set, and so on. Let us denote the concatenation of $k$ characters $I$ by $I(k)$, for example, $\mathcal{A}_{I(4)}$ will denote $\mathcal{A}_{IIII}$.

For $n=2$, the $\mathcal{A}_{I(2)} = \mathcal{A}_{II}$-density point, $\Phi_{\mathcal{A}_{I(2)}} = \Phi_{\mathcal{A}_{II}}$ operator, $\mathcal{T}_{\mathcal{A}_{I(2)}} = \mathcal{T}_{\mathcal{A}_{II}}$--density topology were defined in the previous section.

Now, we present our induction hypothesis.

Let $n \in \mathbb{N}$, $n > 2$. Suppose we have defined, consecutively for $k = 1, \ldots, n-1$, the notions of $\mathcal{A}_{I(k)}$-density point, the mapping $\Phi_{\mathcal{A}_{I(k)}} : S \to 2^{\mathbb{R}}$, and the topology $\mathcal{T}_{\mathcal{A}_{I(k)}}$. Assume that for $k < n$, the appropriate analogues of Proposition 2, Remark 4 and Theorem 2 given below are valid.

For $k = 1, \ldots, n-1$, the family $\mathcal{A}_{I(k)}$ is a family of subsets of $[-1, 1]$ having the Baire property that have $\mathcal{A}_{I(k-1)}$-density point at 0.

**Definition 4.** For $k = 1, \dots, n-1$, we say that $x$ is an $\mathcal{A}_{I(k)}$-density point of $A \in S$ if, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_{I(k)}$, such that the sequence $\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\}_{m\in\mathbb{N}}$ of characteristic functions converges $I$-a.e. on $B$ to 1.

The set of all $\mathcal{A}_{I(k)}$-density points of $A \in S$ is denoted by $\Phi_{\mathcal{A}_{I(k)}}(A)$.

**Proposition 3.** *For $k = 1, \dots, n-1$, there exists a set $A$ such that*

$$\Phi(A) = \Phi_{\mathcal{A}_{I(k-1)}}(A) \subsetneqq \Phi_{\mathcal{A}_{I(k)}}(A).$$

**Remark 7.** For $k = 1, \dots, n-1$ in the proof of Proposition 2, the set $B \in \mathcal{A}_{I(k-1)}$ associated, as required in Definition 4, with the appropriate subsequence of $\{t_n\}_{n\in\mathbb{N}}$ either equals the interval $[0,1]$ or is a dilation either of the set $D$ or the set $A$, when restricted to the interval $(0,c)$ for some $c \in (0,1]$.

**Theorem 8.** *For $k = 1, \dots, n-1$, the mapping $\Phi_{\mathcal{A}_{I(k)}}\colon S \to 2^{\mathbb{R}}$ has the following properties*:

(0) For each $A \in S$, $\Phi_{\mathcal{A}_{I(k)}}(A) \in S$.

(1) For each $A \in S$, $A \sim \Phi_{\mathcal{A}_{I(k)}}(A)$.

(2) For each $A, B \in S$, if $A \sim B$, then $\Phi_{\mathcal{A}_{I(k)}}(A) = \Phi_{\mathcal{A}_{I(k)}}(B)$.

(3) $\Phi_{\mathcal{A}_{I(k)}}(\emptyset) = \emptyset$, $\Phi_{\mathcal{A}_{I(k)}}(\mathbb{R}) = \mathbb{R}$.

(4) For each $A, B \in S$, $\Phi_{\mathcal{A}_{I(k)}}(A \cap B) = \Phi_{\mathcal{A}_{I(k)}}(A) \cap \Phi_{\mathcal{A}_{I(k)}}(B)$.

Let us consider one more family of sets having the Baire property:

$\mathcal{A}_{I(n)}$: a family of subsets of $[-1,1]$ that have $\mathcal{A}_{I(n-1)}$-density point at 0. $\mathcal{A}^+_{I(n)}$ and $\mathcal{A}^-_{I(n)}$ will denote a family of subsets of $[-1,1]$ having the Baire property that have 0 as a right (left) $\mathcal{A}_{I(n)}$-density point. If $A \in \mathcal{A}^+_{I(n)}$ $(A \in \mathcal{A}^-_{I(n)})$, we say that 0 is an $\mathcal{A}_{I(n)}$-density point of $A$ from the right (left).

**Definition 5.** We say that $x$ is an $\mathcal{A}_{I(n)}$-density point of $A \in S$ if, for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{A}_{I(n)}$, such that the sequence $\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\}_{m\in\mathbb{N}}$ of characteristic functions converges $I$-a.e. on $B$ to 1.

**Proposition 4.**

(a) *For each $A \in S$, $\Phi(A) \subset \Phi_{\mathcal{A}_I}(A) \subset \Phi_{\mathcal{A}_{II}}(A) \subset \dots \subset \Phi_{\mathcal{A}_{I(n-1)}}(A) \subset \Phi_{\mathcal{A}_{I(n)}}(A)$, $n \in \mathbb{N}$.*

(b) *For each $n \in \mathbb{N}$, there exists a set $A$ such that $\Phi(A) \subsetneqq \Phi_{\mathcal{A}_I}(A) \subsetneqq \Phi_{\mathcal{A}_{II}}(A) \subsetneqq \dots \subsetneqq \Phi_{\mathcal{A}_{I(n-1)}}(A) \subsetneqq \Phi_{\mathcal{A}_{I(n)}}(A)$.*

Proof. Part (a) follows from the fact that the requirements on $A$ to have $x$ as an $\mathcal{A}_{I(n)}$-density point are weaker than the requirements on the set to have $x$ as an $\mathcal{A}_{I(n-1)}$-density point.

As previously, the proof of (b) is analogous to the proof presented in [WO4] for the measure case with the appropriate changes. We present it here in a complete form for the convenience of the reader.

To prove (b), for given $n \in \mathbb{N}$, we extend the construction of the set $H$ described in Proposition 2 up to $n$ constructive steps. Again, we shall start with the right-side notion of the $\mathcal{A}_{I(n)}$-density. We shall define a set $E \in S$ such that:

(1) 0 is not an $\mathcal{A}_{I(n-1)}$-density point of $E$ from the right.

(2) 0 is an $\mathcal{A}_{I(n)}$-density point of $E$ from the right.

Let $D^{(1)} \in \mathcal{A}_I = \mathcal{A}_{I(1)}$ be an open set such that $[0,1] \setminus D^{(1)} \in S \setminus I$, and $\{{}^{(1)}c_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, ${}^{(1)}c_1 < 1$, such that $\lim_{i\to\infty} \frac{{}^{(1)}c_{i+1}}{{}^{(1)}c_i} = 0$. We define the set $D^{(2)} \in S$ as

$$D^{(2)} = \bigcup_{i=1}^{\infty} \Big[ \big( {}^{(1)}c_i \cdot D^{(1)} \big) \cap \big( {}^{(1)}c_{i+1}, {}^{(1)}c_i \big) \Big] \cap (0,1] \,.$$

By definition of $D^{(2)}$,

$$\left( \frac{1}{{}^{(1)}c_i} \cdot D^{(2)} \right) \cap \left( \frac{{}^{(1)}c_{i+1}}{{}^{(1)}c_i}, 1 \right) = D^{(1)} \cap \left( \frac{{}^{(1)}c_{i+1}}{{}^{(1)}c_i}, 1 \right),$$

$D^{(2)} \in \mathcal{A}_{I(2)}$ (by the proof of Proposition 2), i.e., 0 is an $\mathcal{A}_{I(1)}$-density point of $D^{(2)}$ from the right.

Now, let $\{{}^{(2)}c_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, ${}^{(2)}c_1 < 1$, such that $\lim_{i\to\infty} \frac{{}^{(2)}c_{i+1}}{{}^{(2)}c_i} = 0$. We define the set $D^{(3)} \in S$ as

$$D^{(3)} = \bigcup_{i=1}^{\infty} \Big[ \big( {}^{(2)}c_i \cdot D^{(2)} \big) \cap \big( {}^{(2)}c_{i+1}, {}^{(2)}c_i \big) \Big] \cap (0,1] \,.$$

Assume that we have consecutively defined the sets $D^{(k)}$, $k = 1, 2, \ldots, n$ in the above manner with the use of sequences of real numbers (respectively) $\{{}^{(k)}c_i\}_{i\in\mathbb{N}}$, $k = 1, 2, \ldots, n{-}1$, decreasing to 0, ${}^{(k)}c_1 < 1$, such that $\lim_{i\to\infty} \frac{{}^{(k)}c_{i+1}}{{}^{(k)}c_i} = 0$, such that $D^{(k)} \in \mathcal{A}_{I(k)}$, i.e., 0 is an $\mathcal{A}_{I(k-1)}$-density point of $D^{(k)}$ from the right. Let $\{{}^{(n)}c_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, ${}^{(n)}c_1 < 1$, such that $\lim_{i\to\infty} \frac{{}^{(n)}c_{i+1}}{{}^{(n)}c_i} = 0$. We define a set $E = D^{(n+1)} \in S$ as

$$E = D^{(n+1)} = \bigcup_{i=1}^{\infty} \Big[ \big( {}^{(n)}c_i \cdot D^{(n)} \big) \cap \big( {}^{(n)}c_{i+1}, {}^{(n)}c_i \big) \Big] \cap (0,1] \,.$$

Now, let $\{t_i\}_{i\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. We can find some subsequences

$$\{t_i^{(1)}\}_{i\in\mathbb{N}} \quad \text{and} \quad \{{}^{(n)}c_i^{(1)}\}_{i\in\mathbb{N}} \text{ of } \{t_i\}_{i\in\mathbb{N}} \quad \text{and} \quad \left\{{}^{(n)}c_i\right\}_{i\in\mathbb{N}}, \quad \text{respectively,}$$

such that ${}^{(n)}c_i^{(1)} \le t_i^{(1)}$, and there are no elements of $\{t_i\}_{i\in\mathbb{N}}$ nor of $\{{}^{(n)}c_i\}_{i\in\mathbb{N}}$ between ${}^{(n)}c_i^{(1)}$ and $t_i^{(1)}$. Let us consider the sequence $\{\frac{{}^{(n)}c_i^{(1)}}{t_i^{(1)}}\}_{i\in\mathbb{N}} \subset (0,1]$. We can find its convergent subsequence $\{\frac{{}^{(n)}c_i^{(2)}}{t_i^{(2)}}\}_{i\in\mathbb{N}}$. There are two possibilities:

(a) $\lim_{n\to\infty} \frac{{}^{(n)}c_i^{(2)}}{t_i^{(2)}} = a \neq 0$, i.e., $\lim_{n\to\infty} \frac{{}^{(n)}c_i^{(2)}}{at_i^{(2)}} = 1$.

By an argument similar to that used in the proof of Proposition 2 (a), we prove that

$$\left\{\chi_{\left(\left(\frac{1}{t_i^{(2)}}\cdot E\right)\cap[0,a]\right)}\right\}_{i\in\mathbb{N}}$$

converges $I$-a.e. to $\chi_{\left(a\cdot D^{(n-1)}\right)\cap[0,a]}$ on $[0,a]$.

We can consequently find one more subsequence $\{\frac{{}^{(n)}c_i^{(3)}}{t_i^{(3)}}\}_{i\in\mathbb{N}}$ such that

$$\left\{\chi_{\left(\left(\frac{1}{t_i^{(3)}}\cdot E\right)\cap[0,a]\right)}\right\}_{i\in\mathbb{N}}$$

converges $I$-a.e. to $\chi_{\left(a\cdot D^{(n)}\right)\cap[0,a]}$ on $[0,a]$, and thus, $I$-a.e. on $\left(a\cdot D^{(n)}\right)\cap[0,a]$ to 1. Since $\left(a\cdot D^{(n)}\right)\cap[0,a]\in\mathcal{A}_{I(n)}$, we put

$$B = \left(a\cdot D^{(n)}\right)\cap[0,a] \quad \text{and} \quad B\in\mathcal{A}^+_{I(n)}.$$

(b) $\lim_{n\to\infty} \frac{{}^{(n)}c_i^{(2)}}{t_i^{(2)}} = 0$.

Again, there are two possibilities:

(b1) The sequence $\{\frac{{}^{(n)}c_i^{(2)-1}}{t_i^{(2)}}\}_{i\in\mathbb{N}}$ has a subsequence $\{\frac{{}^{(n)}c_i^{(3)-1}}{t_i^{(3)}}\}_{i\in\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{{}^{(n)}c_i^{(3)-1}}{t_i^{(3)}} = b < \infty$, then we proceed similarly as in $(a)$.

We clearly have $b\geq 1$ and, by an argument similar to that used in part (a),

$$\left\{\chi_{\left(\frac{1}{t_i^{(3)}}\cdot E\right)\cap[0,b]}\right\}_{i\in\mathbb{N}}$$

is convergent $I$-a.e. on $[0,1]$ to $\chi_{\left(b\cdot D^{(n)}\right)\cap[0,1]}$, and thus, convergent $I$-a.e. on $\left(b\cdot D^{(n)}\right)\cap[0,1]$ to 1. Since $\left(b\cdot D^{(n)}\right)\cap[0,1]\in\mathcal{A}_{I(n)}$, we put

$$B=\left(b\cdot D^{(n)}\right)\cap[0,1] \quad \text{and} \quad B\in\mathcal{A}^{+}_{I(n)}.$$

(b2) The sequence $\left\{\frac{{}^{(n)}c_i^{(2)-1}}{t_i^{(2)}}\right\}_{i\in\mathbb{N}}$ tends to infinity. By the definition of $E$, for every $0<\varepsilon<\frac{1}{2}$, there exists $i_0\in\mathbb{N}$ such that for all $i>i_0$ we have

$$\left(\frac{1}{t_i^{(2)}}\cdot E\right)\cap\left(\varepsilon,\frac{1}{\varepsilon}\right)=\left(\frac{{}^{(n)}c_i^{(2)-1}}{t_i^{(2)}}\cdot D^{(n)}\right)\cap\left(\varepsilon,\frac{1}{\varepsilon}\right).$$

Then, according to Remark 3 and assuming, in induction hypothesis, Remark 7, the sequence

$$\left\{\chi_{\left(\left(\frac{1}{t_i^{(2)}}\cdot E\right)\cap[0,a]\right)}\right\}_{i\in\mathbb{N}}$$

is convergent $I$-a.e. on the interval $[0,1]$ to the characteristic function of a set $B$ from $S$, which is either the interval $[0,1]$ or a set that, if restricted to interval $(0,c)$ for some $c\le 1$, is the image under a dilation of some of $D^{(k)}$, $k=1,2,\dots,n-1$. We may proceed as above and set $B\in\mathcal{A}^{+}_{I(n)}$.

Clearly, 0 is not an $\mathcal{A}_{I(n-1)}$-density point of $E$ from the right and 0 is an $\mathcal{A}_{I(n)}$-density point of $E$ from the right, i.e., $0\in\Phi_{\mathcal{A}_{I(n)}}(-E\cup E)\setminus\Phi_{\mathcal{A}_{I(n-1)}}(-E\cup E)$.

Let $\left\{(a_n,b_n)\right\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint open subintervals of $[0,1]$. In each $(a_k,b_k)$, $k\le n+1$, we inscribe the image under a dilation of the set $-D^{(k)}\cup D^{(k)}$ with the center of dilation at point 0 and a scale factor of $\frac{b_k-a_k}{2}$ to obtain a required set

$$A=\bigcup_{k\le n+1}\left[\left(\frac{b_k-a_k}{2}\cdot\left(-D^{(k)}\cup D^{(k)}\right)\right)+\frac{a_k+b_k}{2}\right].$$

For $k=1,\dots,n+1$, the point $\frac{a_k+b_k}{2}\in\Phi_{\mathcal{A}_{I(k-1)}}(A)\setminus\Phi_{\mathcal{A}_{I(k-2)}}(A)$; we assume here $\Phi_{\mathcal{A}_{I(0)}}=\Phi\left(A\right)$ and $\Phi_{\mathcal{A}_{I(-1)}}=\emptyset$. □

**Theorem 9.** *For $n\in\mathbb{N}$, the mapping $\Phi_{\mathcal{A}_{I(n)}}:S\to 2^{\mathbb{R}}$ has the following properties:*

(0) *For each $A\in S$, $\Phi_{\mathcal{A}_{I(n)}}(A)\in S$.*

(1) *For each $A\in S$, $A\sim\Phi_{\mathcal{A}_{I(n)}}(A)$.*

(2) *For each $A,B\in S$, if $A\sim B$, then $\Phi_{\mathcal{A}_{I(n)}}(A)=\Phi_{\mathcal{A}_{I(n)}}(B)$.*

(3) *$\Phi_{\mathcal{A}_{I(n)}}(\emptyset)=\emptyset$, $\Phi_{\mathcal{A}_{I(n)}}(\mathbb{R})=\mathbb{R}$.*

(4) *For each $A,B\in S$, $\Phi_{\mathcal{A}_{I(n)}}(A\cap B)=\Phi_{\mathcal{A}_{I(n)}}(A)\cap\Phi_{\mathcal{A}_{I(n)}}(B)$.*

Proof. (4) follows directly from Theorem 8 (4). □

**Remark 8.** It is an immediate consequence of (0), (1), and (2) of Theorem 9 that $\Phi_{\mathcal{A}_{I(n)}}$ is idempotent, i.e., $\Phi_{\mathcal{A}_{I(n)}}(A) = \Phi_{\mathcal{A}_{I(n)}}\big(\Phi_{\mathcal{A}_{I(n)}}(A)\big)$. We also have $\Phi_{\mathcal{A}_{I(n)}}(A) \cap \Phi_{\mathcal{A}_{I(n)}}(\mathbb{R} \setminus A) = \emptyset$.

**Theorem 10.** *For every $n \in \mathbb{N}$, the family $\mathcal{T}_{\mathcal{A}_{I(n)}} = \big\{A \in S\colon A \subset \Phi_{\mathcal{A}_{I(n)}}(A)\big\}$ is a topology stronger than the $\mathcal{T}_{\mathcal{A}_{I(n-1)}}$-density topology.*

Proof. By Theorem 9, we have $\emptyset, \mathbb{R} \in \mathcal{T}_{\mathcal{A}_{I(n)}}$, and the family $\mathcal{T}_{\mathcal{A}_{I(n)}}$ is closed under finite intersections. To prove that $\mathcal{T}_{\mathcal{A}_{I(n)}}$ is closed under arbitrary unions, observe that from Theorem 9 (1), $\Phi_{\mathcal{A}_{I(n)}}(A) \setminus A \in I$ for each $A \in S$. Take a family $\{A_t\}_{t \in T} \subset \mathcal{T}_{\mathcal{A}_{I(n)}}$. We have $A_t \subset \Phi_{\mathcal{A}_{I(n)}}(A)$ for each $t \in T$. Choose a sequence $\{t_n\}_{n \in \mathbb{N}}$ such that, for each $t \in T$, we have $A_t \setminus \bigcup_{n \in \mathbb{N}} A_{t_n} \in I$. It is possible by the CCC property of $(S, I)$. Then,

$$\Phi_{\mathcal{A}_{I(n)}}(A_t) = \Phi_{\mathcal{A}_{I(n)}}\left(\left(A_t \cap \bigcup_{n \in \mathbb{N}} A_{t_n}\right) \cup \left(A_t \setminus \bigcup_{n \in \mathbb{N}} A_{t_n}\right)\right) \subset \Phi_{\mathcal{A}_{I(n)}}\left(\bigcup_{n \in \mathbb{N}} A_{t_n}\right)$$

for each $t \in T$. Hence,

$$\bigcup_{n \in \mathbb{N}} A_{t_n} \subset \bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \Phi_{\mathcal{A}_{I(n)}}(A_t) \subset \Phi_{\mathcal{A}_{I(n)}}\left(\bigcup_{n \in \mathbb{N}} A_{t_n}\right).$$

The first and the last set in the above sequence of inclusions differ in a set of first category and both belong to $S$, so $\bigcup_{t \in T} A_t \in S$. Also $\bigcup_{t \in T} A_t \subset \Phi_{\mathcal{A}_{I(n)}}\big(\bigcup_{t \in T} A_t\big)$, by central inclusion and the monotonicity of $\Phi_{\mathcal{A}_{I(n)}}$, so finally $\bigcup_{t \in T} A_t \in \mathcal{T}_{\mathcal{A}_{I(n)}}$.

The set $(-E \cup E) \cup \{0\}$, where $E$ is defined in Proposition 4, belongs to $\mathcal{T}_{\mathcal{A}_{I(n)}}$ but not to $\mathcal{T}_{\mathcal{A}_{I(n-1)}}$. □

Thus, we now have

$$\mathcal{T} \subsetneqq \mathcal{T}_{\mathcal{A}_{I}} \subsetneqq \mathcal{T}_{\mathcal{A}_{II}} \subsetneqq \cdots \subsetneqq \mathcal{T}_{\mathcal{A}_{I(n-1)}} \subsetneqq \mathcal{T}_{\mathcal{A}_{I(n)}},$$

where $\mathcal{T}$ denotes the natural topology on the real line.

Similarly as in the case of $\mathcal{T}_{\mathcal{A}_{II}}$, relying on results from [WO3], one can obtain the following results.

**Theorem 11.** *For an arbitrary set $A \subset \mathbb{R}$*

$$\mathrm{Int}_{\mathcal{T}_{\mathcal{A}_{\mathrm{I(n)}}}}(\mathrm{A}) = \mathrm{A} \cap \Phi_{\mathcal{A}_{\mathrm{I(n)}}}(\mathrm{B}),$$

*where $B \in S$ is such that $B \subset A$ and $A \setminus B \in I$.*

**Theorem 12.** *A set $A \in \mathcal{T}_{\mathcal{A}_{I(n)}}$ is $\mathcal{T}_{\mathcal{A}_{I(n)}}$-regular open if and only if $A = \Phi_{\mathcal{A}_{I(n)}}(A)$.*

**Theorem 13.** *The following equalities hold:*

$$\begin{aligned} I &= \{A \subset \mathbb{R} \colon A \text{ is a } \mathcal{T}_{\mathcal{A}_{I(n)}}-\text{ nowhere dense set}\} \\ &= \{A \subset \mathbb{R} \colon A \text{ is a } \mathcal{T}_{\mathcal{A}_{I(n)}}-\text{ first category set}\} \\ &= \{A \subset \mathbb{R} \colon A \text{ is a } \mathcal{T}_{\mathcal{A}_{I(n)}}-\text{ closed and } \mathcal{T}_{\mathcal{A}_{I(n)}}-\text{ discrete set}\}. \end{aligned}$$

**Theorem 14.**

(a) *The $\sigma$-algebra of $\mathcal{T}_{\mathcal{A}_{I(n)}}$ – Borel sets coincides with $S$.*

(b) *If $E \subset \mathbb{R}$ is a $\mathcal{T}_{\mathcal{A}_{I(n)}}$ – compact set, then $E$ is finite.*

(c) *The space $(\mathbb{R}, \mathcal{T}_{\mathcal{A}_{I(n)}})$ is neither first countable, nor second countable, nor Lindelöf, nor separable. It is a Baire space.*

## A final generalization

**Definition 6.** We shall say that $x$ is an $\mathcal{A}_{I(\omega)}$-density point of $A \in S$ if it is an $\mathcal{A}_{I(n)}$-density point of $A$ for some $n \in \mathbb{N}$.

**Remark 9.** For any $A \in S$, we have

$$\Phi_{\mathcal{A}_{I(\omega)}}(A) = \bigcup_{n\in\mathbb{N}} \Phi_{\mathcal{A}_{I(n)}}(A)\,.$$

There exists a set $A \in S$ such that

$$\Phi_{\mathcal{A}_{I(n-1)}}(A) \subsetneqq \Phi_{\mathcal{A}_{I(n)}}(A) \subsetneqq \Phi_{\mathcal{A}_{I(\omega)}}(A)\,, \quad n \in \mathbb{N}.$$

**Proposition 5.**

(a) *For each $A \in S$, $\Phi(A) \subset \Phi_{\mathcal{A}_I}(A) \subset \Phi_{\mathcal{A}_{II}}(A) \subset \dots \subset \Phi_{\mathcal{A}_{I(n-1)}}(A) \subset \Phi_{\mathcal{A}_{I(n)}}(A) \subset \dots \subset \Phi_{\mathcal{A}_{I(\omega)}}(A)$.*

(b) *There exists a set $A$ such that $\Phi(A) \subsetneqq \Phi_{\mathcal{A}_I}(A) \subsetneqq \Phi_{\mathcal{A}_{II}}(A) \subsetneqq \dots \subsetneqq \Phi_{\mathcal{A}_{I(n-1)}}(A) \subsetneqq \Phi_{\mathcal{A}_{I(n)}}(A) \subsetneqq \dots \subsetneqq \Phi_{\mathcal{A}_{I(\omega)}}(A)$.*

Proof. Part (a) is a consequence of Remark 9.

(b) Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint open subintervals of $[0,1]$. We take the sequence of sets $D^{(k)}$, $k = 1, 2, 3, \dots n+1$, defined in Proposition 4 and extend the sequence in similar manner for all $k > n+1$. Then, we define

$$A = \bigcup_{k\in\mathbb{N}} \left[\left(\frac{b_k - a_k}{2} \cdot \left(-D^{(k)} \cup D^{(k)}\right)\right) + \frac{a_k + b_k}{2}\right].$$

□

**Theorem 15.** *The mapping $\Phi_{\mathcal{A}_{I(\omega)}}\colon S \to 2^{\mathbb{R}}$ has the following properties:*

(0) For each $A \in S$, $\Phi_{\mathcal{A}_{I(\omega)}}(A) \in S$.

(1) For each $A \in S$, $A \sim \Phi_{\mathcal{A}_{I(\omega)}}(A)$.

(2) For each $A, B \in S$, if $A \sim B$, then $\Phi_{\mathcal{A}_{I(\omega)}}(A) = \Phi_{\mathcal{A}_{I(\omega)}}(B)$.

(3) $\Phi_{\mathcal{A}_{I(n)}}(\emptyset) = \emptyset$, $\Phi_{\mathcal{A}_{I(\omega)}}(\mathbb{R}) = \mathbb{R}$.

(4) For each $A, B \in S$, $\Phi_{\mathcal{A}_{I(\omega)}}(A \cap B) = \Phi_{\mathcal{A}_{I(\omega)}}(A) \cap \Phi_{\mathcal{A}_{I(\omega)}}(B)$.

P r o o f. (4) follows from the fact that if $x \in \Phi_{\mathcal{A}_{I(\omega)}}(A) \cap \Phi_{\mathcal{A}_{I(\omega)}}(B)$, then there exists $k, l \in \mathbb{N}$ such that

$$x \in \Phi_{\mathcal{A}_{I(k)}}(A) \quad \text{and} \quad x \in \Phi_{\mathcal{A}_{I(l)}}(B)\,.$$

Then,

$$x \in \Phi_{\mathcal{A}_{I(n)}}(A) \cap \Phi_{\mathcal{A}_{I(n)}}(B)\,,$$

where $n = \max\{k, l\}$, and $x \in \Phi_{\mathcal{A}_{I(n)}}(A \cap B) \subset \Phi_{\mathcal{A}_{I(\omega)}}(A \cap B)$ follows directly from Theorem 9 (4) and Remark 9. □

**Theorem 16.** *The family $\mathcal{T}_{\mathcal{A}_{I(\omega)}} = \{A \in S\colon A \subset \Phi_{\mathcal{A}_{I(\omega)}}(A)\}$ is a topology stronger than the $\mathcal{T}_{\mathcal{A}_{I(n)}}$-density topology for every $n \in \mathbb{N}$.*

P r o o f. Obvious. □

Thus, we have now

$$\mathcal{T} \subsetneqq \mathcal{T}_{\mathcal{A}_I} \subsetneqq \mathcal{T}_{\mathcal{A}_{II}} \subsetneqq \dots \subsetneqq \mathcal{T}_{\mathcal{A}_{I(n-1)}} \subsetneqq \mathcal{T}_{\mathcal{A}_{I(n)}} \dots \subsetneqq \mathcal{T}_{\mathcal{A}_{I(\omega)}}.$$

**Theorem 17.** *For an arbitrary set $A \subset \mathbb{R}$,*

$$\operatorname{Int}_{\mathcal{T}_{\mathcal{A}_{I(\omega)}}}(A) = A \cap \Phi_{\mathcal{A}_{I(\omega)}}(B)\,,$$

*where $B \in S$ is such that $B \subset A$ and $A \setminus B \in I$.*

**Theorem 18.** *We have $\bigcup_{n\in\mathbb{N}} \mathcal{T}_{\mathcal{A}_{I(n)}} \subsetneqq \mathcal{T}_{\mathcal{A}_{I(\omega)}}$.*

P r o o f. Since $\Phi_{\mathcal{A}_{I(\omega)}}(A) = \bigcup_{n\in\mathbb{N}} \Phi_{\mathcal{A}_{I(n)}}(A)$, $A \subset \Phi_{\mathcal{A}_{I(n)}}(A)$ implies $A \subset \Phi_{\mathcal{A}_{I(\omega)}}(A)$. The set $A$ defined in proposition 5 verifies that $\bigcup_{n\in\mathbb{N}} \mathcal{T}_{\mathcal{A}_{I(n)}} \neq \mathcal{T}_{\mathcal{A}_{I(\omega)}}$. □

**Theorem 19.** *There is a set $E \in S$ such that, for every $n \in \mathbb{N}$, the difference $\Phi_{\mathcal{A}_{I(n+1)}}(E) \setminus \Phi_{\mathcal{A}_{I(n)}}(E)$ is a countable set.*

P r o o f. In each interval $[k, k+1]$, $k \in \mathbb{N}$, we put a copy of the set $A$ from Proposition 5 obtaining $E = \bigcup_{k\in\mathbb{N}}(A + k)$. Thus, in every interval $[k, k+1]$, $k \in \mathbb{N}$, we have a point in $\Phi_{\mathcal{A}_{I(n+1)}}(E) \setminus \Phi_{\mathcal{A}_{I(n)}}(E)$. □

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