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QUASICONTINUOUS FUNCTIONS, DENSELY CONTINUOUS FORMS AND COMPACTNESS

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ABSTRACT. Let X be a locally compact space. A subfamily $\mathcal{F}$ of the space $D^\star(X,\mathbb{R})$ of densely continuous forms with nonempty compact values from X to $\mathbb{R}$ equipped with the topology τ_{UC} of uniform convergence on compact sets is compact if and only if $\{\sup(F) : F \in \mathcal{F}\}$ is compact in the space $Q(X,\mathbb{R})$ of quasicontinuous functions from X to $\mathbb{R}$ equipped with the topology τ_{UC}.

1. Introduction

Quasicontinuous functions were introduced by Kempisty in 1932 in [14]. They are important in many areas of mathematics. They found applications in the study of minimal USCO and minimal CUSCO maps [7], [8], in the study of topological groups [3], [16], [18], in proofs of some generalizations of Michael's selection theorem [5], in the study of extensions of densely defined continuous functions [6], in the study of dynamical systems [4]. The quasicontinuity is also used in the study of CHART groups [17].

Densely continuous forms were introduced by Hammer and McCoy in [12]. Densely continuous forms can be considered as set-valued mappings from a topological space X into a topological space Y which have a kind of minimality property found in the theory of minimal USCO mappings. In particular, every minimal USCO mapping from a Baire space into a metric space is a densely continuous form. There is also a connection between differentiability properties of convex functions and densely continuous forms as expressed via the subdifferentials of convex functions, which are a kind of convexification of minimal USCO mappings [12].

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In [12] the authors proved the Ascoli-type theorem for densely continuous forms from a locally compact space to a boundedly compact metric space. In our paper we present new characterizations of compact subsets of $\big(D^\star\big(X,\mathbb{R}\big),\tau_{UC}\big)$ via quasicontinuous selections of elements of $D^\star(X,\mathbb{R})$.

2. Preliminaries

In what follows let X,Y be Hausdorff topological spaces, $\mathbb{Z}^+$ be the set of positive integers, $\mathbb{R}$ be the space of real numbers with the usual metric. The symbol $\overline{A}$ will stand for the closure of the set A in a topological space.

A function $f\colon X\to Y$ is quasicontinuous [19] at $x\in X$ if for every open set $V\subset Y$, $f(x)\in V$ and every open set $U\subset X$, $x\in U$ there is a nonempty open set $W\subset U$ such that $f(W)\subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

Denote by $F(X,Y)$ the set of all functions from X to Y and by $Q(X,Y)$ the set of all quasicontinuous functions in $F(X,Y)$.

By 2^Y we denote the space of all closed subsets of Y and by $CL(Y)$ the space of all nonempty closed subsets of Y. By $K(Y)$ we denote the space of all nonempty compact subsets of Y. The space of all functions from X to 2^Y we denote by $F(X,2^Y)$. We also call the functions from $F(X,2^Y)$ set-valued functions, or multifunctions, from X to Y. In our paper, we will identify functions and set-valued functions with their graphs.

To define a densely continuous form from X to Y [12], denote by $DC(X,Y)$ the set of all functions $f\in F(X,Y)$ such that the set $C(f)$ of points of continuity of f is dense in X. We call such functions densely continuous.

Of course, $DC(X,Y)$ contains the set $C(X,Y)$ of all continuous functions from X to Y. If $Y=\mathbb{R}$ and X is a Baire space, then all upper and lower semicontinuous functions on X belongs to $DC(X,Y)$ and if X is a Baire space and Y is a metric space, then every quasicontinuous function $f\colon X\to Y$ has a dense G_δ-set $C(f)$ of the points of continuity of f [19]; i.e., $Q(X,Y)\subset DC(X,Y)$. Notice that points of continuity and quasicontinuity of functions are studied in [2].

For every $f\in DC(X,Y)$, we denote by $\overline{f\restriction C(f)}$ the closure of the graph of $f\restriction C(f)$ in $X\times Y$. If D is any dense subset of $C(f)$, then $\overline{f\restriction D}$, the closure of the graph of $f\restriction D$ in $X\times Y$ is equal to $\overline{f\restriction C(f)}$. We define the set $D(X,Y)$ of densely continuous forms by

$$D(X,Y)=\big\{\overline{f\restriction C(f)}\colon f\in DC(X,Y)\big\}.$$

Densely continuous forms from X to Y may be considered as set-valued mappings, where for each $x\in X$ and $F\in D(X,Y)$, $F(x)=\big\{y\in Y\colon (x,y)\in F\big\}$.

Definition 2.1 ([19])**.** Let X, Y be topological spaces and F be a set-valued mapping from X to Y. F is lower quasicontinuous at a point $x \in X$ if for every open set G intersecting $F(x)$ and every open set H containing x there is a nonempty open set $U \subset H$ such that $G \cap F(u) \neq \emptyset$ for any $u \in U$. F is lower quasicontinuous if it is lower quasicontinuous at any point of X.

We will study relations between quasicontinuous functions and densely continuous forms.

The following two propositions are clear.

Proposition 2.2. *Let X, Y be topological spaces and F be a densely continuous form from X to Y. Then F is lower quasicontinuous.*

Proposition 2.3. *Let X, Y be topological spaces. Let f be a function from X to Y. If $f \subset \overline{f \upharpoonright C(f)}$, then f is a quasicontinuous function and $f \in DC(X, Y)$.*

Remark 2.4. It is easy to see that if $f \in Q(X, Y)$ and D is a dense subset of X, then $\overline{f} = \overline{f \upharpoonright D}$.

Denote by

$$D^{\bullet}(X, Y) = \{F \in D(X, Y) \colon F(x) \neq \emptyset \text{ for every } x \in X\}.$$

We have the following characterizations of the elements of $D^{\bullet}(X, Y)$ [7]. For a reader's convenience we will prove it.

Theorem 2.5. *Let X be a Baire space and Y be a metric space. Let F be a set-valued mapping from X to Y. The following are equivalent:*

(1) $F \in D^{\bullet}(X, Y)$.

(2) There is a quasicontinuous function $f \colon X \to Y$ such that $\overline{f} = F$.

(3) Every selection f of F is quasicontinuous and $\overline{f} = F$.

Proof. (1) $\Rightarrow$ (3) Let f be a selection of F. There is $g \in DC(X, Y)$ such that $F = \overline{g \upharpoonright C(g)}$. Of course $F(x) = \{g(x)\}$ for every $x \in C(g)$; i.e., $f(x) = g(x)$ for every $x \in C(g)$. It is easy to verify that $C(g) \subset C(f)$. (Let $x \in C(g)$. Suppose $x \notin C(f)$. There is an open neighbourhood V of $f(x)$ such that for every open neighbourhood U of x there is $x_U \in U$ with $f(x_U) \notin V$. Let H be an open neighbourhood of $f(x)$ such that $\overline{H} \subset V$. The continuity of g at x implies that there is an open neighbourhood O of x such that $g(O) \subset H$. Then $O \times (Y \setminus \overline{H})$ is a neighbourhood of $(x_O, f(x_O))$ which has an empty intersection with the graph of g, a contradiction, since $f \subset \overline{g \upharpoonright C(g)}$.)

Since $f \subset F = \overline{g \upharpoonright C(g)} \subset \overline{f \upharpoonright C(f)}$, by Proposition 2.3 we have that f is quasicontinuous. Of course $F = \overline{f}$.

(3) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (1) is also trivial since if $f \in Q(X, Y)$, then $C(f)$ is a dense G_{δ}-set by [19] and by Remark 2.4 $\overline{f} = \overline{f \upharpoonright C(f)} = F$. ☐

Notice that closures of graphs of quasicontinuous functions were studied also in [15].

Let (Y, d) be a metric space. The open d-ball with center $z_0 \in Y$ and radius $\epsilon > 0$ will be denoted by $S_\epsilon(z_0)$ and the ϵ-enlargement $\bigcup_{a \in A} S_\epsilon(a)$ for a subset A of Y will be denoted by $S_\epsilon(A)$.

If $A \in CL(Y)$, the distance functional $d(., A) \colon Y \mapsto [0, \infty)$ is described by the familiar formula

$$d(z, A) = \inf\{d(z, a) \colon a \in A\}.$$

Let A and B be nonempty subsets of (Y, d). The excess of A over B with respect to d is defined by the formula

$$e_d(A, B) = \sup\{d(a, B) \colon a \in A\}.$$

The Hausdorff (extended-valued) metric H_d on $CL(Y)$ [1] is defined by

$$H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$

We will often use the following equality on $CL(Y)$:

$$H_d(A, B) = \inf\{\varepsilon > 0 \colon A \subset S_\varepsilon(B) \text{ and } B \subset S_\varepsilon(A)\}.$$

If (Y, d) is a complete metric space, then both $\big(CL(Y), H_d\big)$ and $\big(K(Y), H_d\big)$ are complete [1].

The topology generated by H_d is called the Hausdorff metric topology.

Following [12] we will define the topology τ_p of pointwise convergence on $F(X, 2^Y)$. The topology τ_p of pointwise convergence on $F(X, 2^Y)$ is induced by the uniformity $\mathfrak{U}_p$ of pointwise convergence which has a base consisting of sets of the form

$$W(A, \varepsilon) = \Big\{(\Phi, \Psi) \colon \forall\ x \in A \quad H_d\big(\Phi(x), \Psi(x)\big) < \varepsilon\Big\},$$

where A is a finite set in X and $\varepsilon > 0$.

We will define the topology τ_{UC} of uniform convergence on compact sets on $F(X, 2^Y)$ [12]. This topology is induced by the uniformity $\mathfrak{U}_{UC}$ which has a base consisting of sets of the form

$$W(K, \varepsilon) = \Big\{(\Phi, \Psi) \colon \forall\ x \in K \ \ H_d\big(\Phi(x), \Psi(x)\big) < \varepsilon\Big\},$$

where $K \in K(X)$ and $\varepsilon > 0$. The general τ_{UC}-basic neighborhood of $\Phi \in F(X, 2^Y)$ will be denoted by $W(\Phi, K, \varepsilon)$, i.e.,

$$W(\Phi, K, \varepsilon) = W(K, \varepsilon)[\Phi] = \Big\{\Psi \colon H_d\big(\Phi(x), \Psi(x)\big) < \varepsilon \text{ for every } x \in K\Big\}.$$

Finally we will define the topology τ_U of uniform convergence on $F(X, 2^Y)$ [12]. Let e be the (extended-valued) metric on $F(X, 2^Y)$ defined by

$$e(\Phi, \Psi) = \sup\Big\{H_d\big(\Phi(x), \Psi(x)\big) \colon x \in X\Big\}$$

for each $\Phi, \Psi \in F(X, 2^Y)$. Then the topology of uniform convergence for the space $F(X, 2^Y)$ is the topology generated by the metric e.

We use the symbols τ_p, τ_{UC} and τ_U also for the topology of pointwise convergence, the topology of uniform convergence on compacta and the topology of uniform convergence on the space of all functions from X to Y, respectively.

3. Densely continuous forms and quasicontinuous functions

Let X and Y be topological spaces. Define the mapping

$$\phi\colon F(X,Y) \to F(X, 2^Y) \text{ as } \phi(f) = \overline{f}.$$

We say that a metric space (Y, d) is boundedly compact [1] if every closed bounded subset is compact. Therefore (Y, d) is a locally compact, separable metric space and d is complete. In fact, any locally compact, separable metric space has a compatible metric d such that (Y, d) is a boundedly compact space [20].

Proposition 3.1. *Let X be a topological space and (Y, d) be a boundedly compact metric space. The mapping $\phi\colon \big(F(X,Y), \tau_U\big) \to \big(F(X, 2^Y), \tau_U\big)$ is continuous.*

Proof. Let $\{f_\sigma : \sigma \in \Sigma\}$ be a net in $F(X,Y)$ which uniformly converges to $f \in F(X,Y)$. We show that $\{\overline{f}_\sigma : \sigma \in \Sigma\}$ uniformly converges to $\overline{f}$ in $F(X, 2^Y)$. Suppose that $\{\overline{f}_\sigma : \sigma \in \Sigma\}$ fails to converge uniformly to $\overline{f}$. There exists $\epsilon > 0$ such that

$$\begin{aligned} &\text{for every } \sigma \in \Sigma \text{ there are } \beta_\sigma \geq \sigma \text{ and } a_\sigma \in X \\ &\text{such that } H_d\big(\overline{f}(a_\sigma), \overline{f}_{\beta_\sigma}(a_\sigma)\big) > \epsilon. \end{aligned} \tag{*}$$

There is $\sigma \in \Sigma$ such that $d\big(f_\eta(x), f(x)\big) < \epsilon/4$ for every $\eta \geq \sigma$ and for every $x \in X$. By (*) there are $\beta_\sigma \geq \sigma$ and $a_\sigma \in X$ such that

$$H_d\big(\overline{f}(a_\sigma), \overline{f}_{\beta_\sigma}(a_\sigma)\big) > \epsilon.$$

So either there exists $s \in \overline{f}(a_\sigma)$ such that $S_\epsilon(s) \cap \overline{f}_{\beta_\sigma}(a_\sigma) = \emptyset$ or there is $t \in \overline{f}_{\beta_\sigma}(a_\sigma)$ such that $S_\epsilon(t) \cap \overline{f}(a_\sigma) = \emptyset$.

Suppose there is $s \in \overline{f}(a_\sigma)$ such that $S_\epsilon(s) \cap \overline{f}_{\beta_\sigma}(a_\sigma) = \emptyset$.

Then there exists a net $\{x_\lambda : \lambda \in \Lambda\}$ converging to a_σ such that $f_{\beta_\sigma}(x_\lambda) \in S_{\epsilon/2}(s)$ for every $\lambda \in \Lambda$. Since the space (Y, d) is boundedly compact there is a point $u \in \overline{S_{\epsilon/2}(s)}$ which is a cluster point of the net $\big\{f_{\beta_\sigma}(x_\lambda)\colon \lambda \in \Lambda\big\}$ and so $S_\epsilon(s) \cap \overline{f}_{\beta_\sigma}(a_\sigma) \neq \emptyset$, a contradiction.

The other case is similar. □

The following proposition follows from the previous one.

Proposition 3.2. *Let X be a locally compact topological space and (Y,d) be a boundedly compact metric space. Then the mapping $\phi : \big(F(X,Y),\tau_{UC}\big) \to \big(F(X,2^Y),\tau_{UC}\big)$ is continuous.*

Proposition 3.3. *Let X be a locally compact topological space and (Y,d) be a boundedly compact metric space. Let $\mathcal{C}$ be a compact subset of $\big(Q(X,Y),\tau_{UC}\big)$. Then $\mathcal{D} = \big\{\overline{f} : f \in \mathcal{C}\big\}$ is a compact subset of $\big(D(X,Y),\tau_{UC}\big)$.*

P r o o f. The proof follows from Proposition 3.2. □

The following example shows that Proposition 3.2 does not work for the pointwise topology.

Example 3.4. Let $X = [0,1]$ with the usual topology. Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = 1$ for each $x \in X$ and the functions $f_n : X \to \mathbb{R}$, $n \in \mathbb{Z}^+$ defined by

$$f_n(x) = \begin{cases} \cos\frac{1}{x}, & x \in \big(0, \frac{1}{2n\pi}\big]; \\ 1, & x \in \{0\} \cup \big(\frac{1}{2n\pi}, 1\big]. \end{cases}$$

Then the sequence $\{f_n : n \in \mathbb{Z}^+\}$ pointwise converges to f, but $\{\overline{f}_n : n \in \mathbb{Z}^+\}$ does not pointwise converge to $\overline{f}$.

Denote by $D^\star(X,\mathbb{R})$ the set of all densely continuous forms with nonempty compact values in $\mathbb{R}$. Define the mapping $\sup : D^\star(X,\mathbb{R}) \to Q(X,\mathbb{R})$ as follows

$$\sup(F)(x) = \sup F(x).$$

We have the following proposition.

Proposition 3.5. *Let X be a Baire space. The mapping $\sup : D^\star(X,\mathbb{R}) \to Q(X,\mathbb{R})$ is injective.*

P r o o f. Let $F, G \in D^\star(X,\mathbb{R})$ be such that $F \neq G$. By Theorem 2.5 $\overline{\sup(F)} = F$ and $\overline{\sup(G)} = G$.

Let $(x,y) \in \overline{\sup(F)} \setminus \overline{\sup(G)}$. Let U, V be open sets in X and $\mathbb{R}$, respectively, such that $x \in U$, $y \in V$ and $(U \times V) \cap \overline{\sup(G)} = \emptyset$. Let $\big(z, \sup(F)(z)\big) \in U \times V$. Thus $\sup F(z) \neq \sup G(z)$. □

Remark 3.6. It is easy to see that if A and B are nonempty compact subsets of $\mathbb{R}$, then $d(\sup A, \sup B) \leqq H_d(A,B)$.

Theorem 3.7. *Let X be a locally compact space. The spaces $\big(D^\star(X,\mathbb{R}),\tau_{UC}\big)$ and $\big(\sup(D^\star(X,\mathbb{R})),\tau_{UC}\big)$ are uniformly isomorphic.*

Proof. From Remark 3.6 follows that the mapping $\sup : \big(D^\star(X,\mathbb{R}),\tau_{UC}\big) \to \big(Q(X,\mathbb{R}),\tau_{UC}\big)$ is uniformly continuous.

To prove that also $\sup^{-1}$ is uniformly continuous let $K \in K(X)$ and $\varepsilon > 0$. The local compactness of X implies that there is an open set V in X such that $K \subset V$ and $\overline{V}$ is compact. Let $F, G \in D^\star(X,\mathbb{R})$ be such that $d\big(\sup(F)(x), \sup(G)(x)\big) < \varepsilon$ for every $x \in \overline{V}$. We prove that $H_d(F,G) \le \varepsilon$ for every $x \in K$.

By Theorem 2.5 we have $\overline{\sup(F)} = F$ and $\overline{\sup(G)} = G$. Let $x_0 \in K$ and let $r \in F(x_0)$. Then there is a net $\{x_\sigma : \sigma \in \Sigma\}$ in V such that $\{\sup(F)(x_\sigma) : \sigma \in \Sigma\}$ converges to r. The net $\{\sup(G)(x_\sigma) : \sigma \in \Sigma\}$ has a cluster point $s \in G(x_0)$. From this follows that $d\big(r, G(x_0)\big) \le \varepsilon$. Similarly, we can show that if $s \in G(x_0)$, then $d\big(s, F(x_0)\big) \le \varepsilon$. Hence $H_d\big(F(x_0), G(x_0)\big) \le \varepsilon$. □

The following Lemma will be useful in the proof of the next theorem.

Lemma 3.8. *Let X be a locally compact space and (Y,d) be a metric space. Then $Q(X,Y)$ is a closed subset of $\big(F(X,Y),\tau_{UC}\big)$.*

Proof. It is known that the uniform limit of quasicontinuous functions with values in a metric space is quasicontinuous [19]. □

Theorem 3.9. *Let X be a locally compact topological space. Then the spaces $\big(D^\star(X,\mathbb{R}),\mathfrak{U}_{UC}\big)$ and $\big(\sup(D^\star(X,\mathbb{R})),\mathfrak{U}_{UC}\big)$ are complete uniform spaces.*

Proof. Let $\{F_\sigma : \sigma \in \Sigma\}$ be a Cauchy net in $\big(D^\star(X,\mathbb{R}),\mathfrak{U}_{UC}\big)$. By Remark 3.6 the corresponding net $\{sup(F_\sigma) : \sigma \in \Sigma\}$ is Cauchy in $\big(Q(X,\mathbb{R}),\mathfrak{U}_{UC}\big)$. Since $\mathbb{R}$ with the Euclidean metric d is complete, by [13] $\{sup(F_\sigma) : \sigma \in \Sigma\}$ τ_{UC}-converges to a function $f : X \to \mathbb{R}$. By Lemma 3.8 f is quasicontinuous. By Proposition 3.2 $\{F_\sigma : \sigma \in \Sigma\}$ τ_{UC}-converges to $\overline{f}$. Since the space $\big(K(\mathbb{R}),H_d\big)$ is complete, by [13] $\{F_\sigma : \sigma \in \Sigma\}$ τ_{UC}-converges to a $F : X \to K(\mathbb{R})$. It is easy to verify that $F = \overline{f}$. By Theorem 2.5 F is densely continuous form, i.e., $\big(D^\star(X,\mathbb{R}),\mathfrak{U}_{UC}\big)$ is complete. By Theorem 3.7 $\big(\sup(D^\star(X,\mathbb{R})),\mathfrak{U}_{UC}\big)$ is complete. □

Theorem 3.10. *Let X be a locally compact topological space. A subset $\mathcal{F} \subseteq D^\star(X,\mathbb{R})$ is compact in $\big(D^\star(X,\mathbb{R}),\tau_{UC}\big)$ if and only if $\{\sup(F) : F \in \mathcal{F}\}$ is compact in $\big(Q(X,\mathbb{R}),\tau_{UC}\big)$.*

Let $\mathcal{E} \subset F(X,Y)$ and let $x \in X$, denote by $\mathcal{E}[x]$ the set $\{f(x) \in Y;\ f \in \mathcal{E}\}$. We say that a subset $\mathcal{E}$ of $F(X,Y)$ is pointwise bounded [9] provided for every $x \in X$, $\overline{\mathcal{E}[x]}$ is compact in (Y,d).

If X is a locally compact space and (Y, d) is a metric space, the Ascoli theorem [13] says that a subset $\mathcal{E}$ of $\big(C(X, Y), \tau_{UC}\big)$ is compact if and only if it is closed in $\big(C(X, Y), \tau_{UC}\big)$, pointwise bounded and $\mathcal{E}$ is equicontinuous, where a subset $\mathcal{E}$ of $C(X, Y)$ is equicontinuous provided that for each $x \in X$ and $\epsilon > 0$ there is a neighbourhood U of x with $d\big(f(x), f(z)\big) < \epsilon$ for all $z \in U$ and $f \in \mathcal{E}$.

In [11] it is proved the Ascoli-type theorem for quasicontinuous locally bounded functions, in [9] we proved Ascoli-type theorems for quasicontinuous subcontinuous functions and in [10] we proved Ascoli-type theorems for quasicontinuous functions. To present our characterizations of compact subsets of $\big(Q(X, Y), \tau_{UC}\big)$ we need the following definition, which was introduced in [11] in the context of locally bounded functions from $F(X, Y)$.

Definition 3.11. Let X be a topological space and (Y, d) be a metric space. We say that a subset $\mathcal{E}$ of $F(X, Y)$ is densely equiquasicontinuous at a point x of X provided that for every $\epsilon > 0$, there exists a finite family $\mathcal{B}$ of subsets of X which are either open or nowhere dense such that $\cup\mathcal{B}$ is a neighbourhood of x and such that for every $f \in \mathcal{E}$, for every $B \in \mathcal{B}$ and for every $p, q \in B$, $d\big(f(p), f(q)\big) < \epsilon$. Then $\mathcal{E}$ is densely equiquasicontinuous provided that it is densely equiquasicontinuous at every point of X.

Remark 3.12. It is easy to prove that if $\mathcal{E}$ is a densely equiquasicontinuous subset of $F(X, Y)$, then closure of $\mathcal{E}$ with respect to the topology τ_p is also densely equiquasicontinuous.

We say that a system $\mathcal{E} \subset F(X, Y)$ is supported at $x \in X$ [10] if for every $\epsilon > 0$ there exists a neighbourhood $U(x)$ of x and a finite family $\{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n\}$ of nonempty subsets of $\mathcal{E}$ such that $\bigcup_{i=1}^{n} \mathcal{E}_i = \mathcal{E}$ and for every $z \in U(x)$, every $i \in \{1, 2, \dots, n\}$, and every $f, g \in \mathcal{E}_i$, $d\big(f(z), g(z)\big) < \epsilon$.

Theorem 3.13 ([10])**.** *Let X be a locally compact topological space and (Y, d) be a boundedly compact metric space. A subset $\mathcal{E} \subset \big(Q(X, Y), \tau_{UC}\big)$ is compact if and only if $\mathcal{E}$ is closed, pointwise bounded, there is a dense open set M such that $\mathcal{E}$ is densely equiquasicontinuous at each $x \in M$ and $\mathcal{E}$ is supported at each point $x \in X \setminus M$.*

Theorem 3.14 ([10])**.** *Let X be a locally compact topological space and (Y, d) be a boundedly compact metric space. A subset $\mathcal{E} \subset \big(Q(X, Y), \tau_{UC}\big)$ is compact if and only if $\mathcal{E}$ is closed, pointwise bounded, there is a dense G_δ-set G such that $\mathcal{E}$ is equicontinuous at each point $x \in G$ and $\mathcal{E}$ is supported at each point $x \in X \setminus G$.*

In [12] the authors proved the Ascoli-type theorem for densely continuous forms from a locally compact space to a boundedly compact metric space. In our paper we present new characterizations of compact subsets of $\big(D^\star(X, \mathbb{R}), \tau_{UC}\big)$ via quasicontinuous selections of elements of $D^\star(X, \mathbb{R})$.

Theorem 3.15. *Let X be a locally compact topological space. A subset $\mathcal{F} \subseteq \big(D^\star(X,\mathbb{R}), \tau_{UC}\big)$ is compact if and only if $\big\{\sup(F) : F \in \mathcal{F}\big\}$ is closed in $\big(Q(X,\mathbb{R}), \tau_{UC}\big)$, pointwise bounded, there is a dense open set M such that $\big\{\sup(F) : F \in \mathcal{F}\big\}$ is densely equiquasicontinuous at each $x \in M$ and is supported at each point $x \in X \setminus M$.*

Theorem 3.16. *Let X be a locally compact topological space. A subset $\mathcal{F} \subseteq \big(D^\star(X,\mathbb{R}), \tau_{UC}\big)$ is compact if and only if $\big\{\sup(F) : F \in \mathcal{F}\big\}$ is closed in $\big(Q(X,\mathbb{R}), \tau_{UC}\big)$, pointwise bounded, there is a dense G_δ-set G such that $\big\{\sup(F) : F \in \mathcal{F}\big\}$ is equicontinuous at each point $x \in G$ and is supported at each point $x \in X \setminus G$.*

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