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ON A LINDENBAUM COMPOSITION THEOREM

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ABSTRACT. We discuss several questions about Borel measurable functions on a topological space. We show that two Lindenbaum composition theorems [Lindenbaum, A. *Sur les superpositions des fonctions représentables analytiquement*, Fund. Math. **23** (1934), 15–37] proved for the real line hold in perfectly normal topological space as well. As an application, we extend a characterization of a certain class of topological spaces with hereditary Jayne-Rogers property for perfectly normal topological space. Finally, we pose an interesting question about lower and upper Δ^0_2-measurable functions.

1. Introduction

Let X be a topological space and let $f, g : X \to [0,1]$ be functions. If both f and g are continuous, then their composition $f \circ g$ is such as well. However, it may be more complex if we take f and g further in the Baire or Young hierarchies. For instance, if f and g are lower semicontinuous then $f \circ g$ need not be even in the first Baire class. One can show that there is still an upper bound and $f \circ g$ is a function from the second Young class. A. Lindenbaum [13] in his extensive paper about compositions of functions from various classes of Borel measurable functions showed that this upper bound is best possible for functions on the real line. In Section 4, we show that Lindenbaum's result holds in perfectly normal topological space. The technique of proof follows J. Cichoń, M. Morayne, J. Pawlikowski and S. Solecki [6], who extended the result for Polish spaces.

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B. Tsaban and L. Zdomskyy [17] investigated hereditary Hurewicz spaces including a survey on the topic. By their main result, and by L. Bukovský, I. Recław and M. Repický [3], we have that a perfectly normal topological space X is a QN-space if and only if X has Hurewicz property hereditarily and every Δ^0_2-measurable function on X is a discrete limit of continuous functions. The latter property is called Jayne-Rogers property in [4], since J. E. Jayne and C. A. Rogers [9] showed that any analytic subset of a Polish space has Jayne-Rogers property. However, it seems that the question which topological spaces possess Jayne-Rogers property is an open problem. In Section 5, extending the result in [16], we show that every lower semicontinuous function on a perfectly normal topological space X is a discrete limit of continuous functions if and only if X has Jayne-Rogers property and X is a σ-set. Moreover, in the same section, we are investigating possible candidates for convergence like characterization of lower and upper Δ^0_2-measurable functions.

The paper begins with a preliminary Section 2 containing necessary notation and introducing a proof tool from [6]. Section 3 contains a summary on Borel functions hierarchies of our interest in perfectly normal topological space. Section 4 is devoted to a proof of two Lindenbaum composition theorems for perfectly normal topological space with our own observation of a case of a limit ordinal. The last section was described in the preceding paragraph.

2. Preliminaries

All topological spaces are assumed to be infinite and Hausdorff. Basic set--theoretical and topological terminology follows mainly [2] and [8].

Let X, Y and Z be sets. Then, ${}^XY$ denotes the set of all functions from X to Y. For functions $f : X \to Y$ and $g : Y \to Z$, by the composition of these functions $g \circ f : X \to Z$, we mean $(g \circ f)(x) = g\big(f(x)\big)$. Let $\mathbf{A} \subseteq {}^XY, \mathbf{B} \subseteq {}^YZ$ be families of functions, then

$$\mathbf{B} \circ \mathbf{A} = \left\{c \in {}^XZ : (\exists a \in \mathbf{A})(\exists b \in \mathbf{B})\ b \circ a = c\right\}.$$

If a is a single function, we abuse the notation and by $a \circ \mathbf{B}$ we denote the set $\{a\} \circ \mathbf{B}$.

A family of real-valued functions is called a lattice of functions if it is closed under the pointwise maximum and minimum operation. As most of our investigation is concerned with functions attaining values in $[0, 1]$, we define pointwise addition, multiplication by a non-negative real constant c, and the additive inverse for such functions as follows

$$(f \oplus g)(x) := \min\{1, (f+g)(x)\},$$
$$(c \odot g)(x) := \min\{1, (c \cdot g)(x)\},$$
$$(\ominus g)(x) := 1 - g(x).$$

Let $(f_i)_{i\in\omega}$ be a sequence of real valued functions, then

- $f_i \to f$ denotes the fact that the sequence converges to f pointwise, i.e., the set $\{i \in \omega : |f_i(x) - f(x)| \geq \varepsilon\}$ is finite for every $x \in X$ and $\varepsilon > 0$;
- $f_i \nearrow f$ denotes a pointwise convergence of non-decreasing sequence, i.e., we have $f_i \leq f_{i+1}$, and similarly for $f_i \searrow f$;
- another type of convergence is a discrete convergence, where we require that the following set $\{i \in \omega : f_i(x) \neq f(x)\}$ is finite for all $x \in X$.

If $\mathbf{A}$ is a family of functions, then $\mathbf{A}\uparrow$ denotes the set of all functions that are nondecreasing limits of functions from $\mathbf{A}$, analogously the class $\mathbf{A}\downarrow$.

We shall use a tool developed in [6][1] to represent numbers from a particular dense subset of the unit interval $[0,1]$ as sequences of numbers of another particular dense subset of $[0,1]$. Indeed, let

$$\mathbf{S} = \left([0,1] \setminus \left\{\sum_{i=0}^{n} \frac{a_i}{2^{i+1}} : n \in \omega \wedge a_i \in \{0,1\}\right\}\right) \cup \{0\}.$$

By Φ we will denote the function from ${}^{\omega}\mathbf{S}$ to $[0,1]$ defined as

$$\Phi\left((a_i)_{i\in\omega}\right) = \sum_{n=0}^{\infty} \frac{a(\pi^{-1}(n))}{2^{n+1}},$$

where

$$a_n = \sum_{i=0}^{\infty} \frac{a(n,i)}{2^{i+1}}, \quad n < \omega,$$

and $\pi : \omega^2 \to \omega$ is the bijection defined by

$$\pi(n,k) = \frac{(n+k+1)(n+k)}{2} + k.$$

Let $\mathbf{T} = \Phi\left({}^{\omega}\mathbf{S}\right) \subseteq [0,1]$. We shall recall the following assertion from [6][1] which justifies to interpret ${}^{\omega}\mathbf{S}$ equipped with Tychonoff product topology as a set of reals.

Lemma 2.1. *The set* $\mathbf{T}$ *is dense in* $[0,1]$, $0 \in \mathbf{T} \subseteq \mathbf{S} \subseteq [0,1]$, *and* $\Phi : {}^{\omega}\mathbf{S} \to \mathbf{T}$ *is an homeomorphism.*

[1] See the proof of Theorem 4.4 in [6]

Let $\mathcal{A}$ be a family of subsets of X. A function $f : X \to [0,1]$ is $\mathcal{A}$-measurable if $f^{-1}(U) \in \mathcal{A}$ for every $U \subseteq [0,1]$. A function $f : X \to [0,1]$ is lower, upper $\mathcal{A}$-measurable if $f^{-1}\big((r,1]\big) \in \mathcal{A}$, $f^{-1}\big([0,r)\big) \in \mathcal{A}$, for every $r \in [0,1]$, respectively. The class of all $\mathcal{A}$-measurable, lower $\mathcal{A}$-measurable, and upper $\mathcal{A}$-measurable functions, respectively, is denoted by

$$\mathsf{M}\mathcal{A}, \ \underline{\mathsf{M}}\mathcal{A} \quad \text{and} \quad \overline{\mathsf{M}}\mathcal{A}\,.$$ [2]

It is easy to see that if $\mathcal{A}$ is a σ-topology[3], then

$$\mathsf{M}\mathcal{A} = \underline{\mathsf{M}}\mathcal{A} \cap \overline{\mathsf{M}}\mathcal{A}\,.$$

$f : X \to {}^{\omega}\mathbf{S}$ is lower $\mathcal{A}$-measurable and upper $\mathcal{A}$-measurable if the function

$$\Phi \circ f : X \to \mathbf{T}$$

has the same property, respectively, i.e., we inherit the order from reals.

3. Hierarchies

Hierarchies of Borel measurable functions are part of standard textbooks in topology or descriptive set theory such as [10,12]. However, the usual exposition takes care of Polish spaces. Therefore, we recall the hierarchies with special focus on more general spaces including the corresponding references. Useful sources in the realm of perfectly normal topological space are [7,15].

Let X be a topological space. The Baire hierarchy was introduced by R. Baire in his dissertation in 1899 [1], for more see [2,10,12]. We set $\mathbf{B}_0(X) = \mathbf{C}(X)$, where $\mathbf{C}(X)$ is the set of all continuous functions from X to $[0,1]$, and for all $0 < \alpha < \omega_1$,

$$\mathbf{B}_\alpha(X) = \Big\{ f \in {}^{X}[0,1] : f_n \to f \wedge f_n \in \bigcup_{\beta<\alpha} \mathbf{B}_\beta(X) \Big\}.$$

Next, there is the hierarchy we are most interested in, since in perfectly normal spaces it generalizes semicontinuous functions. Young hierarchy was introduced by W. H. Young [18,19]. Let $\mathbf{L}_0(X) = \mathbf{U}_0(X) = \mathbf{C}(X)$, then for every $0 < \alpha < \omega_1$,

$$\mathbf{L}_\alpha(X) = \big(\mathbf{U}_{<\alpha}(X)\big)\uparrow, \quad \mathbf{U}_\alpha(X) = \big(\mathbf{L}_{<\alpha}(X)\big)\downarrow, \ \text{where} \ \ \mathbf{U}_{<\alpha}(X) = \bigcup_{\beta<\alpha} \mathbf{U}_\beta(X),$$

and similarly for $\mathbf{L}_{<\alpha}(X)$.

[2] We follow [6].

[3] I.e., $\mathcal{A}$ is closed under countable union, finite intersection, contains $\emptyset$, X, see [2].

Basic properties are summarized in the following assertion which, for arbitrary topological space, is proved in a monograph by R. Sikorski [15], mostly in Theorem V.10.1.[4] In what follows for clarity, we do not denote the topological space.

Theorem 3.1. *Let X be a topological space, $\alpha, \beta < \omega_1$. Then*

(1) $f \in \mathbf{L}_\alpha$ *if and only if* $\ominus f \in \mathbf{U}_\alpha$.

(2) $\mathbf{B}_\alpha = \mathbf{L}_{\alpha+1} \cap \mathbf{U}_{\alpha+1}$.

(3) *If* $\beta < \alpha$, *then* $\mathbf{U}_\beta \subset \mathbf{L}_\alpha$, $\mathbf{L}_\beta \subset \mathbf{U}_\alpha$, $\mathbf{U}_\beta \subset \mathbf{U}_\alpha$, $\mathbf{L}_\beta \subset \mathbf{L}_\alpha$, $\mathbf{L}_\beta \subset \mathbf{B}_\alpha$, $\mathbf{U}_\beta \subset \mathbf{B}_\alpha$, $\mathbf{B}_\beta \subset \mathbf{B}_\alpha$. *Thus* $\mathbf{U}_{\alpha+1} = \mathbf{L}_\alpha \downarrow$ *and* $\mathbf{L}_{\alpha+1} = \mathbf{U}_\alpha \uparrow$.

(4) $\mathbf{L}_\alpha$ *and* $\mathbf{U}_\alpha$ *are lattices of functions and*
   (a) $\mathbf{U}_\alpha = \mathbf{U}_\alpha \downarrow$,
   (b) $\mathbf{L}_\alpha = \mathbf{L}_\alpha \uparrow$,
   (c) $\mathbf{U}_{\alpha+1} = \mathbf{B}_\alpha \downarrow$,
   (d) $\mathbf{L}_{\alpha+1} = \mathbf{B}_\alpha \uparrow$.

(5) $\mathbf{L}_\alpha$, $\mathbf{U}_\alpha$ *and* $\mathbf{B}_\alpha$ *are closed under taking uniform limits.*

(6) *If* $\{f_i : i \in \omega\} \subseteq \mathbf{L}_\alpha(X)$, *then* $\sup \{f_i : i \in \omega\} \in \mathbf{L}_\alpha(X)$.

The standard relations between Baire and Young functions are depicted in the following diagram:[5]

$$
\begin{tikzcd}
 & \mathbf{L}_1 \arrow[dr] & & \mathbf{L}_2 \arrow[dr] & & & \mathbf{L}_\alpha \arrow[dr] & & \\
\mathbf{C}(X) = \mathbf{B}_0 \arrow[ur] \arrow[dr] & & \mathbf{B}_1 \arrow[ur] \arrow[dr] & & \mathbf{B}_2 & \cdots & & \mathbf{B}_\alpha & \cdots \\
 & \mathbf{U}_1 \arrow[ur] & & \mathbf{U}_2 \arrow[ur] & & & \mathbf{U}_\alpha \arrow[ur] & &
\end{tikzcd}
$$

$\mathbf{L}_\alpha(X)$ and $\mathbf{U}_\alpha(X)$ are lattices of functions closed under addition, and multiplication by non-negative real numbers, $\mathbf{B}_\alpha(X)$ is a lattice of functions and a vector space over $\mathbb{R}$, see [7, 15].

[4]We deal with functions taking values only in the closed unit interval as opposed to R. Sikorski [15], who proved the results for real valued functions. This, however, is not a drawback, since the results for our functions follow from the real valued ones, only a minor change is needed in defining addition.

[5]An arrow indicates inclusion.

The notation for Borel hierarchy follows [2, 10], i.e., $\Sigma^0_0(X) = \Pi^0_0(X)$ are clopen sets of a topological space X, $\Sigma^0_1(X)$, $\Pi^0_1(X)$ are open and closed, respectively, and for $\alpha > 1$ we denote

$$\Sigma^0_\alpha(X) = \Big\{ \bigcup_{n\in\omega} A_n : \ A_n \in \bigcup_{\beta<\alpha} \Pi^0_\beta(X) \Big\},$$

$$\Pi^0_\alpha(X) = \Big\{ \bigcap_{n\in\omega} A_n : \ A_n \in \bigcup_{\beta<\alpha} \Sigma^0_\beta(X) \Big\}.$$

Moreover, we put $\Delta^0_\alpha(X) = \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X)$. In an arbitrary topological space, we have

$$\begin{tikzcd}
\Sigma^0_1 \arrow[dr] & \Sigma^0_2 \arrow[r] \arrow[dr] & \Sigma^0_3 \arrow[r] \arrow[dr] & \Sigma^0_4 \ \cdots \\
\Pi^0_1 \arrow[ur] & \Pi^0_2 \arrow[r] \arrow[ur] & \Pi^0_3 \arrow[r] \arrow[ur] & \Pi^0_4 \ \cdots
\end{tikzcd}$$

The missing horizontal arrows are present in any perfectly normal topological space. Functions in $\underline{\mathsf{M}}\Sigma^0_1(X)$, $\overline{\mathsf{M}}\Sigma^0_1(X)$ are called lower, upper, semicontinuous, respectively. $\mathbf{B}$ will denote the family $\mathsf{M}(\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha(X))$, i.e., the Borel measurable functions.

The following theorem is usually considered for metric spaces, see [2,6,10,12]. The third part is commonly attributed to H. Lebesgue, F. Hausdorff or S. Banach. For perfectly normal spaces, the result follows from [15], Theorem V.10.1, using Tong's theorem characterizing perfectly normal spaces, see [8, 1.7.15.(c)], and also partially from [7], Proposition 3.14.

Theorem 3.2. *Let X be a perfectly normal topological space, $0 < \alpha < \omega_1$. Then, the following equalities*

$$\begin{aligned}
\mathbf{L}_\alpha(X) &= \underline{\mathsf{M}}\,\Sigma^0_\alpha(X)\,,\\
\mathbf{U}_\alpha(X) &= \overline{\mathsf{M}}\,\Sigma^0_\alpha(X)\,,\\
\mathbf{B}_\alpha(X) &= \mathsf{M}\Sigma^0_{\alpha+1}(X)
\end{aligned}$$

hold.

A proof of the following can be found in [6], Lemma 4.3. Although the assertion stated there is for Polish spaces, the proof for perfectly normal spaces is the same.

Lemma 3.3. *Let X be a perfectly normal topological space, $\alpha < \omega_1$, $f \in \mathbf{L}_\alpha(X)$, $H \subseteq [0,1]$ dense and $0 \in H$. Then, there exists a sequence $(f_i)_{i\in\omega}$ of functions from $\mathbf{U}_{<\alpha}(X, H)$ such that $f_n \nearrow f$.*

Let the set ${}^{\omega}[0,1]$ be equipped with the Tychonoff product topology. The projection function

$$\mathsf{pr}_n : {}^{\omega}[0,1] \to [0,1]$$

defined by $\mathsf{pr}_n\big((t_i)_{i\in\omega}\big) = t_n$ is in $\mathbf{C}({}^{\omega}[0,1])$ and the supremum function

$$\mathsf{s} : {}^{\omega}[0,1] \to [0,1]$$

defined by $\mathsf{s}\big((t_i)_{i\in\omega}\big) = \sup\{t_i : i \in \omega\}$ is in $\mathbf{L}_1({}^{\omega}[0,1])$.

4. Composition of functions and convergence

We begin with a simple fact that the equalities

$$\mathbf{U}_\alpha(X) \circ \mathbf{C}(X,X) = \mathbf{U}_\alpha(X), \quad \mathbf{L}_\alpha(X) \circ \mathbf{C}(X,X) = \mathbf{L}_\alpha(X)$$

hold in an arbitrary topological space. One can easily think out an example showing that the composition in reversed order need not preserve the equality. However, restricting to the class of non-decreasing continuous functions on $\mathbb{R}$ would preserve the equalities even if the order of the composition was reversed.

Let X be a set. The coding function $\boldsymbol{F}$ is a function

$$\boldsymbol{F} : {}^{\omega}\Big({}^{X}[0,1]\Big) \to {}^{X}\Big({}^{\omega}[0,1]\Big)$$

defined by $\boldsymbol{F}_{(f_i)_{i\in\omega}}(x) = (f_i(x))_{i\in\omega}$. Exactly the same coding function $\boldsymbol{F}$ was used in [6] since it does not raise the complexity of functions of the input sequence. The following was proved for Polish spaces in [6].[1] We state it for classes $\mathbf{L}_\alpha$ although it is valid for $\mathbf{U}_\alpha$ as well.

Lemma 4.1. *Let X be a perfectly normal topological space and $\alpha < \omega_1$.*

(1) *If $(f_i)_{i\in\omega}$ is a sequence of functions from $\mathbf{L}_\alpha(X,\mathbf{S})$, then*

$$\boldsymbol{F}_{(f_i)_{i\in\omega}} \in \mathbf{L}_\alpha(X, {}^{\omega}\mathbf{S}).$$

(2) *If $(f_i)_{i\in\omega}$ is a sequence of functions from $\mathbf{L}_{<\alpha}(X,\mathbf{S})$, then*

$$\boldsymbol{F}_{(f_i)_{i\in\omega}} \in \mathsf{M}\Sigma^0_\alpha(X, {}^{\omega}\mathbf{S}).$$

P r o o f. (1) We define a function $T_n : \mathbf{S} \to \mathbf{T}$ for every $n \in \omega$ and $c_i \in \{0,1\}$ as

$$T_n\left(\sum_{i=0}^{\infty} c_i 2^{-i}\right) = \sum_{i=0}^{\infty} c_i 2^{-\pi(n,i)}.$$

T_n is increasing and continuous, thus $T_n \circ f_n \in \mathbf{L}_\alpha(X)$. We also have

$$\left(\Phi \circ \boldsymbol{F}_{(f_i)_{i\in\omega}}\right)(x) = \sum_{n=0}^{\infty} (T_n \circ f_n)(x) \quad \text{for every} \quad x \in X.$$

Using the Weierstrass M-test for the series $\sum_{n=0}^{\infty}(T_n \circ f_n)$ on X, it is easy to see that this sum is uniformly convergent, whence $\Phi \circ \boldsymbol{F}_{(f_i)_{i\in\omega}} \in \mathbf{L}_\alpha(X)$.

The proof of (2) is almost the same as of (1). Using the inclusion $\mathbf{L}_{<\alpha}(X) \subseteq \mathsf{M}\Sigma^0_\alpha(X)$ and the fact that classes of measurable functions are closed under taking uniform limits, we readily get that $\Phi \circ \boldsymbol{F}_{(f_i)_{i\in\omega}} \in \mathsf{M}\Sigma^0_\alpha(X)$. $\square$

The main essence of the proof of Lindenbaum theorem lies in a simple fact that if X is a topological space and $f \in {}^X[0,1]$, then for any sequence $(f_i)_{i\in\omega}$ of functions such that $f_n \nearrow f$ we have

$$f = \mathsf{s} \circ \boldsymbol{F}_{(f_i)_{i\in\omega}}.$$

Thus, any function $f \in \mathbf{L}_{\alpha+1}(X)$ can be represented as a composition of the lower semicontinuous function s and a function $\boldsymbol{F}_{(f_i)_{i\in\omega}} \in \mathbf{U}_\alpha(X, {}^\omega\mathbf{S})$, even though $\mathsf{s} \circ \mathbf{L}_\alpha(X, {}^\omega\mathbf{S}) = \mathbf{L}_\alpha(X, {}^\omega\mathbf{S})$. Consequently, we obtain

Theorem 4.2 (A. Lindenbaum). *Let X be a perfectly normal topological space, $\alpha < \omega_1$. Then, there exists a $g \in \mathbf{L}_1([0,1])$ such that*

(1) $\mathbf{L}_{\alpha+1}(X) \subseteq g \circ \mathbf{U}_\alpha(X, \mathbf{T})$,

(2) $\mathbf{L}_\alpha(X) \subseteq g \circ \mathsf{M}\Sigma^0_\alpha(X, \mathbf{T})$.

Proof. To construct g, we simply extend the function $\mathsf{s} \circ \Phi^{-1}$ on $[0,1]$, e.g., by Proposition 1.1 in [5]. We shall prove part (1), the proof of part (2) is analogous using the second assertion from Lemma 4.1.

(1) Let $f \in \mathbf{L}_{\alpha+1}(X)$. Using Lemma 3.3, we get a sequence of functions $(f_i)_{i\in\omega}$ such that $f_i \in \mathbf{U}_\alpha(X)$, $\operatorname{rng}(f) \subseteq \mathbf{S}$ and $f_i \nearrow f$. Thus,

$$f = s \circ \Phi^{-1} \circ \Phi \circ \boldsymbol{F}_{(f_i)_{i\in\omega}} = g \circ \left(\Phi \circ \boldsymbol{F}_{(f_i)_{i\in\omega}}\right), \quad \text{and} \quad \Phi \circ \boldsymbol{F}_{(f_i)_{i\in\omega}} \in \mathbf{U}_\alpha(X). \quad \square$$

In the part concerning the decomposition of $\mathbf{L}_\alpha$, for α limit, we cannot straightforwardly generalize the first part, i.e., in Polish spaces,

$$\mathbf{L}_\alpha(X) \nsubseteq \mathbf{L}_1 \circ \mathbf{U}_{<\alpha}(X) \qquad \text{holds.}$$

To see this, suppose otherwise, and let $f \in \mathbf{L}_\alpha(X) \setminus \mathbf{L}_{<\alpha}(X)$. We would get $f = g \circ h$, where $g \in \mathbf{L}_1$ and $h \in \mathbf{U}_\beta(X)$, for some $\beta < \alpha$. Thus, $f \in \mathbf{L}_{\beta+1}(X)$, a contradiction.

Before stating Corollary 4.3, we stress that inclusion $\mathbf{L}_\alpha([0,1]) \circ \mathbf{L}_\beta(X) \subseteq \mathbf{L}_{\beta+\alpha}(X)$ holds in an arbitrary topological space X.

Corollary 4.3. *Let X be a perfectly normal topological space, $\alpha < \omega_1$. Then, we have*

$$\mathbf{L}_{\alpha+1}(X) = \mathbf{L}_1([0,1]) \circ \mathbf{U}_\alpha(X) = \mathbf{L}_1([0,1]) \circ \mathbf{L}_\alpha(X) = \mathbf{L}_1([0,1]) \circ \mathbf{B}_\alpha(X),$$

$$\mathbf{U}_{\alpha+1}(X) = \mathbf{U}_1([0,1]) \circ \mathbf{L}_\alpha(X) = \mathbf{U}_1([0,1]) \circ \mathbf{U}_\alpha(X) = \mathbf{U}_1([0,1]) \circ \mathbf{B}_\alpha(X),$$

$$\mathbf{L}_\alpha(X) = \mathbf{L}_1([0,1]) \circ \mathsf{M}\Sigma^0_\alpha(X), \quad \mathbf{U}_\alpha(X) = \mathbf{U}_1([0,1]) \circ \mathsf{M}\Sigma^0_\alpha(X).$$

Theorem 4.4 (A. Lindenbaum). *Let X be a perfectly normal topological space, $0 < \alpha, \beta < \omega_1$. Then,*

$$\mathbf{L}_{\beta+\alpha}(X) = \mathbf{L}_\alpha([0,1]) \circ \mathbf{U}_\beta(X) = \mathbf{L}_\alpha([0,1]) \circ \mathbf{L}_\beta(X),$$
$$\mathbf{U}_{\beta+\alpha}(X) = \mathbf{U}_\alpha([0,1]) \circ \mathbf{U}_\beta(X) = \mathbf{U}_\alpha([0,1]) \circ \mathbf{L}_\beta(X).$$

P r o o f. Using a transfinite induction on α, we shall prove

$$\mathbf{L}_{\beta+\alpha}(X) \subseteq \mathbf{L}_\alpha([0,1]) \circ \mathbf{L}_\beta(X, \mathbf{T}).$$

The reversed inclusion holds in arbitrary topological space. The case $\alpha = 1$ is Theorem 4.2.

Let $\alpha = \gamma + 1$. By Corollary 4.3 and the induction hypothesis, we have

$$\begin{aligned}\mathbf{L}_{\beta+\alpha}(X) = \mathbf{L}_1([0,1]) \circ \mathbf{L}_{\beta+\gamma}(X) &\subseteq \mathbf{L}_1([0,1]) \circ \mathbf{L}_\gamma([0,1]) \circ \mathbf{L}_\beta(X, \mathbf{T}) \\ &= \mathbf{L}_\alpha([0,1]) \circ \mathbf{L}_\beta(X, \mathbf{T}).\end{aligned}$$

Now, let $\alpha = \sup\{\alpha_i : i \in \omega\}$ and $f \in \mathbf{L}_{\beta+\alpha}(X)$. There exists $(p_i)_{i\in\omega}$ such that $p_i \in \mathbf{L}_{\beta+\alpha_i}(X, \mathbf{S})$ and $p_i \nearrow f$. Using the induction hypothesis on functions p_i, we get a decomposition

$$p_i = g_i \circ h_i,$$

where

$$g_i \in \mathbf{L}_{\alpha_i}(\mathbf{T}_i, \mathbf{S}) \quad \text{and} \quad h_i \in \mathbf{L}_\beta(X, \mathbf{T}), \quad \mathbf{T}_i = \mathrm{rng}\ \mathrm{h}_i.$$

Note that $g_i \circ \mathsf{pr}_i \in \mathbf{L}_{\alpha_i}(\prod_{j\in\omega} \mathbf{T}_j, \mathbf{S})$. Thus, $\boldsymbol{F}_{(g_i \circ\, \mathsf{pr}_i)_{i\in\omega}} \in \mathbf{L}_\alpha(\prod_{i\in\omega} \mathbf{T}_i, {}^\omega S)$, and consequently, $\mathsf{s} \circ \boldsymbol{F}_{(g_i \circ\, \mathsf{pr}_i)_{i\in\omega}} \in \mathbf{L}_\alpha(\prod_{i\in\omega} \mathbf{T}_i, [0,1])$. Since

$$\boldsymbol{F}_{(h_i)_{i\in\omega}} \in \mathbf{L}_\beta(X, {}^\omega\mathbf{T}) \quad \text{and} \quad f = (\mathsf{s} \circ \boldsymbol{F}_{(g_i \circ\, \mathsf{pr}_i)_{i\in\omega}}) \circ \boldsymbol{F}_{(h_i)_{i\in\omega}},$$

we are almost done except for the domain of $\mathsf{s} \circ \boldsymbol{F}_{(g_i \circ \mathsf{pr}_i)_{i\in\omega}}$ is $\prod_{i\in\omega} \mathbf{T}_i \subseteq {}^\omega\mathbf{S}$. To that end, we can take an element $g \in \mathbf{L}_\alpha([0,1],[0,1])$ to be an extension of

$$\mathsf{s} \circ \boldsymbol{F}_{(g_i \circ\, \mathsf{pr}_i)_{i\in\omega}} \circ \Phi^{-1} \quad \text{and} \quad h = \Phi \circ \boldsymbol{F}_{(h_i)_{i\in\omega}} \in \mathbf{L}_\beta(X, \mathbf{T}).$$

One can see that $f = g \circ h$. □

5. Jayne-Rogers property

In addition to pointwise and pointwise monotone limits, this section treats several questions related to discrete limits and (lower, upper) Δ^0_2-measurable functions. We can construct the following diagram of relations in an arbitrary topological space.

$$\begin{tikzcd}
 & \overline{\mathsf{M}}\Sigma^0_1 \arrow[r] & \overline{\mathsf{M}}\Delta^0_2 \arrow[dr] & \\
\mathbf{C} \arrow[ur] \arrow[r] \arrow[dr] & \mathsf{M}\Delta^0_2 \arrow[ur] \arrow[dr] & & \mathsf{M}\Sigma^0_2 \\
 & \underline{\mathsf{M}}\Sigma^0_1 \arrow[r] & \underline{\mathsf{M}}\Delta^0_2 \arrow[ur] &
\end{tikzcd}$$

By Theorem 3.2, we have $\underline{\mathsf{M}}\Sigma^0_1 = \mathbf{L}_1$, $\overline{\mathsf{M}}\Sigma^0_1 = \mathbf{U}_1$, $\mathsf{M}\Sigma^0_2 = \mathbf{B}_1$ in any perfectly normal space.

Let $\mathbf{B}_1^{(\mathrm{d})}(X)$ denote the family of all discrete limits of continuous functions on X. Note that the terminology may differ, see [11] for the survey on the topic. J. E. Jayne and C. A. Rogers [9] investigate piecewise continuous functions, i.e., f is piecewise continuous if there is a sequence $\langle F_n;\ n \in \omega\rangle$ of closed subsets of X such that $f|F_n$ is continuous on F_n for any $n \in \omega$ and $X = \bigcup_{n\in\omega} F_n$. However, if X is a normal space then

$$f \in \mathbf{B}_1^{(\mathrm{d})} \text{ if and only if } f \text{ is piecewise continuous.}$$ [6]

One can easily see that for a topological space X we have

$$\mathbf{B}_1^{(\mathrm{d})} \subseteq \mathsf{M}\Delta^0_2.$$

$\mathsf{M}\Delta^0_2 = \mathbf{B}_1^{(\mathrm{d})}$ in every analytic subset of a Polish space by J. E. Jayne and C. A. Rogers [9] and in any perfectly normal QN-space by L. Bukovský, I. Recław and M. Repický [3] and B. Tsaban and L. Zdomskyy [17].[7] It seems that in general the equality is an open problem.

Question 1. Is there $X \subseteq \mathbb{R}$ such that $\mathsf{M}\Delta^0_2(X) \neq \mathbf{B}_1^{(\mathrm{d})}(X)$?

Inspired by Question 1, the property of topological space "any function from family $\mathcal{F}$ is a discrete limit of a sequence of functions from family $\mathcal{G}$" is studied in [16].[8] Naturally, for an arbitrary topological space X we have

$$\mathbf{L}_1 \subseteq \mathbf{B}_1^{(\mathrm{d})} \Leftrightarrow \mathbf{U}_1 \subseteq \mathbf{B}_1^{(\mathrm{d})}, \quad \mathbf{U}_1 \subseteq \mathbf{B}_{1,\mathbf{L}_1}^{(\mathrm{d})} \Leftrightarrow \mathbf{L}_1 \subseteq \mathbf{B}_{1,\mathbf{U}_1}^{(\mathrm{d})}.$$ [9]

Moreover, both properties are hereditary and a topological space with any of them is a σ-set, see Corollary 4.6 and Proposition 4.7 in [16]. Let us recall that a topological space X is a σ-set if $\Pi^0_2(X) = \Sigma^0_2(X)$.

[6] For more, see Proposition 4.3 in [16].

[7] The equality $\mathsf{M}\Delta^0_2 = \mathbf{B}_1^{(\mathrm{d})}$ is called Jayne-Rogers property in [4, 16].

[8] The property is denoted as $\mathrm{DL}(\mathcal{F}, \mathcal{G})$. For instance, X has $\mathrm{DL}\big(\mathsf{M}\Delta^0_2, \mathbf{C}(X)\big)$ if and only if $\mathbf{B}_1^{(\mathrm{d})}(X) = \mathsf{M}\Delta^0_2(X)$.

[9] Discrete limits of lower and upper semicontinuous functions, respectively.

Part (b) of Theorem 4.1 in [16] has been proved for separable metric spaces. However, the only constraint in a proof for a more general topological space was Lindenbaum's Theorem which was known to be valid in separable metric spaces. Due to our Theorem 4.2, we can strengthen part (b) of Theorem 4.1 in [16] to perfectly normal spaces.

Theorem 5.1. *Let X be a perfectly normal space. Then,*

$$\mathbf{L}_1 \subseteq \mathbf{B}^{(\mathrm{d})}_{1,\mathbf{U}_1} \Leftrightarrow \mathbf{L}_1 \subseteq \mathbf{B}^{(\mathrm{d})}_1 \Leftrightarrow \mathbf{B}_1 = \mathbf{B}^{(\mathrm{d})}_1 \Leftrightarrow \mathbf{B} = \mathbf{B}^{(\mathrm{d})}_1$$
$$\Leftrightarrow \big(\mathsf{M}\Delta^0_2 = \mathbf{B}^{(\mathrm{d})}_1 \wedge X \text{ is a } \sigma\text{-set}\big).$$

By T. Natkaniec and W. Sieg [14], the family $\mathsf{M}\Delta^0_2$ is large in some sense since they show that if X is complete metric space then the family of all pointwise limits of sequences of functions from $\mathsf{M}\Delta^0_2(X)$ is the family $\mathbf{B}_2(X)$.

In the following, we focus on a question whether also lower and upper Δ^0_2-measurable functions admit nice characterizations using convergence. We introduce a new type of convergence for a sequence $(f_i)_{i\in\omega}$ of functions on X by

$$f_n \nearrow_* f \Leftrightarrow (\forall x \in X)(\forall \varepsilon > 0)(\exists n_0)(\forall n \in \omega)$$
$$\big(n \geq n_0 \to f(x) - \varepsilon < f_n(x) \leq f(x)\big),$$
$$f_n \searrow^* f \Leftrightarrow (\forall x \in X)(\forall \varepsilon > 0)(\exists n_0)(\forall n \in \omega)$$
$$\big(n \geq n_0 \to f(x) \leq f_n(x) < f(x) + \varepsilon\big).$$

We begin with an observation.

Lemma 5.2. *If f is a discrete limit of a sequence $(f_i)_{i\in\omega}$, $g_i \nearrow g$ and $h_i \searrow h$, then $f_i \oplus g_i \nearrow_* f \oplus g$ and $f_i \oplus h_i \searrow^* f \oplus h$.*

Proof. Let the assumption of assertion be true, and $x \in X$, $\varepsilon > 0$. There is $n_{f,g}$ such that

$$f_n(x) = f(x) \quad \text{and} \quad g(x) - \varepsilon < g_n(x) \leq g(x) \quad \text{for } n \geq n_{f,g}\,.$$

Thus,

$$f(x) + g(x) - \varepsilon < f_n(x) + g_n(x) \leq f(x) + g(x) \quad \text{for } n \geq n_{f,g}\,.$$

Similarly for the other case. □

We introduce respective classes

$$\mathbf{L}^*_1(X) = \Big\{ f \in {}^X[0,1] : f_n \nearrow_* f \wedge f_n \in \mathbf{C}(X) \Big\},$$

$$\mathbf{U}^*_1(X) = \Big\{ f \in {}^X[0,1] : f_n \searrow^* f \wedge f_n \in \mathbf{C}(X) \Big\}.$$

We summarize their basic properties.

Proposition 5.3. *Let X be a topological space.*

(1) *If X is perfectly normal and not a σ-set, then $\mathbf{U}_1 \not\subseteq \mathbf{L}_1^* \subsetneq \mathbf{B}_1$ and $\mathbf{L}_1 \not\subseteq \mathbf{U}_1^* \subsetneq \mathbf{B}_1$.*

(2) *If X is uncountable Polish, then $\mathbf{B}_1^{(\mathrm{d})} \cup \mathbf{L}_1 \subsetneq \mathbf{L}_1^*$ and $\mathbf{B}_1^{(\mathrm{d})} \cup \mathbf{U}_1 \subsetneq \mathbf{U}_1^*$.*

(3) *If X is perfectly normal, then $\mathbf{L}_1^* \subseteq \underline{\mathsf{M}}\Delta_2^0$ and $\mathbf{U}_1^* \subseteq \overline{\mathsf{M}}\Delta_2^0$.*[10]

(4) *If $f, g \in \mathbf{L}_1^*(X)$, then $\ominus f \in \mathbf{U}_1^*$, $\max\{f,g\} \in \mathbf{L}_1^*$, $\min\{f,g\} \in \mathbf{L}_1^*$, $f \oplus g \in \mathbf{L}_1^*$ and $f \odot g \in \mathbf{L}_1^*$.*

Proof. (1), (2) One can easily see that inclusions

$$\mathbf{B}_1^{(\mathrm{d})} \cup \mathbf{L}_1 \subseteq \mathbf{L}_1^* \subseteq \mathbf{B}_1 \quad \text{and} \quad \mathbf{B}_1^{(\mathrm{d})} \cup \mathbf{U}_1 \subseteq \mathbf{U}_1^* \subseteq \mathbf{B}_1$$

hold in any topological space. To prove that $\mathbf{U}_1 \not\subseteq \mathbf{L}_1^*$ and $\mathbf{L}_1^* \neq \mathbf{B}_1$, we will construct a function $h \in \mathbf{U}_1 \setminus (\mathbf{L}_1^* \cup \mathsf{M}\Delta_2^0)$. Let $A \in \Pi_2^0 \setminus \Sigma_2^0$ and $A = \bigcap_{n\in\omega} A_n$ such that $A_0 = X$ and $(A_i)_{i\in\omega}$ is a decreasing sequence of open sets. We define a function $h : X \to [0,1]$ as follows

$$h(x) = \begin{cases} \frac{1}{n+1}, & x \in A_n \setminus A_{n+1}, \\ 0, & x \in A. \end{cases}$$

One can easily see that $h \in \mathbf{U}_1 \setminus (\mathbf{L}_1 \cup \mathsf{M}\Delta_2^0)$. We shall show that $h \notin \mathbf{L}_1^*$. Indeed, let $h_n \to h$. Put

$$D_m = \{x \in X : \ (\forall n \geq m)\ h_n(x) \leq 0\}.$$

D_m is a closed set for every $m \in \omega$. If $h_n \nearrow_* h$, then $\bigcup_{m\in\omega} D_m = A$, which is a contradiction.

Finally, to prove that $\mathbf{B}_1^{(\mathrm{d})} \cup \mathbf{L}_1 \neq \mathbf{L}_1^*$ and $\mathbf{B}_1^{(\mathrm{d})} \cup \mathbf{U}_1 \neq \mathbf{U}_1^*$ hold, we shall construct a function $f \in \mathbf{U}_1^* \setminus (\mathbf{U}_1 \cup \mathsf{M}\Delta_2^0)$. Let $F \subseteq X$ be an uncountable closed subset of X, $z \in X \setminus F$ an accumulation point of $X \setminus F$, $U, V \subseteq X$ being open and disjoint in X such that $F \subseteq U$ and $z \in V$. There is $A \in \Pi_2^0(F) \setminus \Sigma_2^0(F)$. Since F is closed, we have $A \in \Pi_2^0(X) \setminus \Sigma_2^0(X)$, and $A = \bigcap_{n\in\omega} A_n$ such that $A_0 = X$, $A_1 = U$ and $(A_i)_{i\in\omega}$ is a decreasing sequence of open sets in X. Defining an upper semicontinuous function h in the same manner as before, we obtain a sequence $(h_i)_{i\in\omega}$ of continuous functions such that $h_i \searrow h$. Note that $h_i(x) = 1$ for $x \in X \setminus U$. On the other hand, let us set $g(x) = 0$ for $x \in X \setminus \{z\}$ and $g(z) = 1$. Thus, $h \geq g$. There is a sequence $(g_i)_{i\in\omega}$ of continuous functions converging discretely to g. We may assume that $g_i(x) = 0$ for $x \in X \setminus V$ to obtain $h_i \geq g_i$. Since $(h-g)^{-1}(\{0\}) = A \cup \{z\} \notin \Sigma_2^0(X)$ and $(h-g)^{-1}\big([0,1/2)\big) = A_2 \cup \{z\} \notin \Sigma_1^0(X)$, we have $f = h - g \notin \mathbf{U}_1 \cup \mathsf{M}\Delta_2^0$.

[10] Functions from classes $\mathbf{L}_1^*$ and $\mathbf{U}_1^*$ are obviously Σ_2^0-measurable.

Finally, by Lemma 5.2 and part (4) of this proposition, we have $h_i - g_i \searrow^* h - g$, since $h - g = \ominus\big((\ominus h) \oplus g\big)$ and similarly for $h_i - g_i$.

(3) Let $(f_i)_{i\in\omega}$ be a sequence of continuous functions such that $f_i \nearrow_* f$. We want to show that for any $r \in [0,1]$ we have

$$f^{-1}\big((r,1]\big) \in \Sigma^0_2 \cap \Pi^0_2.$$

Since

$$f^{-1}\big((r,1]\big) = \bigcup_{i\in\omega} \bigcup_{j\in\omega} \bigcap_{n>j} {f_n}^{-1}\big([r+2^{-i},1]\big)$$

and

$$f^{-1}\big([0,r]\big) = \bigcup_{i\in\omega} \bigcap_{n>i} {f_n}^{-1}\big([0,r]\big),$$

we obtain

$$f^{-1}\big((r,1]\big) \in \Sigma^0_2 \quad \text{and} \quad f^{-1}\big([0,r]\big) \in \Sigma^0_2.$$

$\square$

The following relations among convergence classes hold in an arbitrary topological space, compare with the earlier diagram in this section.

$$\begin{tikzcd}
 & \mathbf{U}_1 \arrow[r] & \mathbf{U}_1^* \arrow[dr] & \\
\mathbf{C}(X) \arrow[ur] \arrow[r] \arrow[dr] & \mathbf{B}_1^{(\mathrm{d})} \arrow[ur] \arrow[dr] & & \mathbf{B}_1 \\
 & \mathbf{L}_1 \arrow[r] & \mathbf{L}_1^* \arrow[ur] &
\end{tikzcd}$$

We ask whether the inclusions in part (3) of Proposition 5.3 can be reversed, thus putting the diagrams in a one-to-one correspondence.

Question 2. Are the equalities $\underline{\mathsf{M}}\Delta^0_2 = \mathbf{L}_1^*$ and $\overline{\mathsf{M}}\Delta^0_2 = \mathbf{U}_1^*$ true in any Polish space?

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