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SOME VARIATION ON THE RIEMANN DERANGEMENT THEOREM

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ABSTRACT. In this paper, we discuss a possibility of making the given real series convergent with the aid of multipliers from a given unbounded set.

The Riemann Derangement Theorem sounds as follows.

Theorem 1. *For any conditionally convergent series $\sum_{n=1}^{\infty} a_n$ of real terms and for any $x \in \mathbb{R}$ (more generally: for any nonempty closed interval $I \subset \mathbb{R} \cup \{\pm\infty\}$), there exists a permutation p of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{p(n)} = x$ (in general case the set of limit points of series $\sum_{n=1}^{\infty} a_{p(n)}$ is equal to I).*

In connection with the above theorem, the following problem, in which the role of permutation p of $\mathbb{N}$, appearing in the discussed theorem, is played by the sequence of multipliers from the unbounded set, seems to be interesting. Let us present the problem more precisely.

Let $s > 0$ and $\mathcal{N}_s := \{\pm n^s : n \in \mathbb{N} \cup \{0\}\}$. Let $\{a_n\}_{n=1}^{\infty} \subset (0, \infty)$ be a sequence converging to zero, for brevity called a zero sequence.

Is it true that for every $x \in \mathbb{R}$ (more generally: for any nonempty closed interval $I \subset \mathbb{R} \cup \{\pm\infty\}$) there exists a sequence of multipliers $\{\varepsilon_n\}_{n=1}^{\infty} \subset \mathcal{N}_s$ (which cannot be a one-to-one sequence) such that $\sum_{n=1}^{\infty} \varepsilon_n a_n = x$ (or, in general case, the set of limit points of $\sum_{n=1}^{\infty} \varepsilon_n a_n$ is equal to I)?

The following theorem gives a positive answer to our problem.

Theorem 2. *Let $\{x_n\}_{n=0}^{\infty}$ be an increasing sequence of nonnegative numbers such that $x_0 = 0$ and $\lim_{n\to\infty} x_n = +\infty$. If*

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{x_n} = 0,$$

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then, for every zero sequence $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}$, $a_k \neq 0$ *and for every* $x \in \mathbb{R}$, *there exists a sequence* $\{\varepsilon_k\}_{k=1}^{\infty}$, $\varepsilon_k \in \{\varepsilon x_n : n \geq 0, \varepsilon = \pm 1\}$, $k \in \mathbb{N}$, *such that*

$$\sum_{k=1}^{\infty} \varepsilon_k \, a_k = x.$$

P r o o f. Let us determine the zero sequence $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}$, $a_k \neq 0$ and $x \in \mathbb{R}$. With regard to the way of selecting the sequence $\{\varepsilon_k\}_{k=1}^{\infty}$, one can assume that $a_k > 0$, for every $k \in \mathbb{N}$.

First, we select $n_1 \in \mathbb{N}_0$ and $\varepsilon = \varepsilon(n_1) = \pm 1$ so that

$$\varepsilon x_{n_1} a_1 \leq x \leq \varepsilon x_{n_1+\varepsilon} a_1,$$

and we put $\varepsilon_1 = \varepsilon x_{n_1}$.

If we have determined the numbers $\varepsilon_1 \in \mathbb{R}$ and $\varepsilon_2, \dots, \varepsilon_{k-1} \in [0, \infty)$, and additionally $0 \leq x - \sum_{i=1}^{k-1} \varepsilon_i \, a_i$, then we choose $n_k \in \mathbb{N}_0$ so that

$$x_{n_k} a_k \leq x - \sum_{i=1}^{k-1} \varepsilon_i a_i < x_{n_k+1} a_k \tag{1}$$

and we take $\varepsilon_k = x_{n_k}$.

Our question concerns the convergence of sequence

$$\left\{ x - \sum_{i=1}^{k} \varepsilon_i a_i \right\}_{k=1}^{\infty}.$$

It is easy to verify that this is the nonincreasing sequence and

$$0 \leq x - \sum_{i=1}^{k} \varepsilon_i a_i \overset{(1)}{<} \big(x_{n_k+1} - x_{n_k}\big) a_k = \frac{x_{n_k+1} - x_{n_k}}{x_{n_k}} x_{n_k} a_k, \tag{2}$$

for every $k \in \mathbb{N}$. From the left inequality in (1), we receive that the sequence $\{x_{n_k} a_k\}_{k=1}^{\infty}$ is bounded. If the subsequence of a sequence $\{n_k\}_{k=1}^{\infty}$ convergent to $+\infty$ exists, then, with regard to the assumption about the sequence

$$\left\{ \frac{x_{n+1} - x_n}{x_n} \right\}_{k=1}^{\infty},$$

there exists a subsequence of the sequence

$$\left\{ \frac{x_{n_k+1} - x_{n_k}}{x_{n_k}} \right\}_{k=1}^{\infty}$$

convergent to zero, and from (2), in view of monotonicity of the sequence

$$\left\{ x - \sum_{i=1}^{k} \varepsilon_i a_i \right\}_{k=1}^{\infty},$$

we obtain that

$$\lim_{k\to\infty}\left(x-\sum_{i=1}^{k}\varepsilon_i\,a_i\right)=0.$$

If the sequence $\{n_k\}_{k=1}^{\infty}$ is bounded, then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ will also be bounded, and because of that, so it will be the sequence $\{x_{n_k+1}-x_{n_k}\}_{k=1}^{\infty}$, which, in view of $a_k\to 0$ from (2), gives $x=\sum_{i=1}^{\infty}\varepsilon_i\,a_i$. □

Remark 3. Some technical manipulations in the presented proof enable to strengthen the thesis of Theorem 2, i.e., the number $x\in\mathbb{R}$ from Theorem 2 could be replaced with any nonempty closed interval $I\subset\mathbb{R}\cup\{\pm\infty\}$.

Remark 4. Authors of the paper are convinced that the assumptions of Theorem 2 can be significantly reduced and that there exists some deeper connection between Theorems 2 and 1.

Final comment. In the meantime, first Author of this paper generalized Theorem 2 by indicating its fundamental connection with Theorem 1 (see [1]). It turned out that, in fact, the assumption from Theorem 2 about the limit of quotient can be omitted.

REFERENCES

[1] WITUŁA, R.: *Certain multiplier version of the Riemann derangement theorem,* Demostratio Math. (accepted).

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