ABSTRACT. In this paper, we discuss a possibility of making the given real series convergent with the aid of multipliers from a given unbounded set.

The Riemann Derangement Theorem sounds as follows.

**Theorem 1.** For any conditionally convergent series \( \sum_{n=1}^{\infty} a_n \) of real terms and for any \( x \in \mathbb{R} \) (more generally: for any nonempty closed interval \( I \subseteq \mathbb{R} \cup \{ \pm \infty \} \)), there exists a permutation \( p \) of \( \mathbb{N} \) such that \( \sum_{n=1}^{\infty} a_{p(n)} = x \) (in general case the set of limit points of series \( \sum_{n=1}^{\infty} a_{p(n)} \) is equal to \( I \)).

In connection with the above theorem, the following problem, in which the role of permutation \( p \) of \( \mathbb{N} \), appearing in the discussed theorem, is played by the sequence of multipliers from the unbounded set, seems to be interesting. Let us present the problem more precisely.

Let \( s > 0 \) and \( \mathcal{N}_s := \{ \pm n^s : n \in \mathbb{N} \cup \{0\} \} \). Let \( \{a_n\}_{n=1}^{\infty} \subset (0, \infty) \) be a sequence converging to zero, for brevity called a zero sequence.

Is it true that for every \( x \in \mathbb{R} \) (more generally: for any nonempty closed interval \( I \subseteq \mathbb{R} \cup \{ \pm \infty \} \)) there exists a sequence of multipliers \( \{\varepsilon_n\}_{n=1}^{\infty} \subset \mathcal{N}_s \) (which cannot be a one-to-one sequence) such that \( \sum_{n=1}^{\infty} \varepsilon_n a_n = x \) (or, in general case, the set of limit points of \( \sum_{n=1}^{\infty} \varepsilon_n a_n \) is equal to \( I \))?

The following theorem gives a positive answer to our problem.

**Theorem 2.** Let \( \{x_n\}_{n=0}^{\infty} \) be an increasing sequence of nonnegative numbers such that \( x_0 = 0 \) and \( \lim_{n \to \infty} x_n = +\infty \). If

\[
\lim_{n \to \infty} \frac{x_{n+1} - x_n}{x_n} = 0,
\]

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then, for every zero sequence \( \{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \), \( a_k \neq 0 \) and for every \( x \in \mathbb{R} \), there exists a sequence \( \{\varepsilon_k\}_{k=1}^{\infty} \), \( \varepsilon_k \in \{\varepsilon x_n : n \geq 0, \varepsilon = \pm 1\} \), \( k \in \mathbb{N} \), such that

\[
\sum_{k=1}^{\infty} \varepsilon_k a_k = x.
\]

**Proof.** Let us determine the zero sequence \( \{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \), \( a_k \neq 0 \) and \( x \in \mathbb{R} \). With regard to the way of selecting the sequence \( \{\varepsilon_k\}_{k=1}^{\infty} \), one can assume that \( a_k > 0 \), for every \( k \in \mathbb{N} \).

First, we select \( n_1 \in \mathbb{N}_0 \) and \( \varepsilon = \varepsilon(n_1) = \pm 1 \) so that

\[
\varepsilon x_{n_1} a_1 \leq x \leq \varepsilon x_{n_1} + \varepsilon a_1,
\]

and we put \( \varepsilon_1 = \varepsilon x_{n_1} \).

If we have determined the numbers \( \varepsilon_1 \in \mathbb{R} \) and \( \varepsilon_2, \ldots, \varepsilon_{k-1} \in [0, \infty) \), and additionally \( 0 \leq x - \sum_{i=1}^{k-1} \varepsilon_i a_i \), then we choose \( n_k \in \mathbb{N}_0 \) so that

\[
x_{n_k} a_k \leq x - \sum_{i=1}^{k-1} \varepsilon_i a_i < x_{n_k+1} a_k
\]

and we take \( \varepsilon_k = x_{n_k} \).

Our question concerns the convergence of sequence

\[
\left\{ x - \sum_{i=1}^{k} \varepsilon_i a_i \right\}_{k=1}^{\infty}.
\]

It is easy to verify that this is the nonincreasing sequence and

\[
0 \leq x - \sum_{i=1}^{k} \varepsilon_i a_i < (x_{n_k+1} - x_{n_k}) a_k = \frac{x_{n_k+1} - x_{n_k}}{x_{n_k}} x_{n_k} a_k,
\]

for every \( k \in \mathbb{N} \). From the left inequality in \( \text{(1)} \), we receive that the sequence \( \{x_{n_k} a_k\}_{k=1}^{\infty} \) is bounded. If the subsequence of a sequence \( \{n_k\}_{k=1}^{\infty} \) convergent to \( +\infty \) exists, then, with regard to the assumption about the sequence

\[
\left\{ \frac{x_{n_k+1} - x_{n_k}}{x_{n_k}} \right\}_{k=1}^{\infty},
\]

there exists a subsequence of the sequence

\[
\left\{ \frac{x_{n_k+1} - x_{n_k}}{x_{n_k}} \right\}_{k=1}^{\infty},
\]

convergent to zero, and from \( \text{(2)} \), in view of monotonicity of the sequence

\[
\left\{ x - \sum_{i=1}^{k} \varepsilon_i a_i \right\}_{k=1}^{\infty},
\]

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we obtain that
\[ \lim_{k \to \infty} \left( x - \sum_{i=1}^{k} \varepsilon_i a_i \right) = 0. \]

If the sequence \( \{n_k\}_{k=1}^{\infty} \) is bounded, then the sequence \( \{x_{n_k}\}_{k=1}^{\infty} \) will also be bounded, and because of that, so it will be the sequence \( \{x_{n_k+1} - x_{n_k}\}_{k=1}^{\infty}, \) which, in view of \( a_k \to 0 \) from (2), gives \( x = \sum_{i=1}^{\infty} \varepsilon_i a_i. \)

**Remark 3.** Some technical manipulations in the presented proof enable to strengthen the thesis of Theorem 2 i.e., the number \( x \in \mathbb{R} \) from Theorem 2 could be replaced with any nonempty closed interval \( I \subset \mathbb{R} \cup \{\pm \infty\}. \)

**Remark 4.** Authors of the paper are convinced that the assumptions of Theorem 2 can be significantly reduced and that there exists some deeper connection between Theorems 2 and 1.

**Final comment.** In the meantime, first Author of this paper generalized Theorem 2 by indicating its fundamental connection with Theorem 1 (see [1]). It turned out that, in fact, the assumption from Theorem 2 about the limit of quotient can be omitted.

**REFERENCES**