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PROBABILITY ON HILBERT SPACE EFFECT ALGEBRAS

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ABSTRACT. An important example of an effect algebra (in other terminology a D-poset) has been constructed by linear positive operators on a Hilbert space. In the paper basic notions of probability theory are exposed and the central limit theorem is proved. Some similar results on D-lattices are known, but the effect algebra studied here need not be a lattice.

1. Generalized effect algebras and D-posets

Definition ([4]). A generalized effect algebra $(E, \oplus, 0)$ is a set E with an element $0 \in E$ and a partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ conditions:

(GE1) $x \oplus y = y \oplus x$ if one side is defined,

(GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(GE3) if $x \oplus y = x \oplus z$, then $y = z$,

(GE4) if $x \oplus y = 0$, then $x = y = 0$,

(GE5) $x \oplus 0 = x$ for all $x \in E$.

Theorem. *Define on E a binary relation $\leq$ by*

$$x \leq y \Longleftrightarrow \exists z \in E, \qquad x \oplus z = y.$$

Then $\leq$ is a partial ordering.

Proof.

1. Since $x \oplus 0 = x$, we have $x \leq x$ for any $x \in E$.

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2. Let $x \le y$ and $y \le x$. Then there exist $u, v \in E$ such that

$$y = x \oplus u, \quad x = y \oplus v,$$

hence by (GE2)

$$x = (x \oplus u) \oplus v = x \oplus (u \oplus v).$$

Of course, by (GE3) we obtain

$$u \oplus v = 0,$$

hence by (GE4)

$$u = v = 0$$

and therefore $y = x \oplus 0 = x$.

3. Let $x \le y$, $y \le z$. Then there exist $u, v \in E$ such that

$$y = x \oplus u, \quad z = y \oplus v,$$

hence by (GE2)

$$z = (x \oplus u) \oplus v = x \oplus (u \oplus v).$$

Therefore by the definition

$$x \le z.$$

□

Definition. Let $x \le y$, $y = x \oplus z$ (z is determined uniquely by (GE3)). Then we write $z = y \ominus x$.

Theorem. *The operation $\ominus$ is a partial binary operation satisfying the following properties:*

(i) *If $x \le y$, then $y \ominus x \le y$ and*

$$x = y \ominus (y \ominus x).$$

(ii) *If $x \le y \le z$, then $z \ominus y \le z \ominus x$ and*

$$(z \ominus x) \ominus (z \ominus y) = y \ominus x.$$

Proof. Let $x \le y$. Then $z = y \ominus x$ is defined by the formula

$$y = x \oplus z.$$

Therefore $y \ge z = y \ominus x$. Moreover, by the definition

$$x = y \ominus z = y \ominus (y \ominus x)$$

and hence (i) is proved.

Now let $x \le y \le z$. Then there exist $u, v, w \in E$ such that

$$z = y \oplus u, \quad y = x \oplus w, \quad z = x \oplus v,$$

hence

$$u = z \ominus y, \quad w = y \ominus x, \quad v = z \ominus x.$$

We have

$$z = y \oplus u = (x \oplus w) \oplus u = x \oplus (w \oplus u).$$

On the other hand,

$$z = x \oplus v.$$

Therefore,

$$v = w \oplus u$$

and

$$z \ominus x = v \geq u = z \ominus y.$$

Moreover,

$$(z \ominus x) \ominus (z \ominus y) = v \ominus u = w = y \ominus x,$$

hence also (ii) is proved. □

Definition ([2]). A D-poset is an algebraic structure $(D, \ominus, \leq, 0, u)$ where $\leq$ is a partial ordering with the least element 0 and the greatest element u and $\ominus$ is a partial binary operation satisfying the following properties:

(D1) If $x \leq y$, then $y \ominus x \leq y$ and

$$x = y \ominus (y \ominus x).$$

(D2) If $x \leq y \leq z$, then $z \ominus y \leq z \ominus x$, and

$$(z \ominus x) \ominus (z \ominus y) = y \ominus x.$$

Theorem. *Let $(E, \oplus)$ be a generalized effect algebra, u be an arbitrary positive element on E. Then $(E, \ominus, \leq, 0, u)$ is a D-poset.*

2. States

Definition ([1]). A partial algebra $(E, \oplus, 0, 1)$ is called an effect algebra if $0, 1$ are two distinct elements and $\oplus$ is a partial binary operation on E which satisfies the following conditions for $x, y, z \in E$:

(E1) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(E2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(E3) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$,

(E4) if $1 \oplus x$ is defined, then $x = 0$.

A generalized effect algebra is an effect algebra if there exists an element $u \in E$ such that $x \leq u$ for every $x \in E$. If we take in a generalized effect algebra E a positive element $u > 0$, then the interval $[0, u] = \{x \in E;\ 0 \leq x \leq u\}$ is an effect algebra. We have seen that any effect algebra is a D-poset. It is known that also any D-poset is an effect algebra, where $(a \oplus b) \ominus a = b$, and $(a \oplus b) \ominus b = a$. In [6] the following notion has been introduced.

Definition ([6]). Let E be an effect algebra. A mapping $m\colon E \to [0,1]$ is a state if the following two conditions are satisfied:

(i) $m(u) = 1$,

(ii) if $x \oplus y$ exists, then $m(x \oplus y) = m(x) + m(y)$.

Theorem. *Let E be an effect algebra. Then a mapping $m\colon E \to [0,1]$ is a state if and only if the following property is satisfied:*

(α) *If $x \leq y$, then $m(y \ominus x) = m(y) - m(x)$.*

Proof. Let $m\colon E \to [0,1]$ be a state. Let $x \leq y$, then

$$y = x \oplus (y \ominus x),$$

hence

$$m(y) = m\big(x \oplus (y \ominus x)\big) = m(x) + m(y \ominus x).$$

We see that the condition (α) is satisfied. □

Let (α) be satisfied, and $x \oplus y$ exist. By the definition $x \oplus y \geq x$, and

$$(x \oplus y) \ominus x = y.$$

Then

$$m(y) = m\big((x \oplus y) \ominus x\big) = m(y \ominus x) + m(x).$$

Definition. Let $\mathcal{B}(R)$ be the family of Borel sets. A mapping $x\colon \mathcal{B}(R) \to E$ is an observable if for any disjoint $A, B \in \mathcal{B}(R)$, $x(A) \oplus x(B)$ exists and

$$x(A \cup B) = x(A) \oplus x(B).$$

Theorem. *A mapping $x\colon \mathcal{B}(R) \to E$ is an observable if and only if for any $A, B \in \mathcal{B}(R)$, $A \subset B$, there holds $x(A) \leq x(B)$ and*

$$x(B \setminus A) = x(B) \ominus x(A).$$

The proof is clear.

Definition. A state $m\colon E \to [0,1]$ is continuous, if

$$a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a).$$

An observable $x\colon \mathcal{B}(R) \to E$ is continuous if

$$A_n \nearrow A \Longrightarrow x(A_n) \nearrow x(A).$$

Theorem. *Let $x\colon \mathcal{B}(R) \to E$ be a continuous observable, $\omega\colon E \to [0,1]$ be a continuous state. Then the composite map $\omega o x\colon \mathcal{B}(R) \to [0,1]$ is a probability measure.*

3. Hilbert space effect algebra

Theorem. *Let $\mathcal{H}$ be a complex Hilbert space, and G be the set of all positive linear operators defined on $\mathcal{H}$, $\oplus$ is usual sum of operators, 0 is the null operator. Then $(E, \oplus, 0)$ is a generalized effect algebra.*

Proof. [4, Theorem 3.1.]. □

Theorem. *For any $Q \in \mathcal{H}$ the interval $[0, Q]$ is an effect algebra.*

Proof. [6, Theorem 5]. □

Corollary. *Let $E = [0, Q]$, $m\colon E \to [0,1]$ be a continuous state, $x\colon \mathcal{B}(R) \to E$ be a continuous observable. Then the composite mapping $m_x\colon \mathcal{B}(R) \to [0,1]$, defined by*

$$m_x(A) = m\big(x(A)\big)$$

is a probability measure.

4. Moments of observables

Since the mapping $m_x\colon \mathcal{B}(R) \to [0,1]$ is a probability measure, and probability theory on D-posets has been studied, we want to study it also on a Hilbert effect algebra. The main aim of our investigations in this paper is a central limit theorem. Of course, we need to have moments of observables, the first starting moment $E(x)$ and the second central moment $\sigma^2(x)$.

In the Kolmogorovian probability theory, for a random variable $\xi\colon \Omega \to R$ measurable with respect to a σ-algebra $\mathcal{S}$ we have with the respect to a probability measure $P\colon \mathcal{S} \to [0,1]$

$$E(\xi) = \int_\Omega \xi dP = \int_R t dP_\xi(t),$$

$$\sigma^2(\xi) = \int_\Omega \big(\xi - E(\xi)\big)^2 dP = \int_R \big(t - E(\xi)\big)^2 dP_\xi = \int_R t^2 dP\xi - E(\xi)^2.$$

Here $P_\xi\colon \mathcal{B}(R) \to [0,1]$ is defined on Borel sets by the formula

$$P_\xi(A) = P\big(\xi^{-1}(A)\big).$$

In our case instead of the mapping $A \to \xi^{-1}(A)$ we have an observable

$$x\colon \mathcal{B}(R) \to E, \quad A \to x(A).$$

Instead of the mapping P_ξ we have the probability measure

$$m_x\colon \mathcal{B}(R) \to [0,1], \quad m_x(A) = m\big(x(A)\big).$$

Therefore the following definition is natural.

Definition. An observable $x\colon \mathcal{B}(R) \to E$ is integrable if there exists

$$\int_R t\,dm_x(t);$$

we denote it by

$$E(x) = \int_R t\,dm_x(t).$$

An observable is square integrable if there exists

$$\int_R t^2\,dm_x(t).$$

In this case we write

$$\sigma^2(x) = \int_R t^2\,dm_x(t) - E(x)^2 = \int_R \big(t - E(\xi)\big)^2 dm_x(t).$$

5. Independence

Two random variables $\xi, \eta\colon \Omega \to R$ are independent, if

$$P\big(\xi^{-1}(A) \cap \eta^{-1}(B)\big) = P\big(\xi^{-1}(A)\big).P\big(\eta^{-1}(B)\big)$$

for any $A, B \in \mathcal{B}(R)$. Put

$$T(\omega) = \big(\xi(\omega), \eta(\omega)\big) \in R^2.$$

Then

$$\xi^{-1}(A) \cap \eta^{-1}(B) = T^{-1}(A \times B),$$

hence the independence can be reformulated by the formula

$$P\big(T^{-1}(A \times B)\big) = P\big(\xi^{-1}(A)\big).P\big(\eta^{-1}(B)\big).$$

Definition. Continuous observables $x_1, x_2, \ldots, x_n \colon \mathcal{B}(R^n) \to E$ are independent, if there exists a continuous observable $h \colon \mathcal{B}(R^n) \to E$ such that

$$m\big(h(A_1 \times \cdots \times A_n)\big) = m\big(x_1(A_1)\big) \ldots m\big(x(A_n)\big)$$

for any $A_1, \ldots, A_n \in \mathcal{B}(R)$.

Of course, continuous joint observable $h \colon \mathcal{B}(R^n) \to E$ is a mapping such that

$$h(R^n) = u,$$

$h(A) \oplus h(B)$ exists for disjoint $A, B \in \mathcal{B}(R^n)$ and

$$h(A \oplus B) = h(A) \oplus h(B).$$

Let $g \colon R^n \to R$ be a function (e.g., $g(u_1, \ldots, u_n) = \frac{1}{n}\Sigma_{i=1}^n u_i$), $T(\omega) = \xi_1(\omega), \ldots, \xi_n(\omega) \colon \Omega \to R^n$,

$$k(\omega) = g\big(\xi_1(\omega) \ldots, \xi_n(\omega)\big) = g\big(T(\omega)\big)$$

(e.g., $k = \frac{1}{n}\Sigma_{i=1}^n \xi_i$). Then

$$k^{-1}(A) = T^{-1}\big(g^{-1}(A)\big)$$

for any $A \in \mathcal{B}(R)$.

Definition. Let $x_1, \ldots, x_n \colon \mathcal{B}(R) \to E$ be independent continuous observables, $h \colon \mathcal{B}(R^n) \to E$ be their joint observable, $g \colon R^n \to R$ be a Borel function. Then we define

$$g(x_1, \ldots, x_n)(A) = h\big(g^{-1}(A)\big), \qquad A \in \mathcal{B}(R).$$

Example. The observable $\frac{1}{n}(x_1 + \cdots + x_n) \colon \mathcal{B}(R) \to E$ is defined by the equality

$$\frac{1}{n}(x_1 + \cdots + x_n)(A) = h\big(g^{-1}(A)\big),$$

where

$$g(u_1, \ldots, u_n) = \frac{1}{n}(u_1 + \cdots + u_n).$$

6. Central limit theorem

Theorem. *Let $m \colon E \to [0,1]$ be a continuous state. Let $(x_n)_{n=1}^\infty$ be an independent sequence of continuous observables with the mean value $E(x_1) = E(x_2) \ldots = a$ and the dispersion $\sigma^2(x_1) = \sigma^2(x_2) = \ldots = \sigma^2$. Then*

$$\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\Sigma_{i=1}^n x_i - a}{\frac{\sigma}{\sqrt{n}}}(-\infty, t)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}}\, du.$$

P r o o f. For every n define

$$P_n\colon \mathcal{B}(R^n) \to [0,1]$$

by the formula

$$P_n(A) = m\big(h_n(A)\big),$$

where $h_n\colon \mathcal{B}(R^n) \to E$ is the joint observable of $x_1, \dots, x_n$. Evidently,

$$P_{n+1}(A \times R) = m\big(h_{n+1}(A \times R)\big) = m\big(h_n(A)\big) = P_n(A).$$

We obtained the consistency condition, hence we can use the Kolmogorov consistency theorem. Consider the space R^N of all sequences of real numbers, $\pi_n\colon R^N \to R^n$ the projection and the set $\mathcal{C}$ of all cylinders in R^N, hence of the sets of the type

$$\pi_n^{-1}(A), A \in \mathcal{B}(R^n), \qquad n \in N,$$

and the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$. Then there exists exactly one probability measure

$$P\colon \sigma(\mathcal{C}) \to [0,1]$$

such that

$$P\big(\pi_n^{-1}(A)\big) = P_n(A) = m\big(h_n(A)\big)$$

for any $A \in \mathcal{B}(R^n)$. Moreover, consider the mappings $\xi_n\colon R^N \to R$ defined by the equality

$$\xi_n\big((u_i)_{i=1}^{\infty}\big) = u_n.$$

Then

$$\begin{aligned} P\big(\xi_n^{-1}(A)\big) &= P\big(\pi_n^{-1}(R \times R \times \dots \times R \times A)\big) \\ &= P_n(R \times R \times \dots \times R \times A) \\ &= m\big(h_n(R \times R \times \dots \times R \times A)\big) \\ &= m\big(x_1(R)\big) \cdot m\big(x_2(R)\big) \times \dots \\ &\quad \dots \times m\big(x_{n-1}(R)\big) \cdot m\big(x_n(A)\big) \\ &= m\big(x_n(A)\big). \end{aligned}$$

Therefore

$$E(\xi_n) = \int_R t dP_{\xi_n}(t) = \int_R t dm_{x_n}(t) = E(x_n) = a, \qquad n = 1, 2, \dots$$

and similarly,

$$\sigma^2(\xi_n) = \sigma^2(x_n) = \sigma^2, \qquad n = 1, 2, \dots$$

The variables $\xi_1, \xi_2, \dots$ are independent. Actually,

$$\begin{aligned}
&P\big(\xi_1^{-1}(A_1) \cap \xi_1^{-1}(A_2) \cap \dots \cap \xi_1^{-1}(A_n)\big)\\
&= P\big(\pi_n^{-1}(A_1 \times \dots \times A_n)\big)\\
&= P_n(A_1 \times \dots \times A_n)\\
&= m\big(h_n(A_1 \times \dots \times A_n)\big)\\
&= m\big(x_1(A_1)\big) \cdot m\big(x_1(A_2)\big) \times \dots \times m\big(x_1(A_n)\big)\\
&= P\big(\xi_1(A_1)\big) \cdot P\big(\xi_1(A_2)\big) \times \dots \times P\big(\xi_1(A_n)\big).
\end{aligned}$$

Therefore the central limit theorem can be applied for the sequence (ξ_n) in the space $\big(R^N, \sigma(\mathcal{C}), P\big)$. Moreover, define $g_n \colon R^n \to R$ by the equality

$$g_n(u_1, \dots, u_n) = \frac{\frac{1}{n}\Sigma_{i=1}^n u_i - a}{\frac{\sigma}{\sqrt{n}}}.$$

Then

$$\begin{aligned}
m\left(\frac{\frac{1}{n}\Sigma_{i=1}^n x_i - a}{\frac{\sigma}{\sqrt{n}}}(-\infty, t)\right) &= m\big(h_n g_n^{-1}(-\infty, t)\big)\\
&= P_n\Big(g_n^{-1}\big((-\infty, t)\big)\Big)\\
&= P\left(\left\{\omega;\ \frac{\frac{1}{n}\Sigma_{i=1}^n \xi_i(\omega) - a}{\frac{\sigma}{\sqrt{n}}} < t\right\}\right),
\end{aligned}$$

therefore

$$\begin{aligned}
\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\Sigma_{i=1}^n x_i - a}{\frac{\sigma}{\sqrt{n}}}(-\infty, t)\right) &= \lim_{n\to\infty} P\left(\left\{\omega;\ \frac{\frac{1}{n}\Sigma_{i=1}^n \xi_i(\omega) - a}{\frac{\sigma}{\sqrt{n}}} < t\right\}\right)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}}\, du.
\end{aligned}$$

□

7. Conclusion

We have considered the space G of linear positive operators in a Hilbert space. Inspired by the papers by Riečanová et al. [6] we considered a partial ordering $\leq$ on G given by the equivalence

$$f \leq g \Leftrightarrow \exists h \in G, \qquad g = f + h.$$

If we take a fixed positive element $Q \in G$, then the interval $E = [0, Q]$ is an effect algebra = a D-poset. We have proved in E the central limit theorem for a continuous state $m\colon E \to [0,1]$ and any sequence of independent continuous observables in E.

We proved the theorem by the local representation of E by a classical probability space and the Kolmogorov consistency theorem. The method is not new, but for the first time the theorem was proved without the assumption that the given algebraic structure is a lattice. Namely, the space of linear positive operators in a Hilbert space need not be a lattice.

Note. It is known that most of the operators that occur in applications of the theory of Hilbert space to differential equations, quantum mechanics, probability theory, and so on, are unbounded. For this reason, we consider, in this case basic definitions and theorems necessary to deal with unbounded operators and also versions of spectral theorem for unbounded operators. Therefore we define at least a type of operator we consider here.

Definition. A linear operator T (in contrast to "on") in a Hilbert space H is a linear transformation on a linear subspace D_T of H into H. D_T is called the domain of T and $R_T = \{Tf : f \in D_T\}$ is the range of T.

T is said bounded if T is bounded as a linear transformation from D_T into H.

Example. Let $H = L_2(-\infty, \infty)$ and let T be defined in H on

$$D_T = \{f \in L_2(-\infty, \infty) : sf(s) \in L_2(-\infty, \infty)\}$$

by $Tf(s) = sf(s)$. Clearly, T is unbounded.

More facts can be found in: [A. Mukherjea and K. Pothoven, *Real and Functional Analysis*, Vol. 6. Plenum Press, New York, 1978].

Analogous to bounded linear operators on H, we can also consider in many cases the adjoint of a linear operator in H whether it be bounded or unbounded by means of the corresponding definition.

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