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ON HASHIMOTO TYPE TOPOLOGIES

Małgorzata Terepeta

ABSTRACT. For topological space $(X, \mathcal{T})$ and two different, proper ideals $\mathcal{I}_1$, $\mathcal{I}_2$, we consider a new topology of Hashimoto type.

Hashimoto type topologies ($\star$ topologies) were discussed by K. Kuratowski [3], N. F. G. Martin [4] and H. Hashimoto [1]. Some improvements of known results and applications of this notion were presented in 1990 by D. Janković and T. R. Hamlett [2].

Let us remind some basic information about Hashimoto type topologies. Let $(X, \mathcal{T})$ be a topological space (topology $\mathcal{T}$ has to be T_1), $\mathcal{I}$ – a proper ideal of subsets of X which contains all singletons. For a set $A \subset X$ we put

$$A^{\mathcal{I}} = \big\{x \in X \colon \forall_{U_x \in \mathcal{T}} \ (U_x \cap A \notin \mathcal{I})\big\},$$

where U_x stands for a neighbourhood of a point $x \in X$. Then (see [1]) the set $A^{\mathcal{I}}$ is closed and contained in $\overline{A}$ (the closure of A in topology $\mathcal{T}$). If we assume that the ideal $\mathcal{I}$ satisfies the condition

$$A \cap A^{\mathcal{I}} = \emptyset \iff A \in \mathcal{I}, \tag{1}$$

then the set $A \setminus A^{\mathcal{I}}$ is small—it belongs to the ideal $\mathcal{I}$. The operator $A^{\mathcal{I}}$ is idempotent $A^{\mathcal{I}\mathcal{I}} = A^{\mathcal{I}}$ and monotone, if $A \subset B$, then $A^{\mathcal{I}} \subset B^{\mathcal{I}}$. The set $\mathrm{Cl}_{\mathcal{I}}(A) = A \cup A^{\mathcal{I}}$ is a closure of A in a new topology $\mathcal{T}_{\mathcal{I}}$ defined as follows

$$\mathcal{T}_{\mathcal{I}} = \{U \setminus N \colon U \in \mathcal{T} \wedge N \in \mathcal{I}\}.$$

Throughout the paper, $\mathcal{C}$ will denote the ideal of countable sets, $\mathcal{N}$ – the ideal of null sets, $\mathcal{K}$ – the ideal of first category sets on $\mathbb{R}$.

Example 1. Let $X = \mathbb{R}$. The ideals $\mathcal{C}$, $\mathcal{N}$, $\mathcal{K}$ fulfil condition (1). If $\mathcal{I} = \mathcal{C}$, then $A^{\mathcal{I}} = \{x \in \mathbb{R} \colon x \text{ is an accumulation point of } A\}$, if $\mathcal{I} = \mathcal{N}$, then $A^{\mathcal{I}} = \{x \in \mathbb{R} \colon \forall_{U_x \in \mathcal{T}} \ U_x \cap A \text{ is of positive outer measure}\}$, if $\mathcal{I} = \mathcal{K}$ the set $A^{\mathcal{I}} = \{x \in \mathbb{R} \colon \forall_{U_x \in \mathcal{T}} \ U_x \cap A \text{ is of second category}\}$.

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In the paper we will consider two different and proper ideals $\mathcal{I}_1$, $\mathcal{I}_2$ fulfilling condition (1) and containing all singletons. For $A \subset X$ and $n \geq 2$, we define the sets

$$A^{\mathcal{I}_{i_1}\ldots\mathcal{I}_{i_n}} = \left(A^{\mathcal{I}_{i_1}\ldots\mathcal{I}_{i_{n-1}}}\right)^{\mathcal{I}_{i_n}},$$

where $\mathcal{I}_{i_m} \in \{\mathcal{I}_1, \mathcal{I}_2\}$ for $m = 1, \ldots, n$.

EXAMPLE 2. $A^{\mathcal{I}_1\mathcal{I}_2}$ and $A^{\mathcal{I}_2\mathcal{I}_1}$ need not be equal. Let $X = \mathbb{R}$, $\mathcal{I}_1 = \mathcal{N}$ and $\mathcal{I}_2 = \mathcal{K}$. Let $A \in \mathcal{I}_1$, $B \in \mathcal{I}_2$ be disjoint sets such that $\mathbb{R} = A \cup B$. Then $A^{\mathcal{I}_1} = \emptyset$, $A^{\mathcal{I}_2} = \mathbb{R}$, hence $A^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$ and $A^{\mathcal{I}_2\mathcal{I}_1} = \mathbb{R}$. Moreover, we have $X = X^{\mathcal{I}_1\mathcal{I}_2}$.

EXAMPLE 3. Let $X = \mathbb{R}$, $\mathcal{T} = \mathcal{T}_o$, $\mathcal{I}_1 = \mathcal{C} \cup (0,1)$ and $\mathcal{I}_2 = \mathcal{C} \cup (1,2)$ Then $\mathcal{I}_1$, $\mathcal{I}_2$ are ideals fulfilling condition (1). For such ideals we obtain $X^{\mathcal{I}_1} = \mathbb{R} \setminus [0,1]$ and $X^{\mathcal{I}_1\mathcal{I}_2} = \mathbb{R} \setminus [0,2]$. Hence $X \setminus X^{\mathcal{I}_1\mathcal{I}_2} \neq \emptyset$.

The main aim of this work is to answer the following questions. Can we introduce the closure operation in X in the same way as it is done for one ideal, do we obtain a topology of Hashimoto type, is this topology comparable to those obtained for ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ separately? What kind of ideal generates this topology?

From [1] and the definition of $A^{\mathcal{I}_1\mathcal{I}_2}$, we immediately have

PROPERTY 4. *For any $A \subset X$*

(1) $A^{\mathcal{I}_1\mathcal{I}_2} \subset \overline{A}$ *and* $A^{\mathcal{I}_1\mathcal{I}_2}$ *is closed in* $\mathcal{T}$,

(2) $(A \cup B)^{\mathcal{I}_1\mathcal{I}_2} = A^{\mathcal{I}_1\mathcal{I}_2} \cup B^{\mathcal{I}_1\mathcal{I}_2}$, $(A \setminus B)^{\mathcal{I}_1\mathcal{I}_2} \supset A^{\mathcal{I}_1\mathcal{I}_2} \setminus B^{\mathcal{I}_1\mathcal{I}_2}$,

(3) $A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2} \in \mathcal{I}_2$, $A^{\mathcal{I}_2} \setminus A^{\mathcal{I}_2\mathcal{I}_1} \in \mathcal{I}_1$,

(4) *if* $A \subset B$, *then* $A^{\mathcal{I}_1\mathcal{I}_2} \subset B^{\mathcal{I}_1\mathcal{I}_2}$.

In some proofs we will make use of [1, Lemma 1], so let us remind it using the notation introduced earlier.

LEMMA 5. *For any open set G and a set $A \subset X$,*

$$(G \cap A)^{\mathcal{I}} = (G \cap A^{\mathcal{I}})^{\mathcal{I}} = \overline{G \cap A^{\mathcal{I}}}.$$

Further, we will prove similar lemma for two ideals $\mathcal{I}_1$, $\mathcal{I}_2$ instead of one ideal $\mathcal{I}$.

THEOREM 6. *If $\mathcal{I}_1 \subset \mathcal{I}_2$, then for any set $A \subset X$ the following inclusions hold:*

$$A^{\mathcal{I}_2} = A^{\mathcal{I}_2\mathcal{I}_1} \subset A^{\mathcal{I}_1\mathcal{I}_2} \subset A^{\mathcal{I}_1} \subset \overline{A}.$$

P r o o f. Let $A \subset X$. If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $A^{\mathcal{I}_2} \subset A^{\mathcal{I}_1}$. From Property 4 we immediately obtain

$$A^{\mathcal{I}_2} \subset A^{\mathcal{I}_1\mathcal{I}_2} \subset A^{\mathcal{I}_1} \subset \overline{A} \quad \text{and} \quad A^{\mathcal{I}_2} \subset A^{\mathcal{I}_2\mathcal{I}_1} \subset A^{\mathcal{I}_1} \subset \overline{A}.$$

We will show that $A^{\mathcal{I}_2\mathcal{I}_1} \subset A^{\mathcal{I}_2}$ for any $A \subset X$. Suppose that there exists a point $x \in A^{\mathcal{I}_2\mathcal{I}_1} \setminus A^{\mathcal{I}_2}$. Then, for any its neighbourhood U_x, we have $U_x \cap A^{\mathcal{I}_2} \notin \mathcal{I}_1$ and there is a neighbourhood V_x of x such that $V_x \cap A \in \mathcal{I}_2$. So $V_x \cap A^{\mathcal{I}_2} \notin \mathcal{I}_1$ and $V_x \cap A \in \mathcal{I}_2$. From Lemma 5 we obtain $\emptyset = (V_x \cap A)^{\mathcal{I}_2} = \overline{V_x \cap A^{\mathcal{I}_2}}$, hence $V_x \cap A^{\mathcal{I}_2} = \emptyset$, and we have a contradiction. $\square$

EXAMPLE 7. Let $X = \mathbb{R}$, $\mathcal{I}_1 = \mathcal{C}$, $\mathcal{I}_2 = \mathcal{N}$ ($\mathcal{I}_2 = \mathcal{K}$). Then there exist sets for which the above inclusions are proper.

Let C denote the ternary Cantor set. Then $C \in \mathcal{I}_2$ and $C^{\mathcal{I}_1} = C$, $C^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$. Therefore, $C^{\mathcal{I}_1} \setminus C^{\mathcal{I}_1\mathcal{I}_2} \neq \emptyset$.

Taking the set C we will construct the set B for which $B^{\mathcal{I}_1\mathcal{I}_2} \setminus B^{\mathcal{I}_2} \neq \emptyset$. For $a, b \in \mathbb{R}$, $a < b$ we put $C(a,b) = \{a + (b-a)x \colon x \in C\}$. The set $C(a,b)$ is perfect, nowhere dense and of measure zero. Let $B = \bigcup_{a,b\in\mathbb{Q},\, a<b} C(a,b)$. Then B is of first category and of measure zero, hence $B^{\mathcal{I}_2} = \emptyset$. We will show that $B^{\mathcal{I}_1} = \mathbb{R}$. Let $x \in \mathbb{R}$ and U_x be its neighbourhood. Then there exist numbers $a, b \in \mathbb{Q}$, $a < b$ such that $C(a,b) \subset U_x$, so $C(a,b) \subset U_x \cap B$. The set $U_x \cap B$ is uncountable, so $U_x \cap B \notin \mathcal{I}_1$. Hence $x \in B^{\mathcal{I}_1}$ and $B^{\mathcal{I}_1} = \mathbb{R}$. From this we have $B^{\mathcal{I}_1\mathcal{I}_2} \setminus B^{\mathcal{I}_2} \neq \emptyset$. $\square$

PROPERTY 8. *If $\mathcal{I}_2$ contains all nowhere dense sets, then $\overline{A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}} \in \mathcal{I}_2$.*

P r o o f. Let $A \subset X$. The sets $A^{\mathcal{I}_1}$ and $A^{\mathcal{I}_1\mathcal{I}_2}$ are closed and

$$\begin{aligned}
\overline{A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}} &= \overline{A^{\mathcal{I}_1} \cap (A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2})} \subset A^{\mathcal{I}_1} \cap \overline{A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}} \\
&= \Big(A^{\mathcal{I}_1} \cap \overline{A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}} \cap A^{\mathcal{I}_1\mathcal{I}_2}\Big) \cup \Big(\big(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big) \cap \overline{A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}}\Big) \\
&= \Big(A^{\mathcal{I}_1} \cap A^{\mathcal{I}_1\mathcal{I}_2} \cap \overline{A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}}\Big) \cup \big(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big) \\
&\subset \Big(A^{\mathcal{I}_1} \cap A^{\mathcal{I}_1\mathcal{I}_2} \cap \overline{X \setminus A^{\mathcal{I}_1\mathcal{I}_2}}\Big) \cup \big(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big) \\
&\subset \big(A^{\mathcal{I}_1} \cap \mathrm{Fr} A^{\mathcal{I}_1\mathcal{I}_2}\big) \cup \big(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big) \subset \mathrm{Fr} A^{\mathcal{I}_1\mathcal{I}_2} \cup \big(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big).
\end{aligned}$$

The set $\mathrm{Fr} A^{\mathcal{I}_1\mathcal{I}_2}$ is nowhere dense, $A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2} \in \mathcal{I}_2$ from Property 4, so

$$\overline{A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}} \in \mathcal{I}_2.$$

$\square$

From now on, we will assume that $\mathcal{I}_2$ contains all nowhere dense subsets of X. Let

$$\mathcal{I}_1 \oplus \overline{\mathcal{I}_2} = \{N \cup M \colon N \in \mathcal{I}_1 \wedge \overline{M} \in \mathcal{I}_2\}.$$

Then $\mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$ is a proper ideal containing all singletons.

The next lemma is a simple observation how small the difference between A and $A^{\mathcal{I}_1\mathcal{I}_2}$ is. It will be mentioned again as a part of Property 23, where an oposite difference is also considered.

Lemma 9. *If $A \subset X$, then $A \setminus A^{\mathcal{I}_1\mathcal{I}_2} \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$.*

Proof. It follows directly from Properties 4 and 8 and the fact that

$$\begin{aligned} A \setminus A^{\mathcal{I}_1\mathcal{I}_2} &= \Big(\big(A \setminus A^{\mathcal{I}_1}\big) \cup \big(A \cap A^{\mathcal{I}_1}\big)\Big) \setminus A^{\mathcal{I}_1\mathcal{I}_2} \\ &= \Big(\big(A \setminus A^{\mathcal{I}_1}\big) \setminus A^{\mathcal{I}_1\mathcal{I}_2}\Big) \cup \Big(A \cap \big(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big)\Big) \\ &\subset \big(A \setminus A^{\mathcal{I}_1}\big) \cup \big(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big). \end{aligned}$$

□

Let us notice that

$$A^{\mathcal{I}_1\mathcal{I}_2} = \emptyset \iff A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}. \tag{2}$$

Indeed, if $A^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$, then, from the above lemma, $A = A \setminus A^{\mathcal{I}_1\mathcal{I}_2} \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$. If $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then $A = N \cup M$, where $N \in \mathcal{I}_1$ and $\overline{M} \in \mathcal{I}_2$. Hence $A^{\mathcal{I}_1\mathcal{I}_2} = (N \cup M)^{\mathcal{I}_1\mathcal{I}_2} = (M^{\mathcal{I}_1})^{\mathcal{I}_2} \subset (\overline{M})^{\mathcal{I}_2} = \emptyset$. Moreover, if $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then for any set $B \subset X$,

$$(B \cup A)^{\mathcal{I}_1\mathcal{I}_2} = B^{\mathcal{I}_1\mathcal{I}_2} \quad \text{and} \quad (B \setminus A)^{\mathcal{I}_1\mathcal{I}_2} = B^{\mathcal{I}_1\mathcal{I}_2}.$$

The operator $A^{\mathcal{I}_1\mathcal{I}_2}$ is idempotent, as the next theorem shows.

Theorem 10. *If $A \subset X$, then $A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1\mathcal{I}_2} = A^{\mathcal{I}_1\mathcal{I}_2}$.*

Proof. Let $A \subset X$. First, we will show that $A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1} \subset A^{\mathcal{I}_1\mathcal{I}_2}$. Suppose that there is a point $x \in A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}$. Then there exists V_x—an open neighbourhood of x for which $V_x \cap A^{\mathcal{I}_1\mathcal{I}_2} \notin \mathcal{I}_1$ and $\overline{V_x \cap A^{\mathcal{I}_1}} \in \mathcal{I}_2$. Therefore, $(V_x \cap A^{\mathcal{I}_1})^{\mathcal{I}_2} = \emptyset$. From Lemma 5 we obtain $\overline{V_x \cap A^{\mathcal{I}_1\mathcal{I}_2}} = (V_x \cap A^{\mathcal{I}_1})^{\mathcal{I}_2} = \emptyset$, so $V_x \cap A^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$ what contradicts the assumption $V_x \cap A^{\mathcal{I}_1\mathcal{I}_2} \notin \mathcal{I}_1$. Hence $A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$ and $A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1} \subset A^{\mathcal{I}_1\mathcal{I}_2}$.

From Property 4 we have $A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1\mathcal{I}_2} \subset A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_2} = A^{\mathcal{I}_1\mathcal{I}_2}$.

Now, we will show that $A^{\mathcal{I}_1\mathcal{I}_2} \subset A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1\mathcal{I}_2}$. From Lemma 9 we have $A \setminus A^{\mathcal{I}_1\mathcal{I}_2} = N \cup M$, where $N \in \mathcal{I}_1$ and $\overline{M} \in \mathcal{I}_2$. Therefore,

$$(A \setminus A^{\mathcal{I}_1\mathcal{I}_2})^{\mathcal{I}_1\mathcal{I}_2} = (N \cup M)^{\mathcal{I}_1\mathcal{I}_2} = (M^{\mathcal{I}_1})^{\mathcal{I}_2} \subset (\overline{M})^{\mathcal{I}_2} = \emptyset. \tag{3}$$

From Property 4 and inclusion $A^{\mathcal{I}_1\mathcal{I}_2} \setminus A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1\mathcal{I}_2} \subset (A \setminus A^{\mathcal{I}_1\mathcal{I}_2})^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$ we obtain $A^{\mathcal{I}_1\mathcal{I}_2} \subset A^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1\mathcal{I}_2}$, which completes the proof. □

From Theorem 10 it follows that the set $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) = A \cup A^{\mathcal{I}_1\mathcal{I}_2}$ is a closure of A. If $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) = A$, then we will say that A is $\mathcal{I}_1\mathcal{I}_2$-closed. The set is $\mathcal{I}_1\mathcal{I}_2$--open, if its complement is $\mathcal{I}_1\mathcal{I}_2$-closed. Moreover, we have

$$x \in \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(A \setminus \{x\}\big) \iff x \in A^{\mathcal{I}_1\mathcal{I}_2},$$

so $A^{\mathcal{I}_1\mathcal{I}_2}$ is a derived set of A.

Theorem 11. *A is $\mathcal{I}_1\mathcal{I}_2$-closed if and only if $A = F \cup N \cup M$, where F is closed in $\mathcal{T}$, $N \in \mathcal{I}_1$ and $\overline{M} \in \mathcal{I}_2$.*

P r o o f. Assume that A is $\mathcal{I}_1\mathcal{I}_2$-closed, so $A^{\mathcal{I}_1\mathcal{I}_2} \subset A$. Then

$$\begin{aligned} A &= \left(A \setminus A^{\mathcal{I}_1}\right) \cup \left(A \cap A^{\mathcal{I}_1}\right) \\ &= \left(A \setminus A^{\mathcal{I}_1}\right) \cup \Big(\left(A \cap A^{\mathcal{I}_1}\right) \cap A^{\mathcal{I}_1\mathcal{I}_2}\Big) \cup \Big(\left(A \cap A^{\mathcal{I}_1}\right) \setminus A^{\mathcal{I}_1\mathcal{I}_2}\Big) \\ &= \left(A \setminus A^{\mathcal{I}_1}\right) \cup \left(A^{\mathcal{I}_1} \cap A^{\mathcal{I}_1\mathcal{I}_2}\right) \cup \Big(A \cap \left(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\right)\Big). \end{aligned}$$

The sets $A^{\mathcal{I}_1}$ and $A^{\mathcal{I}_1\mathcal{I}_2}$ are closed, so does $A^{\mathcal{I}_1} \cap A^{\mathcal{I}_1\mathcal{I}_2}$. Moreover, $A \setminus A^{\mathcal{I}_1} \in \mathcal{I}_1$ and from Property 4 we have

$$A \cap \left(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\right) \in \mathcal{I}_2.$$

Additionally, from the proof of Lemma 9 we obtain $\overline{A \cap \left(A^{\mathcal{I}_1} \setminus A^{\mathcal{I}_1\mathcal{I}_2}\right)} \in \mathcal{I}_2$.

Suppose now that $A = F \cup N \cup M$, where F is closed in $\mathcal{T}$, $N \in \mathcal{I}_1$ and $\overline{M} \in \mathcal{I}_2$. Then from [1] and Property 4

$$A^{\mathcal{I}_1} = F^{\mathcal{I}_1} \cup N^{\mathcal{I}_1} \cup M^{\mathcal{I}_1} = F^{\mathcal{I}_1} \cup M^{\mathcal{I}_1} \subset F^{\mathcal{I}_1} \cup \overline{M}.$$

Hence

$$A^{\mathcal{I}_1\mathcal{I}_2} \subset F^{\mathcal{I}_1\mathcal{I}_2} \cup (\overline{M})^{\mathcal{I}_2} = F^{\mathcal{I}_1\mathcal{I}_2} \subset \overline{F} = F \subset A$$

and $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) = A \cup A^{\mathcal{I}_1\mathcal{I}_2} = A$ which completes the proof. □

Corollary 12. *A is $\mathcal{I}_1\mathcal{I}_2$-open if and only if $A = G \setminus N \setminus M$, where G is open in $\mathcal{T}$, $N \in \mathcal{I}_1$ and $\overline{M} \in \mathcal{I}_2$.*

The topology introduced by the closure operation $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}$ will be denoted by $\mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$. From Theorem 6 we immediately have

Theorem 13. *$\mathcal{T}_{\mathcal{I}_1} \subset \mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$ for any ideals $\mathcal{I}_1$ and $\mathcal{I}_2$. Moreover, if $\mathcal{I}_1 \subset \mathcal{I}_2$, then $\mathcal{T}_{\mathcal{I}_1} \subset \mathcal{T}_{\mathcal{I}_1\mathcal{I}_2} \subset \mathcal{T}_{\mathcal{I}_2\mathcal{I}_1} = \mathcal{T}_{\mathcal{I}_2}$.*

Example 14. For the ideals $\mathcal{I}_1 = \mathcal{C}$ and $\mathcal{I}_2 = \mathcal{N}$ (or $\mathcal{I}_2 = \mathcal{K}$) the above inclusions are proper. If $\mathcal{I}_1 = \mathcal{N}$ and $\mathcal{I}_2 = \mathcal{K}$, then $\mathcal{T}_{\mathcal{I}_2}$ and $\mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$ are incomparable.

For disjoint sets $A \in \mathcal{N}$ and $B \in \mathcal{K}$ such that $\mathbb{R} = A \cup B$ we have $B = \mathbb{R} \setminus A \setminus \emptyset \in \mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$ and $B \notin \mathcal{T}_{\mathcal{I}_2}$.

In the same way as in Example 7 we will construct the set B_α such that it will be of first category and $\overline{B_\alpha} = \mathbb{R}$. Let C_α be a Cantor set of positive measure α. For $a, b \in \mathbb{R}$, $a < b$, put $C_\alpha(a,b) = \{a + (b-a)x \colon x \in C_\alpha\}$. The set $C_\alpha(a,b)$ is perfect, nowhere dense, so $B_\alpha = \bigcup_{a,b \in \mathbb{Q},\, a<b} C_\alpha(a,b)$ is of first category. Hence $\mathbb{R} \setminus B_\alpha \in \mathcal{T}_{\mathcal{I}_2}$. We will show that $\overline{B_\alpha} = \mathbb{R}$. Let $x \in \mathbb{R}$ and U_x be its neighbourhood. Then there exist numbers $a, b \in \mathbb{Q}$, $a < b$ such that $C_\alpha(a,b) \subset U_x$, so $C_\alpha(a,b) \subset U_x \cap B_\alpha$. Hence $U_x \cap B_\alpha \neq \emptyset$ and $x \in \overline{B_\alpha}$. The set $\mathbb{R} \setminus B_\alpha = \mathbb{R} \setminus \emptyset \setminus B_\alpha$ does not belong to the topology $\mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$. Hence $\mathcal{T}_{\mathcal{I}_2} \setminus \mathcal{T}_{\mathcal{I}_1\mathcal{I}_2} \neq \emptyset$. □

For operator $A^{\mathcal{I}_1\mathcal{I}_2}$ we can prove a lemma similar to Lemma 5.

Lemma 15. *For any open set G and a set $A \subset X$*

$$(G \cap A)^{\mathcal{I}_1\mathcal{I}_2} \overset{(i)}{=} \big(G \cap A^{\mathcal{I}_1\mathcal{I}_2}\big)^{\mathcal{I}_1\mathcal{I}_2} \overset{(ii)}{=} \overline{G \cap A^{\mathcal{I}_1\mathcal{I}_2}}.$$

P r o o f. First, we will show that for any open set G and a set $A \subset X$

$$G \cap A^{\mathcal{I}_1\mathcal{I}_2} = G \cap (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}. \tag{4}$$

From inclusion $G \cap A \subset A$ and from Property 4 we have $(G \cap A)^{\mathcal{I}_1\mathcal{I}_2} \subset A^{\mathcal{I}_1\mathcal{I}_2}$. Hence $G \cap (G \cap A)^{\mathcal{I}_1\mathcal{I}_2} \subset G \cap A^{\mathcal{I}_1\mathcal{I}_2}$.

We will show that if $x \notin G \cap (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}$, then $x \notin G \cap A^{\mathcal{I}_1\mathcal{I}_2}$. Assume that $x \notin G \cap (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}$. We consider two cases. First, let $x \notin G$. Then $x \notin G \cap A^{\mathcal{I}_1\mathcal{I}_2}$. If $x \in G$ but $x \notin G \cap (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}$, then $x \notin (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}$. Therefore, there exists its neighbourhood V_x such that $V_x \cap (G \cap A)^{\mathcal{I}_1} \in \mathcal{I}_2$. Hence $V_x \cap G \cap (G \cap A)^{\mathcal{I}_1} \in \mathcal{I}_2$. From [1, Condition 2d] for the ideal $\mathcal{I}_1$, we obtain $V_x \cap G \cap (G \cap A)^{\mathcal{I}_1} = V_x \cap \big(G \cap A^{\mathcal{I}_1}\big) \in \mathcal{I}_2$. So we have an open set $V_x \cap G$, such that $x \in V_x \cap G$ and $(V_x \cap G) \cap A^{\mathcal{I}_1} \in \mathcal{I}_2$, which means $x \notin A^{\mathcal{I}_1\mathcal{I}_2}$. Finally, we have $x \notin G \cap A^{\mathcal{I}_1\mathcal{I}_2}$ which completes the proof of (4).

Now, we will prove (i). From (4) we have $G \cap A^{\mathcal{I}_1\mathcal{I}_2} \subset (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}$. From Theorem 10 it follows that

$$\big(G \cap A^{\mathcal{I}_1\mathcal{I}_2}\big)^{\mathcal{I}_1\mathcal{I}_2} \subset (G \cap A)^{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_1\mathcal{I}_2} = (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}.$$

Simultaneously, from Lemma 9,

$$\begin{aligned}(G \cap A)^{\mathcal{I}_1\mathcal{I}_2} &= \big(G \cap A \setminus A^{\mathcal{I}_1\mathcal{I}_2}\big)^{\mathcal{I}_1\mathcal{I}_2} \cup \big(G \cap A \cap A^{\mathcal{I}_1\mathcal{I}_2}\big)^{\mathcal{I}_1\mathcal{I}_2} \\ &\subset \big(G \cap A \cap A^{\mathcal{I}_1\mathcal{I}_2}\big)^{\mathcal{I}_1\mathcal{I}_2} \subset (G \cap A^{\mathcal{I}_1\mathcal{I}_2})^{\mathcal{I}_1\mathcal{I}_2}\end{aligned}$$

which finishes the proof of (i).

From Property 4 and (i) we obtain

$$\overline{G \cap A^{\mathcal{I}_1\mathcal{I}_2}} \supset \big(G \cap A^{\mathcal{I}_1\mathcal{I}_2}\big)^{\mathcal{I}_1\mathcal{I}_2} = (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}.$$

On the other hand, from (4) we have

$$\overline{G \cap A^{\mathcal{I}_1\mathcal{I}_2}} \subset \overline{(G \cap A)^{\mathcal{I}_1\mathcal{I}_2}} = (G \cap A)^{\mathcal{I}_1\mathcal{I}_2}$$

which completes the proof of (ii). □

Let us notice that if G is open and $G \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then $G \subset X \setminus X^{\mathcal{I}_1\mathcal{I}_2}$. Indeed, by putting $A = X$ in Lemma 15 we obtain

$$(G \cap X)^{\mathcal{I}_1\mathcal{I}_2} = G^{\mathcal{I}_1\mathcal{I}_2} = \overline{G \cap X^{\mathcal{I}_1\mathcal{I}_2}}.$$

$G \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, hence $G^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$. From this, $G \cap X^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$ and $G \subset X \setminus X^{\mathcal{I}_1\mathcal{I}_2}$ (see also Example 3).

Recall that in the paper we assume that $\mathcal{I}_1 \neq \mathcal{I}_2$ and the ideals are proper. Without this assumption the next property is not true.

Property 16. *If G is open in $\mathcal{T}$, then $\overline{G} = \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(G)$.*

Proof. We will show that for any open set G we have $\overline{G} = \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(G)$. From the definition of the closure operation we have $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(G) = G \cup G^{\mathcal{I}_1\mathcal{I}_2} \subset G \cup \overline{G} = \overline{G}$.

Let $x \in \overline{G}$. We will show that $x \in \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(G)$. Let $U_x \in \mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$ be a neighbourhood of x. Then $U_x = U_0 \setminus N \setminus M$, where $U_0 \in \mathcal{T}$, $N \in \mathcal{I}_1$, $\overline{M} \in \mathcal{I}_2$. Consider $U_x \cap G = (U_0 \setminus N \setminus M) \cap G = (U_0 \cap G) \setminus N \setminus M$. The set $U_0 \cap G$ is open and nonempty, so $(U_0 \cap G) \setminus N \setminus M \neq \emptyset$, hence $x \in \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(G)$. □

Proposition 17. *If G is $\mathcal{I}_1\mathcal{I}_2$-open and $A \subset X$, then*

$$\overline{G \cap A^{\mathcal{I}_1\mathcal{I}_2}} = \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(G \cap A^{\mathcal{I}_1\mathcal{I}_2}\big).$$

Proof. Assume that $G = U \setminus N \setminus M$, where U is open, $N \in \mathcal{I}_1$, $\overline{M} \in \mathcal{I}_2$. Then from Lemma 15 *(ii)*

$$\begin{aligned}
&\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(G \cap A^{\mathcal{I}_1\mathcal{I}_2}\big)\\
&= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\Big(\big(U \setminus N \setminus M\big) \cap A^{\mathcal{I}_1\mathcal{I}_2}\Big)\\
&= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(U \cap A^{\mathcal{I}_1\mathcal{I}_2} \setminus N \setminus M\big)\\
&= \big(U \cap A^{\mathcal{I}_1\mathcal{I}_2} \setminus N \setminus M\big) \cup \big(U \cap A^{\mathcal{I}_1\mathcal{I}_2} \setminus N \setminus M\big)^{\mathcal{I}_1\mathcal{I}_2}\\
&= \big(U \cap A^{\mathcal{I}_1\mathcal{I}_2} \setminus N \setminus M\big) \cup \big(U \cap A^{\mathcal{I}_1\mathcal{I}_2}\big)^{\mathcal{I}_1\mathcal{I}_2}\\
&\overset{(ii)}{=} \big(U \cap A^{\mathcal{I}_1\mathcal{I}_2} \setminus N \setminus M\big) \cup \overline{U \cap A^{\mathcal{I}_1\mathcal{I}_2}} \supset \overline{U \cap A^{\mathcal{I}_1\mathcal{I}_2}} \supset \overline{G \cap A^{\mathcal{I}_1\mathcal{I}_2}}.
\end{aligned}$$

On the other hand, we know that $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(G \cap A^{\mathcal{I}_1\mathcal{I}_2}\big) \subset \overline{G \cap A^{\mathcal{I}_1\mathcal{I}_2}}$ which completes the proof. □

If $X^{\mathcal{I}_1\mathcal{I}_2} = X$ and G is $\mathcal{I}_1\mathcal{I}_2$-open, then $\overline{G} = \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(G)$.

We will say that a set is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense ($\mathcal{I}_1\mathcal{I}_2$-scattered...), if it is nowhere dense (scattered...) in topology $\mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$.

Let us notice that any nowhere dense set is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense. If A is nowhere dense, then $\overline{X \setminus \overline{A}} = X$. From Property 16 we obtain

$$\overline{X \setminus \overline{A}} = \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(X \setminus \overline{A})$$

and

$$X = \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(X \setminus \overline{A}) \subset \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big) \subset X,$$

so A is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense.

Property 18. *If A is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense set, then $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$.*

P r o o f. Let A be $\mathcal{I}_1\mathcal{I}_2$-nowhere dense. Then

$$\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big) = X.$$

The set $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)$ is $\mathcal{I}_1\mathcal{I}_2$-closed, so it can be presented in the form $F \cup N \cup M$, where F is closed in $\mathcal{T}$, $N \in \mathcal{I}_1$ and $\overline{M} \in \mathcal{I}_2$. Let G be an open set such that $F = X \setminus G$. Then $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) = (X \setminus G) \cup N \cup M$ and

$$\begin{aligned} X &= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big) \\ &= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\Big(X \setminus \big((X \setminus G) \cup N \cup M\big)\Big) \\ &= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus (X \setminus G) \setminus N \setminus M\big) \\ &= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(G \setminus N \setminus M) \\ &\subset \overline{G \setminus N \setminus M} \subset \overline{G} \subset X. \end{aligned}$$

The set G is open and dense, so $X \setminus G$ is nowhere dense in X, as $\mathcal{I}_2$ contains all nowhere dense sets $\overline{X \setminus G} \in \mathcal{I}_2$. Moreover,

$$A = A \cap \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) = A \cap \big((X \setminus G) \cup N \cup M\big) = \big(A \cap (X \setminus G)\big) \cup (A \cap N) \cup (A \cap M)$$

what finishes the proof. □

Let us notice that if $X^{\mathcal{I}_1\mathcal{I}_2} = X$ and $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then A is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense set. Indeed, if $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then $A^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$ and $\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) = A$. Therefore,

$$\begin{aligned} \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big) &= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(X \setminus A) \\ &= (X \setminus A) \cup (X \setminus A)^{\mathcal{I}_1\mathcal{I}_2} \\ &\supset (X \setminus A) \cup X^{\mathcal{I}_1\mathcal{I}_2} = X \end{aligned}$$

and we obtain the following corollary.

Corollary 19. *Let $X^{\mathcal{I}_1\mathcal{I}_2} = X$. Then A is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense set if and only if $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$.*

Remind that a set is called scattered if it does not contain any dense in itself subset.

Property 20. *A is $\mathcal{I}_1\mathcal{I}_2$-scattered if and only if $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$.*

P r o o f. First, we will show that for any A the set $A \cap A^{\mathcal{I}_1\mathcal{I}_2}$ is $\mathcal{I}_1\mathcal{I}_2$-dense in itself. From Lemma 9 and (2) we have $(A \setminus A^{\mathcal{I}_1\mathcal{I}_2})^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$. So $A^{\mathcal{I}_1\mathcal{I}_2} = (A \cap A^{\mathcal{I}_1\mathcal{I}_2})^{\mathcal{I}_1\mathcal{I}_2}$ and $A \cap A^{\mathcal{I}_1\mathcal{I}_2} \subset (A \cap A^{\mathcal{I}_1\mathcal{I}_2})^{\mathcal{I}_1\mathcal{I}_2}$, what means $A \cap A^{\mathcal{I}_1\mathcal{I}_2}$ is $\mathcal{I}_1\mathcal{I}_2$-dense in itself. Let A be $\mathcal{I}_1\mathcal{I}_2$-scattered, so it does not contain any $\mathcal{I}_1\mathcal{I}_2$-dense in itself subset. Hence $A \cap A^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$ and from Lemma 9 we have $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$.

Assume that $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$. Let $B \subset A$, $B \neq \emptyset$. Then $B \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$ and from (2) we obtain $B \cap B^{\mathcal{I}_1\mathcal{I}_2} = \emptyset$. Therefore, A does not contain any nonempty dense in itself subset and A is $\mathcal{I}_1\mathcal{I}_2$-scattered. □

Property 21. *If $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then $A = G \cup N$, where G is $\mathcal{I}_1\mathcal{I}_2$-open included in $X \setminus X^{\mathcal{I}_1\mathcal{I}_2}$ and N is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense included in $X^{\mathcal{I}_1\mathcal{I}_2}$.*

Proof. Let $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then A is $\mathcal{I}_1\mathcal{I}_2$-scattered. From [3, p. 79] we know that each scattered set can be decomposed into an open set and nowhere dense set, hence $A = G \cup N$, G is $\mathcal{I}_1\mathcal{I}_2$-open (but also from $\mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$ as a subset of A), N is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense. As G is $\mathcal{I}_1\mathcal{I}_2$-open, it is a subset of certain open set G_0. Then $G_0 \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$ and $G_0 \subset X \setminus X^{\mathcal{I}_1\mathcal{I}_2}$. Hence $G \subset X \setminus X^{\mathcal{I}_1\mathcal{I}_2}$.

We will show that $\mathcal{I}_1\mathcal{I}_2$-nowhere dense N is included in $X^{\mathcal{I}_1\mathcal{I}_2}$. Consider the set $N_1 = N \cap (X \setminus X^{\mathcal{I}_1\mathcal{I}_2})$. Suppose that the set N_1 is nonempty. Then $N_1 \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$ and $N_1 \subset X \setminus X^{\mathcal{I}_1\mathcal{I}_2}$. Hence

$$(X \setminus N_1)^{\mathcal{I}_1\mathcal{I}_2} \cap N_1 = X^{\mathcal{I}_1\mathcal{I}_2} \cap N_1 = \emptyset.$$

N_1 is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense, so

$$\begin{aligned} N_1 \subset X &= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(N_1)\big) \\ &= \big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(N_1)\big) \cup \big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(N_1)\big)^{\mathcal{I}_1\mathcal{I}_2} \\ &= \Big(X \setminus \big(N_1 \cup N_1^{\mathcal{I}_1\mathcal{I}_2}\big)\Big) \cup \Big(X \setminus \big(N_1 \cup N_1^{\mathcal{I}_1\mathcal{I}_2}\big)\Big)^{\mathcal{I}_1\mathcal{I}_2} \\ &= \big(X \setminus N_1\big) \cup \big(X \setminus N_1\big)^{\mathcal{I}_1\mathcal{I}_2}. \end{aligned}$$

Thus $N_1 \subset (X \setminus N_1)^{\mathcal{I}_1\mathcal{I}_2}$ which is a contradiction to the definition of N_1. Hence $N_1 = \emptyset$ and $N \subset X^{\mathcal{I}_1\mathcal{I}_2}$. □

Property 22. *If $X = X^{\mathcal{I}_1\mathcal{I}_2}$ and $A \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$, then $X \setminus A$ is dense in X.*

Proof. From the previous property we have that $A = G \cup N$, where $G \subset X \setminus X^{\mathcal{I}_1\mathcal{I}_2}$ is $\mathcal{I}_1\mathcal{I}_2$-open and $N \subset X^{\mathcal{I}_1\mathcal{I}_2}$ is $\mathcal{I}_1\mathcal{I}_2$-nowhere dense. Hence $G = \emptyset$ and $X \setminus A = X \setminus N$. Moreover,

$$X = \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(N)\big) \subset \overline{X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(N)} \subset \overline{X \setminus N}$$

which means that $X = \overline{X \setminus A}$. □

The next property contains the result of Lemma 9 and shows us how much the set A differs from its derive set in topology $\mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$.

Property 23. *For any $A \subset X$*

1. $A \setminus A^{\mathcal{I}_1\mathcal{I}_2} \in \mathcal{I}_1 \oplus \overline{\mathcal{I}_2}$,
2. *$A^{\mathcal{I}_1\mathcal{I}_2} \setminus A$ is the $\mathcal{I}_1\mathcal{I}_2$-boundary set.*

P r o o f. Consider the set

$$
\begin{aligned}
&\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\Big(X \setminus \big(A^{\mathcal{I}_1\mathcal{I}_2} \setminus A\big)\Big)\\
&= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus (\mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) \setminus A)\big)\\
&= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\Big(\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big) \cup (X \cap A)\Big)\\
&= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\Big(\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big) \cup A\Big)\\
&= \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) \cup \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}\big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big)\\
&\supset \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A) \cup \big(X \setminus \mathrm{Cl}_{\mathcal{I}_1\mathcal{I}_2}(A)\big) = X.
\end{aligned}
$$

Hence $A^{\mathcal{I}_1\mathcal{I}_2} \setminus A$ is a boundary set in topology $\mathcal{T}_{\mathcal{I}_1\mathcal{I}_2}$. □

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Center of Mathematics and Physics
Łódź University of Technology
al. Politechniki 11
PL–90-924 Łódź
POLAND
E-mail: terepeta@p.lodz.pl