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MAXIMUMS OF INTERNALLY QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. In this paper, we characterize maximums of internally quasi-continuous functions, Darboux internally quasi-continuous functions, internally strong Świątkowski functions, and lattices generated by these families of functions. Moreover, we examine maximal classes with respect to maximums for internally quasi-continuous functions and Darboux internally quasi-continuous functions.

1. Preliminaries

We mostly use the standard terminology and notation. The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The word function denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. Functions will be identified with their graphs. The symbols $\mathrm{I}(a,b)$ and $\mathrm{I}[a,b]$ denote the open and the closed interval with endpoints a and b, respectively. For each $A \subset \mathbb{R}$, we use the symbol $\operatorname{int} A$ to denote its interior.

We say that f is a *Darboux function* ($f \in \mathcal{D}$), if it maps connected sets onto connected sets. We say that f is *quasi-continuous* in the sense of Kempisty [5] ($f \in \mathcal{Q}$), if, for all $x \in \mathbb{R}$ and open sets $U \ni x$ and $V \ni f(x)$, the set $\operatorname{int}\big(U \cap f^{-1}(V)\big)$ is nonempty. The symbols $\mathcal{C}(f)$ and $\mathcal{Q}(f)$ will stand for the set of points of continuity of f and the set of points of quasi-continuity of f, respectively. We say that f is *internally quasi-continuous* [8] ($f \in \mathcal{Q}_i$), if it is quasi-continuous and its set of points of discontinuity is nowhere dense; equivalently, f is internally quasi-continuous if $f \restriction \operatorname{int} \mathcal{C}(f)$ is dense in f. We say that x_0 is a point of internal quasi-continuity of f if and only if there is a sequence $(x_n) \subset \operatorname{int} \mathcal{C}(f)$ such that $x_n \to x_0$ and $f(x_n) \to f(x_0)$ (see [8]).

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Denote by $\mathcal{Q}_i(f)$ the set of all points of internally quasi-continuity of f. We say that f is a *strong Świątkowski function* [6] ($f \in \acute{\mathcal{S}}_s$), if, whenever $a, b \in \mathbb{R}$, $a < b$, and $y \in \mathrm{I}\big(f(a), f(b)\big)$, there is an $x_0 \in (a,b) \cap \mathcal{C}(f)$ such that $f(x_0) = y$. The symbol $\mathcal{U}(f)$ denotes $\bigcup\big\{(a,b)\colon f{\restriction}(a,b) \in \acute{\mathcal{S}}_s\big\}$. We say that f is an *internally strong Świątkowski function* [8] ($f \in \acute{\mathcal{S}}_{si}$), if, whenever $a, b \in \mathbb{R}$, $a < b$, and $y \in \mathrm{I}\big(f(a), f(b)\big)$, there is an $x_0 \in (a,b) \cap \operatorname{int} \mathcal{C}(f)$ such that $f(x_0) = y$. We say that $f \in \mathcal{C}onst$ if and only if $f[\mathbb{R}]$ is a singleton.

If $A \subset \mathbb{R}$ and x is a limit point of A, then let

$$\lim(f, A, x) = \lim_{\substack{t \to x,\\ t \in A}} f(t).$$

Similarly, we define $\overline{\lim}(f, A, x)$, $\underline{\lim}(f, A, x^+)$, etc. Moreover, we write $\overline{\lim}(f, x)$ instead of $\overline{\lim}(f, \mathbb{R}, x)$, etc. If $\mathcal{L}$ and $\mathcal{F}$ are families of real functions, then we will write $\mathcal{L}\mathcal{F}$ instead of $\mathcal{L} \cap \mathcal{F}$. We say that $\mathcal{L}$ is a *lattice*, if $\max\{f, g\}, \min\{f, g\} \in \mathcal{L}$ for all functions $f, g \in \mathcal{L}$. Moreover, we define the maximal class with respect to maximums for $\mathcal{L}$ as follows

$$\mathcal{M}_{\max}(\mathcal{L}) = \left\{ f\colon \underset{g \in \mathcal{L}}{\forall} \max\{f, g\} \in \mathcal{L} \right\}.$$

We can easily see that the following inclusions are satisfied:

$$\acute{\mathcal{S}}_{si} \subset \acute{\mathcal{S}}_s \subset \mathcal{D}\mathcal{Q} \subset \mathcal{D}, \quad \mathcal{D}\mathcal{Q} \subset \mathcal{Q}, \quad \text{and} \quad \acute{\mathcal{S}}_{si} \subset \mathcal{D}\mathcal{Q}_i \subset \mathcal{Q}_i \subset \mathcal{Q}.$$

2. Introduction

In 1992, T. Natkaniec proved the following result [9, Proposition 3].

Theorem 2.1. *For every function f the following conditions are equivalent*:

a) *there are functions $g_1, g_2 \in \mathcal{Q}$ with $f = \max\{g_1, g_2\}$,*

b) *the set $\mathbb{R} \setminus \mathcal{Q}(f)$ is nowhere dense and $f(x) \le \overline{\lim}\big(f, \mathcal{C}(f), x\big)$ for each $x \in \mathbb{R}$.*

(In 1996 this theorem was generalized by J. Borsík for functions defined on regular second countable topological spaces [1].) He also remarked that if a function f can be written as the maximum of Darboux quasi-continuous functions, then

$$f(x) \le \min\Big\{\overline{\lim}\big(f, \mathcal{C}(f), x^-\big), \overline{\lim}\big(f, \mathcal{C}(f), x^+\big)\Big\} \qquad \text{for each} \quad x \in \mathbb{R}, \tag{1}$$

and asked whether the following conjecture is true [9, Remark 3].

Conjecture 2.2. If f is a function such that $\mathbb{R} \setminus \mathcal{Q}(f)$ is nowhere dense and condition (1) holds, then there are Darboux quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$.

In 1999, A. Maliszewski showed that this conjecture is false and proved some facts about the maximums of Darboux quasi-continuous functions [7]. In 2002, P. Szczuka proved the following theorem [11, Theorem 4.1].

Theorem 2.3. *For every function f the following conditions are equivalent*:

a) *there are functions $g_1, g_2 \in \acute{\mathcal{S}}_s$ with $f = \max\{g_1, g_2\}$,*

b) *the set $\mathcal{U}(f)$ is dense in $\mathbb{R}$ and*

$$f(x) \le \min\Big\{\overline{\lim}\big(f, \mathcal{C}(f), x^+\big), \overline{\lim}\big(f, \mathcal{C}(f), x^-\big)\Big\} \qquad \textit{for each} \quad x \in \mathbb{R}.$$

In this paper, we examine three new interesting subfamilies of quasi-continuous functions, namely families of internally quasi-continuous functions, Darboux quasi-continuous functions, and internally strong Świątkowski functions. We show some theorems quite analogous to Theorems 2.1 and 2.3. Note that although the problem of the characterization of maximums of Darboux quasi-continuous functions is still open, when replacing 'quasi-continuity' with 'internally quasi-continuity', we obtain a fine characterization (Theorem 3.2). Moreover, we find the smallest lattice containing $\mathcal{Q}_i$, $\mathcal{DQ}_i$, and $\acute{\mathcal{S}}_{si}$ (Theorem 3.3 and Corollary 3.4). The obtained characterizations are similar to those ones of lattices generated by families of quasi-continuous functions and strong Świątkowski functions (See [3, Theorem 1] and [11, Teorem 4.2].) Finally, we examine maximal classes with respect to maximums for families $\mathcal{Q}_i$ and $\mathcal{DQ}_i$.

3. Main results

Theorem 3.1. *For every function f, the following conditions are equivalent*:

a) *there exist functions $g_1, g_2 \in \mathcal{Q}_i$ with $f = \max\{g_1, g_2\}$,*

b) *the set $\operatorname{int}\mathcal{C}(f)$ is dense in $\mathbb{R}$ and $f(x) \le \overline{\lim}\big(f, \mathcal{C}(f), x\big)$ for each $x \in \mathbb{R}$.*

Proof. a) $\Rightarrow$ b). Assume that there are internally quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$. Then clearly, $\operatorname{int}\mathcal{C}(g_1) \cap \operatorname{int}\mathcal{C}(g_2) \subset \operatorname{int}\mathcal{C}(f)$. Since the sets $\operatorname{int}\mathcal{C}(g_1)$ and $\operatorname{int}\mathcal{C}(g_2)$ are dense in $\mathbb{R}$, the set $\operatorname{int}\mathcal{C}(f)$ is dense in $\mathbb{R}$, too. Recall that each internally quasi-continuous function is quasi-continuous. Hence by Theorem 2.1, $f(x) \le \overline{\lim}\big(f, \mathcal{C}(f), x\big)$ for each $x \in \mathbb{R}$.

b) $\Rightarrow$ a). If the function f is continuous we can set $g_1 = g_2 = f$. Clearly, each continuous function is internally quasi-continuous. In the opposite case, write $\operatorname{int}\mathcal{C}(f)$ as the union of a family $\mathcal{I}$ consisting of nonoverlapping compact intervals, such that for each $x \in \operatorname{int}\mathcal{C}(f)$, there are $I_1, I_2 \in \mathcal{I}$ with $x \in \operatorname{int}(I_1 \cup I_2)$. Fix an $I \in \mathcal{I}$ and let $I = [x_1, x_2]$. Define

$$r_I = \operatorname{dist}\big(I, \mathbb{R} \setminus \operatorname{int}\mathcal{C}(f)\big) \quad \text{and} \quad M_I = \sup\big\{f(x) : \ x \in I\big\}.$$

Observe that $r_I > 0$. Since f is continuous on I, there is a point $c \in \operatorname{int} I$ such that $f(c) > M_I - r_I$. Choose elements $x_1 < c_1 < d_1 < c < d_2 < c_2 < x_2$. For $i \in \{1,2\}$, define the function $\varphi_{I,i} \colon I \to \mathbb{R}$ as follows

$$\varphi_{I,i}(x) = \begin{cases} 0 & \text{if } x = x_i \text{ or } x \in \mathrm{I}[d_i, x_{3-i}], \\ -\frac{1}{r_I} - |M_I| & \text{if } x = c_i, \\ \text{linear} & \text{in intervals } \mathrm{I}[x_i, c_i] \text{ and } \mathrm{I}[c_i, d_i]. \end{cases}$$

Now, for $i \in \{1,2\}$, define the function g_i by formula

$$g_i(x) = \begin{cases} f(x) & \text{if } x \notin \operatorname{int} \mathcal{C}(f), \\ f(x) + \varphi_{I,i}(x) & \text{if } x \in I,\ I \in \mathcal{I}. \end{cases}$$

Then clearly, $f = \max\{g_1, g_2\}$ and $\operatorname{int} \mathcal{C}(f) \subset \operatorname{int} \mathcal{C}(g_1) \cap \operatorname{int} \mathcal{C}(g_2)$. Moreover,

$$\left[-\frac{1}{r_I}, M_I - r_I\right] \subset g_i[I] \qquad \text{for each } \ I \in \mathcal{I} \quad \text{ and } \quad i \in \{1,2\}. \tag{2}$$

Fix an $i \in \{1,2\}$. We will show that g_i is internally quasi-continuous.

Let $x_0 \in \mathbb{R}$ and $\delta \in (0,1)$. It is sufficient to find an element

$$t \in (x_0 - \delta, x_0 + \delta) \cap \operatorname{int} \mathcal{C}(g_i)$$

such that

$$g_i(t) \in \big(g_i(x_0) - \delta, g_i(x_0) + \delta\big).$$

We can assume that $x_0 \notin \operatorname{int} \mathcal{C}(g_i)$. Therefore, $x_0 \notin \operatorname{int} \mathcal{C}(f)$. Define

$$\delta' = \min\left\{\frac{\delta}{4}, \frac{1}{|f(x_0)| + 1}\right\}.$$

It is easy to see that

$$\overline{\lim}\big(f, \mathcal{C}(f), x\big) = \overline{\lim}\big(f, \operatorname{int} \mathcal{C}(f), x\big).$$

Hence, using assumptions, we obtain that

$$g_i(x_0) = f(x_0) \le \overline{\lim}\big(f, \operatorname{int} \mathcal{C}(f), x\big).$$

So, there is an $I \in \mathcal{I}$ such that $I \subset (x_0 - \delta', x_0 + \delta')$ and $M_I > f(x_0) - \frac{\delta}{4}$. Since $r_I < \delta'$, we have

$$M_I - r_I > f(x_0) - \frac{\delta}{2} \quad \text{ and } \quad -\frac{1}{r_I} < -|f(x_0)| - 1 < f(x_0) - \frac{\delta}{2}.$$

Therefore, by (2), $f(x_0) - \frac{\delta}{2} \in g_i[I]$. Hence, there is a $t \in I \subset (x_0 - \delta, x_0 + \delta)$ such that

$$g_i(t) = f(x_0) - \frac{\delta}{2} \in \big(f(x_0) - \delta, f(x_0) + \delta\big) = \big(g_i(x_0) - \delta, g_i(x_0) + \delta\big).$$

Finally, since $I \subset \operatorname{int} \mathcal{C}(f) \subset \operatorname{int} \mathcal{C}(g_i)$, we have $t \in \operatorname{int} \mathcal{C}(g_i)$. It follows that g_i is internally quasi-continuous. $\square$

Theorem 3.2. *For every function f, the following conditions are equivalent*:

a) *there are functions* $g_1, g_2 \in \acute{\mathcal{S}}_{si}$ *with* $f = \max\{g_1, g_2\}$,

b) *there are functions* $g_1, g_2 \in \mathcal{DQ}_i$ *with* $f = \max\{g_1, g_2\}$,

c) *the set* $\operatorname{int} \mathcal{C}(f)$ *is dense in* $\mathbb{R}$ *and*

$$f(x) \le \min\Big\{\overline{\lim}\big(f, \mathcal{C}(f), x^+\big), \overline{\lim}\big(f, \mathcal{C}(f), x^-\big)\Big\} \qquad \text{for each} \quad x \in \mathbb{R}.$$

P r o o f. The implication a) $\Rightarrow$ b) is evident.

b) $\Rightarrow$ c). Assume that there are Darboux internally quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$. It is easy to see that the set $\operatorname{int} \mathcal{C}(f)$ is dense in $\mathbb{R}$. Since each internally quasi-continuous function is quasi-continuous, by (1), $f(x) \le \min\big\{\overline{\lim}(f, \mathcal{C}(f), x^+), \overline{\lim}(f, \mathcal{C}(f), x^-)\big\}$ for each $x \in \mathbb{R}$.

c) $\Rightarrow$ a). If the function f is continuous we can set $g_1 = g_2 = f$. Clearly, each continuous function is internally strong Świątkowski. In the opposite case write $\operatorname{int} \mathcal{C}(f)$ as the union of a family $\mathcal{I}$ consisting of nonoverlapping compact intervals, such that for each $x \in \operatorname{int} \mathcal{C}(f)$, there are $I_1, I_2 \in \mathcal{I}$ with $x \in \operatorname{int}(I_1 \cup I_2)$.

Let the functions g_1 and g_2 be constructed as in the proof of Theorem 3.1. Then, $f = \max\{g_1, g_2\}$, $\operatorname{int} \mathcal{C}(f) \subset \operatorname{int} \mathcal{C}(g_1) \cap \operatorname{int} \mathcal{C}(g_2)$, and condition (2) holds. Fix an $i \in \{1, 2\}$. We will show that g_i is internally strong Świątkowski.

Let $\alpha < \beta$ and $y \in \mathrm{I}\big(g_i(\alpha), g_i(\beta)\big)$. We can assume that $g_i(\alpha) < g_i(\beta)$. (The case $g_i(\alpha) > g_i(\beta)$ is analogous.) If $[\alpha, \beta] \subset \operatorname{int} \mathcal{C}(f)$, then $[\alpha, \beta] \subset \operatorname{int} \mathcal{C}(g_i)$, whence there is an $x_0 \in (\alpha, \beta) \cap \operatorname{int} \mathcal{C}(g_i)$ such that $g_i(x_0) = y$. So, assume that $[\alpha, \beta] \setminus \operatorname{int} \mathcal{C}(f) \ne \emptyset$. We consider two cases.

Case 1. $\beta \notin \operatorname{int} \mathcal{C}(f)$.

By assumptions,

$$y < g_i(\beta) = f(\beta) \le \overline{\lim}\big(f, \mathcal{C}(f), \beta^-\big) = \overline{\lim}\big(f, \operatorname{int} \mathcal{C}(f), \beta^-\big). \tag{3}$$

Define $\delta = \frac{\overline{\lim}(f, \mathcal{C}(f), \beta^-) - y}{2} > 0$ and let $\delta' = \min\big\{\beta - \alpha, \delta, \frac{1}{|y|+1}\big\}$. By (3), there is an $I \in \mathcal{I}$ such that $I \subset (\beta - \delta', \beta)$ and $M_I > y + \delta$. Since $r_I < \delta'$, we have

$$M_I - r_I > y \quad \text{and} \quad -\tfrac{1}{r_I} < -|y| - 1 < y.$$

Therefore, by (2), $y \in g_i[I]$. Hence there is an $x_0 \in I \subset (\alpha, \beta)$ with $g_i(x_0) = y$. Moreover, $I \subset \operatorname{int} \mathcal{C}(f) \subset \operatorname{int} \mathcal{C}(g_i)$ implies $x_0 \in \operatorname{int} \mathcal{C}(g_i)$.

Case 2. $\beta \in \operatorname{int} \mathcal{C}(f)$.

Put $\gamma = \max\big\{[\alpha, \beta] \setminus \operatorname{int} \mathcal{C}(f)\big\}$. Then $\gamma < \beta$ and $\gamma \notin \operatorname{int} \mathcal{C}(f)$. Observe that, if $r_I = \operatorname{dist} I, \mathbb{R} \setminus \operatorname{int} \mathcal{C}(f) \to 0^+$, then $-\frac{1}{r_I} \to -\infty$. So, there is an $\eta \in (\gamma, \beta)$ such that $g_i(\eta) < y$ and $[\eta, \beta] \subset \operatorname{int} \mathcal{C}(f) \subset \operatorname{int} \mathcal{C}(g_i)$. Thus, $g_i(x_0) = y$ for some $x_0 \in (\eta, \beta) \cap \operatorname{int} \mathcal{C}(g_i) \subset (\alpha, \beta) \cap \operatorname{int} \mathcal{C}(g_i)$.

It follows that $g_i \in \acute{\mathcal{S}}_{si}$, which completes the proof. $\square$

Theorem 3.3. *The smallest lattice containing all internally strong Świątkowski functions is the family $\mathcal{L}$ consisting of all functions f such that the set* $\operatorname{int}\mathcal{C}(f)$ *is dense in* $\mathbb{R}$.

Proof. Let $g_1, g_2 \in \mathcal{L}$. Then the sets $\operatorname{int}\mathcal{C}(g_1)$ and $\operatorname{int}\mathcal{C}(g_2)$ are dense in $\mathbb{R}$. Since $\operatorname{int}\mathcal{C}(g_1) \cap \operatorname{int}\mathcal{C}(g_2) \subset \operatorname{int}\mathcal{C}\big(\max\{g_1, g_2\}\big)$, the set $\operatorname{int}\mathcal{C}\big(\max\{g_1, g_2\}\big)$ is dense in $\mathbb{R}$, too. It proves that $\max\{g_1, g_2\} \in \mathcal{L}$. Moreover,

$$\min\{g_1, g_2\} = -\max\{-g_1, -g_2\} \in \mathcal{L}.$$

So, $\mathcal{L}$ is a lattice. Since $\mathcal{L}$ contains $\acute{\mathcal{S}}_{si}$, it contains the smallest lattice containing $\acute{\mathcal{S}}_{si}$ as well.

Now, we will show the opposite inclusion. Let $h \in \mathcal{L}$. If the function h is continuous, then h belongs to each lattice containing $\acute{\mathcal{S}}_{si}$. In the opposite case, define $f = -h$. Then clearly the set $\operatorname{int}\mathcal{C}(f)$ is dense in $\mathbb{R}$. Write $\operatorname{int}\mathcal{C}(f)$ as the union of a family $\mathcal{I}$ consisting of nonoverlapping compact intervals, such that for each $x \in \operatorname{int}\mathcal{C}(f)$, there are $I_1, I_2 \in \mathcal{I}$ with $x \in \operatorname{int}(I_1 \cup I_2)$.

Let the functions g_1 and g_2 be constructed as in the proof of Theorem 3.1. Then, $f = \max\{g_1, g_2\}$, $\operatorname{int}\mathcal{C}(f) \subset \operatorname{int}\mathcal{C}(g_1) \cap \operatorname{int}\mathcal{C}(g_2)$, and condition (2) holds. Observe that, if $r_I = \operatorname{dist}\big(I, \mathbb{R}\setminus\operatorname{int}\mathcal{C}(f)\big) \to 0$, then $-\frac{1}{r_I} \to -\infty$. So, for $i \in \{1, 2\}$ and each $x \notin \operatorname{int}\mathcal{C}(f)$,

$$\underline{\lim}\big(g_i, \mathcal{C}(g_i), x^+\big) = \underline{\lim}\big(g_i, \operatorname{int}\mathcal{C}(g_i), x^+\big) = -\infty$$

and

$$\underline{\lim}\big(g_i, \mathcal{C}(g_i), x^-\big) = \underline{\lim}\big(g_i, \operatorname{int}\mathcal{C}(g_i), x^-\big) = -\infty.$$

Since the functions $-g_1$ and $-g_2$ fulfill condition c) of Theorem 3.2, there are functions $g_{11}, g_{12}, g_{21}, g_{22} \in \acute{\mathcal{S}}_{si}$ such that $-g_1 = \max\{g_{11}, g_{12}\}$ and $-g_2 = \max\{g_{21}, g_{22}\}$. Hence,

$$h = -f = \min\{-g_1, -g_2\} = \min\big\{\max\{g_{11}, g_{12}\}, \max\{g_{21}, g_{22}\}\big\}.$$

Consequently the function h belongs to each lattice containing $\acute{\mathcal{S}}_{si}$. This completes the proof. □

Note that $\acute{\mathcal{S}}_{si} \subset \mathcal{DQ}_i \subset \mathcal{Q}_i \subset \mathcal{L}$. So, using Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *The smallest lattice containing all internally quasi-continuous functions and all Darboux internally quasi-continuous functions is the family $\mathcal{L}$ consisting of all functions f such that the set* $\operatorname{int}\mathcal{C}(f)$ *is dense in* $\mathbb{R}$.

By Theorems 3.1 and 3.2, maximal classes with respect to maximums for families $\mathcal{Q}_i$, $\mathcal{DQ}_i$, and $\acute{\mathcal{S}}_{si}$ are not closed with respect to maximums. It was already shown that $\mathcal{M}_{\max}(\mathcal{D}) = \mathcal{D}usc$ [2], $\mathcal{M}_{\max}(\mathcal{Q}) = \mathcal{C}$ [4], $\mathcal{M}_{\max}(\mathcal{DQ}) = \mathcal{DQ}usc$ [10], $\mathcal{M}_{\max}(\acute{\mathcal{S}}_s) = \mathcal{C}onst$ [12], and $\mathcal{M}_{\max}(\acute{\mathcal{S}}_{si}) = \mathcal{C}onst$ [8].

Now, we will examine the maximal classes with respect to maximums for families $\mathcal{Q}_i$ and $\mathcal{DQ}_i$. We start with the following obvious assertion.

Proposition 3.5. *If f is continuous and g is internally quasi-continuous, then* $\max\{f, g\}$ *is internally quasi-continuous.*

Before stating the next result, we have to prove a simple technical lemma.

Lemma 3.6. *Let f be a function, $x_0 \in \mathcal{Q}_i(f)$, and assume that the set U is dense in $\mathbb{R}$. There is a sequence $(x_n) \subset U$ such that $x_n \to x_0$ and $f(x_n) \to f(x_0)$.*

Proof. Since $x_0 \in \mathcal{Q}_i(f)$, there is a sequence $(t_n) \subset \operatorname{int} \mathcal{C}(f)$ with $t_n \to x_0$ and $f(t_n) \to f(x_0)$. Hence, for each $n \in \mathbb{N}$, there exists an open interval I_n such that $|I_n| < \frac{1}{n}$, $t_n \in I_n \subset \operatorname{int} \mathcal{C}(f)$, and $f(I_n) \subset \left(f(x_0) - \frac{1}{n}, f(x_0) + \frac{1}{n}\right)$. Since the set U is dense in $\mathbb{R}$, for each $n \in \mathbb{N}$, we can choose an $x_n \in U \cap I_n$. Clearly, the sequence (x_n) fulfills all our requirements. □

Proposition 3.7. *If f is upper semicontinuous internally quasi-continuous and g is Darboux internally quasi-continuous, then* $\max\{f, g\}$ *is internally quasi-continuous.*

Proof. Assume that f is upper semicontinuous internally quasi-continuous, g is Darboux internally quasi-continuous, and $h = \max\{f, g\}$. It is easy to see that $\operatorname{int} \mathcal{C}(f) \cap \operatorname{int} \mathcal{C}(g) \subset \operatorname{int} \mathcal{C}(h)$ and $\operatorname{int} \mathcal{C}(h)$ is dense in $\mathbb{R}$. We will show that h is internally quasi-continuous.

Fix an $x_0 \in \mathbb{R}$. Observe that $x_0 \in \mathcal{Q}_i(f) \cap \mathcal{Q}_i(g)$. If $\overline{\lim}(g, x) \geq h(x_0)$, then, since $g \in \mathcal{D}$ and $g(x_0) \leq h(x_0)$, there exists a sequence (t_n) such that $t_n \to x_0$ and $g(t_n) \to h(x_0)$. Observe that $t_n \in \mathcal{Q}_i(g)$ for each $n \in \mathbb{N}$. So, by Lemma 3.6, for each $n \in \mathbb{N}$, there is a $x_n \in \operatorname{int} \mathcal{C}(h) \cap \left(t_n - \frac{1}{n}, t_n + \frac{1}{n}\right)$ such that

$$g(x_n) \in \left(g(t_n) - \frac{1}{n}, g(t_n) + \frac{1}{n}\right).$$

Hence, $x_n \to x_0$ and $g(x_n) \to h(x_0)$. Since the function f is upper semicontinuous and $h = \max\{f, g\}$, we have $h(x_n) \to h(x_0)$.

If $\overline{\lim}(g, x) < h(x_0)$, then $h(x_0) = f(x_0)$. (Recall that $g \in \mathcal{D}$.) Using Lemma 3.6, we can choose a sequence $(x_n) \subset \operatorname{int} \mathcal{C}(h)$ such that $x_n \to x_0$ and $f(x_n) \to f(x_0) = h(x_0)$. So, clearly $h(x_n) \to h(x_0)$. Consequently, $x_0 \in \mathcal{Q}_i(h)$, which completes the proof. □

Theorem 3.8. $\mathcal{M}_{\max}(\mathcal{Q}_i) = \mathcal{C}$.

Proof. The inclusion $\mathcal{C} \subset \mathcal{M}_{\max}(\mathcal{Q}_i)$ follows by Proposition 3.5. In [4, Theorem 2], Z. Grande and L. Sołtysik, for a fixed discontinuous function, constructed quasi-continuous functions such that their maximum is not quasi-continuous. In fact, such the functions are also internally quasi-continuous. So, since $\mathcal{Q}_i \subset \mathcal{Q}$, we clearly have $\mathcal{M}_{\max}(\mathcal{Q}_i) \subset \mathcal{C}$. □

Theorem 3.9. $\mathcal{M}_{\max}(\mathcal{DQ}_i) = \mathcal{DQ}_i usc.$

P r o o f. First, we will show that $\mathcal{DQ}_i usc \subset \mathcal{M}_{\max}(\mathcal{DQ}_i)$. Assume that $f \in \mathcal{DQ}_i usc$ and $g \in \mathcal{DQ}_i$. By [2, Theorem 1], $\max\{f,g\} \in \mathcal{D}$ and by Proposition 3.7, $\max\{f,g\} \in \mathcal{Q}_i$. So, $f \in \mathcal{M}_{\max}(\mathcal{DQ}_i)$.

The opposite inclusion follows by [10, Lemma 7]. $\square$

REFERENCES

[1] BORSÍK, J.: *Maxima and minima of simply continuous and quasicontinuous functions*, Math. Slovaca **46** (1996), 261–268.
[2] FARKOVÁ, J.: *About the naximum and the minimum of Darboux functions*, Mat. Čas. **21** (1971), 110–116.
[3] GRANDE, Z.—NATKANIEC, T.: *Lattices generated by $\mathcal{T}$-quasi-continuous functions*, Bull. Polish Acad. Sci. Math. **34** (1986), 525–530.
[4] GRANDE, Z.—SOŁTYSIK, L.: *Some remarks on quasi-continuous real functions*, Problemy Mat. **10** (1988), 79–87.
[5] KEMPISTY, S.: *Sur les fonctions quasicontinues*, Fund. Math. **19** (1932), 184–197.
[6] MALISZEWSKI, A.: *On the limits of strong Świątkowski function*, Zeszyty Nauk. Politech. Łódz., Mat. **27** (1995), 87–93.
[7] MALISZEWSKI, A.: *Maximums of Darboux quasi-continuous functions*, Math. Slovaca **49** (1999), 381–386.
[8] MARCINIAK, M.—SZCZUKA, P.: *On internally strong Świątkowski functions* (preprint).
[9] NATKANIEC, T.: *On the maximum and the minimum of quasi-continuous functions*, Math. Slovaca **42** (1992), 103–110.
[10] NATKANIEC, T.: *On quasi-continuous functions having Darboux property*, Math. Pannon. **3** (1992), 81–96.
[11] SZCZUKA, P.: *Maximums of strong Świątkowski functions*, Math. Slovaca **52** (2002), 541–548.
[12] SZCZUKA, P.: *Maximal classes for the family of strong Świątkowski functions*, Real Anal. Exchange **28** (2002/03), 429–437.

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