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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/v10127-012-0031-y
Tatra Mt. Math. Publ. **52** (2012), 101–113

A NOTE ON DIRECTLY ORDERED SUBSPACES OF $\mathbb{R}^n$

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ABSTRACT. A comprehensive method of determining if a subspace of usually ordered space $\mathbb{R}^n$ is directly-ordered is presented here. Also, it is proven in an elementary way that if a directly-ordered vector space has a positive cone generated by its extreme vectors then the Riesz Decomposition Property implies the lattice conditions. In particular, every directly-ordered subspace of $\mathbb{R}^n$ is a lattice-subspace if and only if it satisfies the Riesz Decomposition Property.

1. Introduction

In this note we deal with the ordered vector spaces over $\mathbb{R}$. Our major reference for all necessary definitions and facts in this area is [2]. Let us recall that V is called an *ordered vector space* if the real vector space V is equipped with a compatible partial order $\le$, i.e., if for any vectors u, v and w from V, if $u \le v$, then $u + w \le v + w$ and for any positive $\alpha \in \mathbb{R}$, $\alpha u \le \alpha v$. In case the partial order is a lattice, V is called a *vector lattice* or a *Riesz space*. The set

$$V^+ = \{u \in V : u \ge 0\}$$

is called a *positive cone* of V. It satisfies the three axioms of a cone:

(i) $K + K \subseteq K$,

(ii) $\mathbb{R}^+ K \subseteq K$,

(iii) $K \cap -K = \{0\}$.

Moreover, any subset of V satisfying the three above conditions is a positive cone of a partial order on V.

An ordered vector space is said to be *directly ordered* if for every two vectors $u, v \in V$ there exist $p, q \in V$ such that $p \le u,\ v \le q$. This condition is equivalent to saying that $V = K - K$, i.e., the positive cone K is *generating*. Every vector lattice is directly ordered.

2010 Mathematics Subject Classification: 06F25.
Keywords: ordered vector space, direct order, lattice-subspace, Riesz Decomposition.

Throughout the paper, by $\mathbb{R}^n$ we will understand the coordinate-wise ordered vector lattice $\bigoplus_{i=1}^n \mathbb{R}$. By a *subspace* V of $\mathbb{R}^n$ we understand any subspace ordered by the order of $\mathbb{R}^n$.

A vector subspace X of a vector lattice V is called a *lattice-subspace* if X equipped with the ordering from V is a vector lattice on its own, i.e., if the least upper bound of any two elements from X exists in X (and automatically does the greatest lower bound of the elements). In their paper [1] the authors studied the lattice-subspaces of $\mathbb{R}^n$ and gave equivalent conditions for a subspace to be a lattice-subspace. Moreover, their method allows one to quickly determine if X is a lattice-subspace based on the vectors generating X. In Section 1 we adopt the spirit of their approach to supply a comprehensive and fast way to determine if a given subspace of $\mathbb{R}^n$ is directly ordered.

Another property enjoyed by the class of vector lattices is the following Riesz Decomposition Property. If $0 \leq u, v, w \in V$ and $w \leq u + v$, then there exist vectors $u', v' \in V$ such that $0 \leq u' \leq u$, $0 \leq v' \leq v$ and $w = u' + v'$. This property is of fundamental importance in ordered vector spaces, partially ordered groups and related areas. Not all ordered vector spaces enjoy the property and not all those that enjoy it are necessarily vector-lattices. For a number of important examples on this topic, see [2, Chapter 1.8]. It is known, however, that every directly ordered finite-dimensional space with a closed cone and satisfying the Riesz Decomposition Property is a vector-lattice. The proof of this fact, even in the finite-dimensional case, requires certain applications of the duality theorems and the Riesz-Kantorowich theorem. In Section 3 we will give a very elementary and short proof that in the case the positive cone of an ordered vector space (not necessarily finite-dimensional) is generated by a set of its extreme vectors then satisfying the Riesz Decomposition Property is equivalent to being a vector lattice. In particular it will immediately follow that every directly ordered subspace of $\mathbb{R}^n$ satisfying the Riesz Decomposition Property is a lattice-subspace.

2. Directly ordered subspaces of $\mathbb{R}^m$

In this section we adopt the motivation similar to that of the algorithm developed in [1] in order to determine whether the subspace V generated by an arbitrary collection of linearly independent vectors has a generating cone. Recall that for the subspace V, $V^+ = V \cap (\mathbb{R}^m)^+$. The subspace V is contained in a minimal coordinate subspace, i.e., a subspace determined by the zero coordinates from a specific set (perhaps empty). It is clear then that $v \in V^+$ is an interior point of V^+ (in the relative topology of V) if and only if v has all the remaining coordinates positive. Below the term *strictly positive vector* $v \in V$ is understood in this sense.

We need the following well-known facts that can be found, e.g., in [2, Lemma 3.2], and [3, Corollary 1 to Theorem 2.9], respectively.

Theorem 2.1. *A cone in $\mathbb{R}^m$ is generating if and only if it has an interior point.*

Theorem 2.2. *For a subspace V of a finite dimensional vector space, exactly one of the following mutually exclusive possibilities holds:*

(1) *V contains a strictly positive vector,*

(2) *$V^\perp$ contains a nonnegative vector, where $V^\perp$ is the orthogonal complement of V.*

Corollary 2.3. A subspace V of $\mathbb{R}^m$ has a generating cone if and only if $V^\perp$ contains no nonnegative vector.

P r o o f. This follows immediately from Theorems 2.1 and 2.2. □

Recall the notation used in [1]. Given a set of n linearly independent vectors $x_1, \ldots, x_n$ in $\mathbb{R}^m$, let $V = \langle x_1, \ldots, x_n \rangle$ be the n-dimensional vector subspace they generate, where $1 \leq n < m$.

As in [1], for $x \in \mathbb{R}^m$, let $x(i)$ denote the ith component of x. Then the matrix whose rows are formed by the x_i can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1(1) & x_1(2) & \cdots & x_1(m) \\ x_2(1) & x_2(2) & \cdots & x_2(m) \\ \vdots & \vdots & \vdots & \vdots \\ x_n(1) & x_n(2) & \cdots & x_n(m) \end{pmatrix}.$$

From here we form the following m vectors of $\mathbb{R}^n$:

$$y_1 = \begin{pmatrix} x_1(1) \\ x_2(1) \\ \vdots \\ x_n(1) \end{pmatrix}, \quad y_2 = \begin{pmatrix} x_1(2) \\ x_2(2) \\ \vdots \\ x_n(2) \end{pmatrix}, \ldots, \quad y_m = \begin{pmatrix} x_1(m) \\ x_2(m) \\ \vdots \\ x_n(m) \end{pmatrix}.$$

Notice that the matrix with rows x_i has rank n, so that among the vectors y_j, $j = 1, \ldots, m$ there exist n linearly independent vectors. If for some j we have $y_j = \mathbf{0}$, then we exclude it from consideration.

We introduce the following definition in analogy to the definition of the *fundamental set of indices* from [1].

Definition 2.4. A set of n indices $\{m_1, \ldots, m_n\}$ is called a *negative fundamental* set of indices for the vectors $x_1, \ldots, x_n \in \mathbb{R}^m$ whenever

(1) the n vectors $y_{m_1}, \dots, y_{m_n}$ are linearly independent; and
(2) for at least one $j \notin \{m_1, \dots, m_n\}$, all the coefficients in the expansion

$$y_j = \sum_{r=1}^{n} \alpha_{j,r} y_{m_r}$$

are non-positive.

As discussed in [3] that a solution ξ to the equation $\sum_{i=1}^{n} \xi_i x_i = b$ is called *basic* if for the set $L = \{i : \xi_i > 0\}$, the set of vectors $\{y_i : i \in L\}$ is linearly independent. For the proof of our main result in this section, we will need the following lemmas.

Lemma 2.5 ([3], Theorem 2.11)**.** *If the equation*

$$\sum_{i=1}^{n} \xi_i x_i = b$$

has a nonnegative solution, then it has a basic nonnegative solution.

Lemma 2.6. *If $\alpha_1 y_1 + \dots \alpha_m y_m = 0$, with $\alpha_i \geq 0$ and not all 0, and all $y_k \neq 0$, then there exists a subset $L \subseteq \{1, \dots, m\}$ such that the set S of vectors $\{y_i : i \in L\}$ is linearly independent and there is $j \notin L$ such that y_j is a negative linear combination of the vectors from S.*

Proof. Without loss of generality, we can assume that $\alpha_1 \neq 0$ and consider the equation

$$\alpha_2 y_2 + \dots + \alpha_m y_m = -\alpha_1 y_1.$$

By assumption, there is a nonnegative solution to this equation and therefore, by Lemma 2.5, there exists a basic nonnegative solution with the set L of indices as above. But then

$$y_1 = -\frac{1}{\alpha_1} \sum_{i \in L} \xi_i y_i.$$

□

Now, we come to the main result of this section.

Theorem 2.7. *The vector subspace V of $\mathbb{R}^m$ is directly ordered (has a generating cone) if and only if the vectors $x_1, \dots, x_n$ do not admit a negative fundamental set of indices $\{m_1, \dots, m_n\}$.*

Proof. First, assume that there exists a negative fundamental set of indices, $\{m_1, \dots, m_n\}$. Without loss of generality, we may assume that $\{m_1, \dots, m_n\} = \{1, \dots, n\} = I$. Then there exists $j \notin I$ such that

$$y_j = \sum_{i=1}^{n} \alpha_i y_i,$$

where $\alpha_i \leq 0$ for each i. This is equivalent to

$$y_j - \sum_{i=1}^{n} \alpha_i y_i = 0$$

which implies the existence of the vector

$$v = \begin{pmatrix} -\alpha_1 \\ -\alpha_2 \\ \vdots \\ -\alpha_n \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

in $V^\perp$, where 1 occurs at the jth component. But $v \geq 0$, so by Corollary 2.3, V^+ is not generating.

For the converse, assume that V^+ is not generating. By Corollary 2.3, there exists $v \in V^\perp$ such that $v \geq 0$. This means, v is of the form

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

with $\alpha_k \geq 0$ for each k. These coefficients α_k and the column vectors y_k satisfy the equation

$$\sum_{k=1}^{m} \alpha_k y_k = 0.$$

Hence by Lemma 2.6 there exists a basic set of indices $\{m_1, \ldots, m_n\} = I$ such that for some $j \notin I$ we have

$$y_j = \sum_{i=1}^{n} \beta_i y_i,$$

where $\beta_i \leq 0$ for each i. The set $\{m_1, \ldots, m_n\}$ is the desired set of negative fundamental indices, and this completes the proof. $\square$

Next, we illustrate Theorem 2.7 with three examples.

Example 2.1. Let

$$x_1 = \begin{pmatrix} -1 \\ 0 \\ 4 \\ 1 \\ -2 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 3 \\ 2 \end{pmatrix}.$$

A quick calculation shows that these vectors are linearly independent. Following the method outlined above, we form the five vectors

$$y_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix},$$

$$y_4 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad y_5 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

Next, we check that there does not exist a negative fundamental set of indices. There are ten possible sets, which give rise to the following equations.

$I_1 = \{1, 2, 3\}$:

$$y_4 = \frac{11}{3}y_1 - \frac{25}{6}y_2 + \frac{7}{6}y_3,$$

$$y_5 = \frac{8}{3}y_1 - \frac{13}{6}y_2 + \frac{1}{6}y_3.$$

$I_2 = \{1, 2, 4\}$:

$$y_3 = -\frac{22}{7}y_1 + \frac{25}{7}y_2 + \frac{6}{7}y_4,$$

$$y_5 = \frac{15}{7}y_1 - \frac{11}{7}y_2 + \frac{1}{7}y_4.$$

$I_3 = \{1, 2, 5\}$:

$$y_3 = -16y_1 + 13y_2 + 6y_5,$$

$$y_4 = -15y_1 + 11y_2 + 7y_5.$$

$I_4 = \{1, 3, 4\}$:

$$y_2 = \frac{22}{25}y_1 + \frac{7}{25}y_3 - \frac{6}{25}y_4,$$

$$y_5 = \frac{19}{25}y_1 - \frac{11}{25}y_3 + \frac{13}{25}y_4.$$

$I_5 = \{1, 3, 5\}$:

$$y_2 = \frac{16}{13}y_1 + \frac{1}{13}y_3 - \frac{6}{13}y_5,$$

$$y_4 = -\frac{19}{13}y_1 + \frac{11}{13}y_3 + \frac{25}{13}y_5,$$

$I_6 = \{1, 4, 5\}$:

$$y_2 = \frac{15}{11}y_1 + \frac{1}{11}y_4 - \frac{7}{11}y_5,$$

$$y_3 = \frac{19}{11}y_1 + \frac{13}{11}y_4 - \frac{25}{11}y_5.$$

$I_7 = \{2, 3, 4\}$:

$$y_1 = \frac{25}{22}y_2 - \frac{7}{22}y_3 + \frac{3}{11}y_4,$$

$$y_5 = \frac{19}{22}y_2 - \frac{15}{22}y_3 + \frac{8}{11}y_4.$$

$I_8 = \{2, 3, 5\}$:

$$y_1 = \frac{13}{16}y_2 - \frac{1}{16}y_3 + \frac{3}{8}y_5,$$

$$y_4 = -\frac{19}{16}y_2 + \frac{15}{16}y_3 + \frac{11}{8}y_5.$$

$I_9 = \{2, 4, 5\}$:

$$y_1 = \frac{11}{15}y_2 - \frac{1}{15}y_4 + \frac{7}{15}y_5,$$

$$y_3 = \frac{19}{15}y_2 + \frac{16}{15}y_4 - \frac{22}{15}y_5,$$

$I_{10} = \{3, 4, 5\}$:

$$y_1 = \frac{19}{11}y_3 - \frac{13}{19}y_4 + \frac{25}{19}y_5,$$

$$y_2 = \frac{15}{19}y_3 - \frac{16}{19}y_4 + \frac{22}{19}y_5.$$

Inspection shows that not one of the I_i, $i = 1, \ldots, 10$ is a negative fundamental set. Therefore V^+ is generating. In fact, notice that

$$x_1 + x_2 + x_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} \in V.$$

Example 2.2. Let

$$x_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -4 \\ 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ -3 \\ -2 \end{pmatrix}.$$

These vectors are linearly independent. We form the vectors

$$y_1 = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

$$y_4 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad y_5 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad y_6 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

Next, we investigate the sets of indices:

$I_1 = \{1, 2, 3\}$:

$$y_4 = -\frac{7}{17}y_1 + \frac{15}{17}y_2 - \frac{13}{176}y_3,$$

$$y_5 = -2y_2 + y_3,$$

$$y_6 = \frac{3}{17}y_1 - \frac{21}{17}y_2 + \frac{8}{17}y_3.$$

$I_2 = \{1, 2, 4\}$:

$$y_3 = -\frac{7}{13}y_1 + \frac{15}{13}y_2 - \frac{17}{13}y_4,$$

$$y_5 = -\frac{7}{13}y_1 - \frac{11}{13}y_2 - \frac{17}{13}y_4,$$

$$y_6 = -\frac{1}{13}y_1 - \frac{9}{13}y_2 - \frac{8}{13}y_4.$$

Hence $I_2 = \{1, 2, 4\}$ is a negative set of fundamental indices. Furthermore, from the equations of y_5 and y_6 in terms of the set I_2, it is clear that the vectors

$$\begin{pmatrix} \frac{7}{13} \\ \frac{11}{13} \\ 0 \\ \frac{17}{13} \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{13} \\ \frac{9}{13} \\ 0 \\ \frac{8}{13} \\ 0 \\ 1 \end{pmatrix}$$

lie in the orthogonal complement of V.

EXAMPLE 2.3. In this example we present a case in which we are able to compress the matrix formed by the x_i. Here, let

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 3 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

These vectors form the matrix

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

which can be compressed by removing rows 2 and 5 to form the matrix

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Now, the relevant vectors formed from the transpose of this matrix are

$$y_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad y_3 = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix},$$

$$y_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad y_5 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

There are ten possibilities of negative fundamental indices:

$I_1 = \{1, 2, 3\}$:

$$y_4 = -\frac{5}{11}y_1 + \frac{12}{11}y_2 + \frac{8}{11}y_3,$$

$$y_5 = y_1 + y_3.$$

$I_2 = \{1, 2, 4\}$:

$$y_3 = \frac{5}{8}y_1 - \frac{3}{2}y_2 + \frac{11}{8}y_4,$$
$$y_5 = \frac{13}{8}y_1 - \frac{3}{2}y_2 + \frac{11}{8}y_4.$$

$I_3 = \{1, 2, 5\}$:

$$y_3 = -y_1 + 6y_5,$$
$$y_4 = -\frac{13}{11}y_1 + \frac{12}{11}y_2 + \frac{8}{11}y_5.$$

$I_4 = \{1, 3, 4\}$:

$$y_2 = \frac{5}{12}y_1 - \frac{2}{3}y_3 + \frac{11}{12}y_4,$$
$$y_5 = y_1 + y_3.$$

$I_5 = \{1, 3, 5\}$:

This set is not linearly independent.

$I_6 = \{1, 4, 5\}$:

$$y_2 = \frac{13}{12}y_1 + \frac{11}{12}y_4 - \frac{2}{3}y_5,$$
$$y_3 = -y_1 + y_5.$$

$I_7 = \{2, 3, 4\}$:

$$y_1 = \frac{12}{5}y_2 + \frac{8}{5}y_3 - \frac{11}{5}y_4,$$
$$y_5 = \frac{12}{5}y_2 + \frac{13}{5}y_3 - \frac{11}{5}y_4.$$

$I_8 = \{2, 3, 5\}$:

$$y_1 = -y_3 + y_5,$$
$$y_4 = \frac{12}{11}y_2 + \frac{13}{11}y_3 - \frac{5}{11}y_5.$$

$I_9 = \{2, 4, 5\}$:

$$y_1 = \frac{12}{13}y_2 - \frac{11}{13}y_4 + \frac{8}{13}y_5,$$
$$y_3 = -\frac{12}{13}y_2 + \frac{11}{13}y_4 + \frac{5}{13}y_5.$$

$I_{10} = \{3, 4, 5\}$:

$$y_1 = -y_3 + y_5,$$

$$y_2 = -\frac{13}{12}y_3 + \frac{11}{12}y_4 + \frac{5}{12}y_5.$$

Thus, no negative set of fundamental indices exists, so V^+ is generating. Notice that

$$x_1 + x_2 + x_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 3 \\ 0 \\ 2 \\ 3 \end{pmatrix} \in V.$$

This vector is an interior point of the cone V^+.

3. Directly ordered subspaces with the Riesz decomposition property

It is easy to see that every directly ordered subspace of $\mathbb{R}^3$ satisfies the Riesz Decomposition Property. However, not all directly ordered subspaces of $\mathbb{R}^n$ have the Riesz Decomposition Property if $n > 3$.

EXAMPLE 3.1. Let $n = 4$ and V be given by the equation

$$x + y - z - t = 0.$$

The subspace V is directly ordered because

$$\begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix} \in V$$

and for the appropriate choice of a it bounds from above any two given vectors from V. We have, however,

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

and all these vectors are in V^+. If we had

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = u + v \quad \text{with} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq u \leq \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq v \leq \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

then clearly $u_3 = v_3 = 0$. But then

$$u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad v = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},$$

which is a contradiction. Thus V does not satisfy the Riesz Decomposition Property.

Let K be a positive cone of an ordered space V. A vector $e \in K$ is called an *extremal vector* if $0 \leq x \leq e$ implies $x = \lambda e$ for some $\lambda \geq 0$ ([2, Chapter 1.6]). Let us denote by $\mathcal{E}$ the set of all extremal vectors of K. We will say that the cone K is *generated by its extremal vectors* if

$$K = \left\{ \sum_{i=1}^{n} \alpha_i \, e_i : n \in \mathbb{N}, e_i \in \mathcal{E}, \ \alpha_i \geq 0, \ i = 1, \ldots, n \right\}.$$

We can state and prove the main theorem.

Theorem 3.1. *If V has a generating cone K which is generated by its extreme vectors and it satisfies the Riesz Decomposition Property, then V is a vector-lattice.*

Proof. We will show that $\mathcal{E}$ is a Hamel basis for V. We can assume that no two distinct vectors from $\mathcal{E}$ are linearly dependent. Since K is generating, it is enough to show that the set $\mathcal{E}$ is linearly independent. Suppose on the contrary that for some $n \in \mathbb{N}$ there exist $e, e_1, \ldots, e_n \in \mathcal{E}$ such that $e = \sum_{i=1}^{n} \alpha_i e_i$ and let n be a minimal number such that e is a linear combination of the vectors $e_1, \ldots, e_n$. Since e is extremal, there must be a negative α_i for some $i = 1, \ldots, n$. Otherwise $a_i > 0$ for $i = 1, \ldots, n$ and we would have $\alpha_1 e_1 \leq \sum_{i=1}^{n} \alpha_i e_i = e$, but then for some λ, $\alpha_1 e_1 = \lambda e$, which is not true. We can assume then that $\alpha_i > 0$ for $i = 1, \ldots, p$ and $\alpha_i < 0$ for $i = p + 1, \ldots, n$ for some $1 < p < n$. So

$$e = \sum_{i=1}^{p} \alpha_i e_i - \sum_{i=p+1}^{n} (-\alpha_i) e_i < \sum_{i=1}^{p} \alpha_i e_i.$$

By the Riesz Decomposition Property

$$e = \sum_{i=1}^{p} f_i, \qquad \text{where} \quad 0 \leq f_i \leq \alpha_i e_i.$$

But since each e_i is an extreme vector, $f_i = \beta_i e_i$, $\beta_i \geq 0$, $i = 1, \dots, p$. So

$$e = \sum_{i=1}^{p} \beta_i e_i.$$

This contradicts the minimality of n. Therefore the set $\mathcal{E}$ is a Hamel basis of V and the positive (or Yudin) basis of K. It is well-known (and straightforward) (see, e.g., [2, Theorem 3.17]) that the resulting ordering makes V a vector-lattice. □

We immediately obtain the characterization of the subspaces of $\mathbb{R}^n$.

Corollary 3.2. A directly ordered subspace of $\mathbb{R}^n$ is a lattice-subspace if and only if it satisfies the Riesz Decomposition Property.

Proof. If V is the subspace of $\mathbb{R}^n$, then its positive cone $V \cap (\mathbb{R}^n)^+$ is a polyhedral cone (for an elementary proof see, e.g., [3, Corollary to Theorem 2.12]) and as such it is generated by its extremal vectors. So by the Theorem 3.1, V is a lattice-subspace. □

REFERENCES

[1] ABRAMOVICH, Y. A.—ALIPRANTIS, C. D.—POLYRAKIS, I. A.: *Lattice-subspaces and positive projections*, Math. Proc. R. Ir. Acad., Sect. A **94** (1994), 237–253.

[2] ALIPRANTIS, C.—TOURKY, R.: *Cones and Duality*, in: Grad. Stud. Math., Vol. 84, Amer. Math. Soc., Providence, RI, 2007.

[3] GALE, D.: *The Theory of Linear Economic Models*. McGraw-Hill, New York, 1960.

Received November 3, 2011

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