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POINTS OF GENERALIZED CONTINUITIES

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ABSTRACT. We characterize points of generalized continuity of functions defined on generalized topological spaces with values in metric spaces.

In the literature, the concept of a topological space is generalized by replacing open sets by other kinds of subsets. In many cases, generalized open sets have the property that the arbitrary unions of themes belongs to the same class of sets. This property is postulated in the concept of generalized topology by Á. Császár in [5]. Further, he introduces the notion of generalized continuous function between generalized topological spaces [7].

We recall some notions. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. We call a class $\mathfrak{g} \subset \mathcal{P}(X)$ a generalized topology [5] (briefly GT), if $\emptyset \in \mathfrak{g}$ and the arbitrary union of elements of $\mathfrak{g}$ belongs to $\mathfrak{g}$. A GT $\mathfrak{g}$ is strong if $X \in \mathfrak{g}$. A set X with a GT $\mathfrak{g}$ is called a generalized topological space (briefly, GTS) and is denoted by $(X, \mathfrak{g})$. By [10], a GT $\mathfrak{g}$ is a quasi-topology, if $A \cap B \in \mathfrak{g}$ for $A, B \in \mathfrak{g}$. For $x \in X$ we denote $\mathfrak{g}(x) = \{A \in \mathfrak{g} : x \in A\}$.

For a GTS $(X, \mathfrak{g})$, the elements of $\mathfrak{g}$ are called $\mathfrak{g}$-open and their complements are $\mathfrak{g}$-closed. For $A \subset X$, we denote by $i_{\mathfrak{g}}(A)$ the union of all $\mathfrak{g}$-open sets contained in A and by $c_{\mathfrak{g}}(A)$ the intersection of all $\mathfrak{g}$-closed sets containing A. A set A is said to be $\mathfrak{g}$-semi-open ($\mathfrak{g}$-pre-open, $\mathfrak{g}$-α-open, $\mathfrak{g}$-β-open), if $A \subset c_{\mathfrak{g}}\big(i_{\mathfrak{g}}(A)\big)$ $\big(A \subset i_{\mathfrak{g}}\big(c_{\mathfrak{g}}(A)\big), A \subset i_{\mathfrak{g}}\big(c_{\mathfrak{g}}(i_{\mathfrak{g}}(A))\big), A \subset c_{\mathfrak{g}}\big(i_{\mathfrak{g}}(c_{\mathfrak{g}}(A))\big)\big)$, respectively [6]. We denote the class of all $\mathfrak{g}$-semi-open ($\mathfrak{g}$-pre-open, $\mathfrak{g}$-α-open, $\mathfrak{g}$-β-open) sets by $\sigma(\mathfrak{g})$ $(\pi(\mathfrak{g}), \alpha(\mathfrak{g}), \beta(\mathfrak{g}))$, respectively. If $(X, \mathfrak{g})$ is a topological space, we obtain the families of semi-open sets $SO(X)$, pre-open sets $PO(X)$, α-open sets $\alpha(X)$ and β-open sets $\beta(X)$. $\sigma(\mathfrak{g})$, $\pi(\mathfrak{g})$, $\alpha(\mathfrak{g})$ and $\beta(\mathfrak{g})$ are GT's [6].

By [7], if $(X, \mathfrak{g})$ and $(Y, \mathfrak{h})$ are GTS's, then a mapping $f\colon X \to Y$ is called $(\mathfrak{g}, \mathfrak{h})$-continuous, if $f^{-1}(V) \in \mathfrak{g}$ for each $V \in \mathfrak{h}$. If $(Y, \mathfrak{h})$ is a topological space, for $\mathfrak{g} = SO(X)$ we have the family of quasicontinuous functions, for $\mathfrak{g} = PO(X)$ we have pre-continuous functions, for $\mathfrak{g} = \alpha(X)$ α-continuous functions and for $\mathfrak{g} = \beta(X)$ we obtain β-continuous functions.

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A function $f\colon X \to Y$ is $(\mathfrak{g},\mathfrak{h})$-continuous at $x \in X$ if for each $V \in \mathfrak{h}\big(f(x)\big)$ there is $U \in \mathfrak{g}(x)$ such that $f(U) \subset V$. By [7], a function f is $(\mathfrak{g},\mathfrak{h})$-continuous if it is such at each point. Denote by $C_{\mathfrak{g},\mathfrak{h}}(f)$ the family of all $(\mathfrak{g},\mathfrak{h})$-continuity points of f. The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ stand for the set of all real, rational and natural numbers, respectively.

In this paper, we will investigate the set $C_{\mathfrak{g},\mathfrak{h}}(f)$. In generally, this set can be arbitrary. However, if (Y,d) is a metric space then we can characterize this set. From now, we will assume that (Y,d) is a metric space. For a set $Z \subset Y$, let $\operatorname{diam}(Z) = \sup\big\{d(u,v) : u,v \in\big\}$ be the diameter of Z. We will use the notion $\mathfrak{g}$-continuity for $(\mathfrak{g},d)$-continuity and $C_{\mathfrak{g}}(f)$ for $C_{\mathfrak{g},d}(f)$.

Definition 1. Let $\mathfrak{g}$ be a GT on X, let (Y,d) be a metric space and let $f\colon X \to Y$ be a function. The function $k_f^{\mathfrak{g}}\colon X \to [0,\infty]$ defined by

$$k_f^{\mathfrak{g}}(x) = \inf\big\{\operatorname{diam} f(A) : A \in \mathfrak{g}(x)\big\} \qquad \text{for} \quad x \in \bigcup \mathfrak{g}$$

and $k_f^{\mathfrak{g}}(x) = \infty$ for $x \in X \setminus \bigcup \mathfrak{g}$, is called the $\mathfrak{g}$-oscillation of f.

Proposition 1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two GT's on X. Then $\mathfrak{g}$ is coarser than $\mathfrak{h}$ (i.e., $\mathfrak{g} \subset \mathfrak{h}$) if and only if $k_f^{\mathfrak{g}} \ge k_f^{\mathfrak{h}}$ for each function $f\colon X \to \mathbb{R}$.*

Proof. Let $\mathfrak{g} \subset \mathfrak{h}$. Let $f\colon X \to \mathbb{R}$, $x \in X$ and $k_f^{\mathfrak{g}}(x) < a$. Then there is $H \in \mathfrak{g}(x)$ with $\operatorname{diam} f(H) < a$. But $H \in \mathfrak{h}(x)$ and so $k_f^{\mathfrak{h}}(x) < a$, therefore $k_f^{\mathfrak{h}}(x) \le k_f^{\mathfrak{g}}(x)$.

Conversely, let $H \in \mathfrak{g}$. Let f be the characteristic function of the set H. For each $x \in H$ we have $H \in \mathfrak{g}(x)$ and $\operatorname{diam} f(H) = 0$, hence $k_f^{\mathfrak{g}}(x) = 0$. We have $k_f^{\mathfrak{h}}(x) \le k_f^{\mathfrak{g}}(x) = 0 < 1/2$. Hence for each $x \in H$ there is $H_x \in \mathfrak{h}(x) \subset \mathfrak{h}$ with $\operatorname{diam} f(H_x) < 1/2$. This yields $f(t) = 1$ for each $t \in H_x$ and $H_x \subset H$. We have $H = \bigcup_{x \in H} H_x \in \mathfrak{h}$ and hence $\mathfrak{g} \subset \mathfrak{h}$. □

Proposition 2. *A function $f\colon X \to Y$ is $\mathfrak{g}$-continuous at x if and only if $k_f^{\mathfrak{g}}(x) = 0$.*

Proof. Let f be $\mathfrak{g}$-continuous at x and $\varepsilon > 0$. Then there is $A \in \mathfrak{g}(x)$ such that $d\big(f(y), f(x)\big) < \varepsilon$ for each $y \in A$. This yields $\operatorname{diam} f(A) \le 2\varepsilon$ and $k_f^{\mathfrak{g}}(x) \le 2\varepsilon$. This is true for each $\varepsilon > 0$ and so $k_f^{\mathfrak{g}}(x) = 0$.

Conversely, let $k_f^{\mathfrak{g}}(x) = 0$ and $\varepsilon > 0$. Then $k_f^{\mathfrak{g}}(x) < \varepsilon$ and there is $A \in \mathfrak{g}(x)$ such that $\operatorname{diam} f(A) < \varepsilon$. Hence for each $y \in A$ we have $d\big(f(y), f(x)\big) < \varepsilon$ and f is $\mathfrak{g}$-continuous at x. □

Definition 2. A function $f\colon X \to [-\infty,\infty]$ is said to be upper $\mathfrak{g}$-continuous [lower $\mathfrak{g}$-continuous] at x if for each $a > f(x)$ [$a < f(x)$] there is a set $A \in \mathfrak{g}(x)$ such that $f(y) < a$ [$f(y) > a$] for each $y \in A$. A function is upper [lower] $\mathfrak{g}$-continuous if it is such at each point. Denote by $uC_{\mathfrak{g}}(f)$ [$lC_{\mathfrak{g}}(f)$] the set of all upper [lower] $\mathfrak{g}$-continuity points of f.

Evidently, $\mathfrak{g}$-continuous function $f\colon X \to \mathbb{R}$ (at x) is upper and lower $\mathfrak{g}$-continuous (at x), however, upper and lower $\mathfrak{g}$-continuous function need not be $\mathfrak{g}$-continuous (e.g., $X = \mathbb{R}$, $\mathfrak{g}$ is the family of all semi-open sets in X, $f(x) = 0$ for $x < 0$, $f(0) = 1$ and $f(x) = 2$ for $x > 0$).

Proposition 3. *A function $f\colon X \to [-\infty, \infty]$ is upper [lower] $\mathfrak{g}$-continuous if and only if $f^{-1}((-\infty, a)) \in \mathfrak{g}$ $[f^{-1}((a, \infty)) \in \mathfrak{g}]$ for each $a \in \mathbb{R}$.*

Proof. Let f be upper $\mathfrak{g}$-continuous and $a \in \mathbb{R}$. For each x with $f(x) < a$ there is $A_x \in \mathfrak{g}(x) \subset \mathfrak{g}$ such that $f(y) < a$ for $y \in A_x$. We have $f^{-1}((-\infty, a)) = \bigcup_{x : f(x) < a} A_x \in \mathfrak{g}$. On the other hand, let $f(x) < a$. We have $A = f^{-1}((-\infty, a)) \in \mathfrak{g}$ and so $A \in \mathfrak{g}(x)$. For each $y \in A$ we have $f(y) < a$. □

By [9], any $\beta \subset \mathcal{P}(X)$ generates the GT $\mathfrak{h}$ on X composed of $\emptyset$ and all sets $H \subset X$ of the form $H = \bigcup_{i\in I} B_i$, where $B_i \in \beta$ and $I \neq \emptyset$ is arbitrary. It is easy to see that

Proposition 4. *A function $f\colon X \to [-\infty, \infty]$ is upper [lower] {upper and lower} $\mathfrak{g}$-continuous if it is $(\mathfrak{g}, \mathfrak{h})$-continuous, where $\mathfrak{h}$ is the GT on $[-\infty, \infty]$ generated by the base β, where $\beta = \{(-\infty, a) : a \in \mathbb{R}\}$ $[\beta = \{(a, \infty)\} : a \in \mathbb{R}\}]$ $\{\beta = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}\}$.*

Definition 3. Let $f\colon X \to [-\infty, \infty]$ be a function. Define $M_f^{\mathfrak{g}}, m_f^{\mathfrak{g}}, \tilde{k}_f^{\mathfrak{g}}\colon X \to [-\infty, \infty]$ as

$M_f^{\mathfrak{g}}(x) = \inf\big\{\sup\{f(y) : y \in A\} : A \in \mathfrak{g}(x)\big\}$ for $x \in \bigcup \mathfrak{g}$ and
$M_f^{\mathfrak{g}}(x) = \infty$ for $x \in X \setminus \bigcup \mathfrak{g}$,

$m_f^{\mathfrak{g}}(x) = \sup\big\{\inf\{f(y) : y \in A\} : A \in \mathfrak{g}(x)\big\}$ for $x \in \bigcup \mathfrak{g}$ and
$m_f^{\mathfrak{g}}(x) = -\infty$ for $x \in X \setminus \bigcup \mathfrak{g}$,

$\tilde{k}_f^{\mathfrak{g}}(x) = M_f^{\mathfrak{g}}(x) - m_f^{\mathfrak{g}}(x)$

(we assume $\infty - \infty = 0$, and, of course, $\infty + a = \infty, \infty - (-\infty) = \infty$).

Proposition 5. *The function $M_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous, $f \le M_f^{\mathfrak{g}}$ and $M_f^{\mathfrak{g}} \le g$ for each upper $\mathfrak{g}$-continuous function g with $f \le g$. The function $m_f^{\mathfrak{g}}$ is lower $\mathfrak{g}$-continuous, $f \ge m_f^{\mathfrak{g}}$ and $m_f^{\mathfrak{g}} \ge g$ for each lower $\mathfrak{g}$-continuous g with $f \ge g$.*

Proof. Evidently $f(x) \le M_f^{\mathfrak{g}}(x)$ for each x. Let $M_f^{\mathfrak{g}}(x) < a$. Then there is $A \in \mathfrak{g}(x)$ such that $\sup\big\{f(y) : y \in A\big\} < a$; therefore $M_f^{\mathfrak{g}}(y) < a$ for each $y \in A$ and $M_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous at x.

Now, let g be upper $\mathfrak{g}$-continuous and $f \le g$. If there is x with $M_f^{\mathfrak{g}}(x) > g(x)$ then $M_f^{\mathfrak{g}}(x) > c > g(x)$ for some $c \in \mathbb{R}$. Then there is $A \in \mathfrak{g}(x)$ such that $g(y) < c$ for each $y \in A$. Further, $\sup\big\{f(y) : y \in A\big\} > c$ and hence for some $z \in A$ we have $g(z) < c < f(z)$, a contradiction. □

Proposition 6. *A function $f\colon X \to [-\infty,\infty]$ is upper [lower] $\mathfrak{g}$-continuous at x if and only if $M_f^{\mathfrak{g}}(x) = f(x)$ $[m_f^{\mathfrak{g}}(x) = f(x)]$.*

Proof. Let f be upper $\mathfrak{g}$-continuous at x and let $f(x) < a$. Then there is a set $A \in \mathfrak{g}(x)$ such that $f(y) < a$ for each $y \in A$. So, $f(x) \le M_f^{\mathfrak{g}}(x) < a$. This is true for each $a > f(x)$ and hence $f(x) = M_f^{\mathfrak{g}}(x)$. Conversely, let $f(x) = M_f^{\mathfrak{g}}(x)$ and $f(x) < a$. By Proposition 5 there is a set $A \in \mathfrak{g}(x)$ such that $M_f^{\mathfrak{g}}(y) < a$ for each $y \in A$. Therefore $f(y) \le M_f^{\mathfrak{g}}(y) < a$ and f is upper $\mathfrak{g}$-continuous at x. □

Proposition 7. *We have $\tilde{k}_f^{\mathfrak{g}} \le k_f^{\mathfrak{g}}$.*

Proof. If $x \notin \bigcup \mathfrak{g}$, then $\tilde{k}_f^{\mathfrak{g}}(x) = \infty = k_f^{\mathfrak{g}}(x)$. If $x \in \bigcup \mathfrak{g}$ then $\tilde{k}_f^{\mathfrak{g}}(x) = M_f^{\mathfrak{g}}(x) - m_f^{\mathfrak{g}}(x) = \inf_{A\in\mathfrak{g}(x)} \sup_{z\in A} f(z) - \sup_{A\in\mathfrak{g}(x)} \inf_{z\in A} f(z) \le \inf_{A\in\mathfrak{g}(x)}(\sup_{z\in A} f(z) - \inf_{z\in A} f(z)) = \inf_{A\in\mathfrak{g}(x)} \operatorname{diam} f(A) = k_f^{\mathfrak{g}}(x)$. □

Proposition 8. *A GT $\mathfrak{g}$ on X is a quasi-topology if and only if $\tilde{k}_f^{\mathfrak{g}} = k_f^{\mathfrak{g}}$ for each $f\colon X \to \mathbb{R}$.*

Proof. Let $\mathfrak{g}$ be a quasi-topology. Let $\tilde{k}_f^{\mathfrak{g}}(z) < c$. Then $z \in \bigcup \mathfrak{g}$. Since $f(z) < \infty$ we have $m_f^{\mathfrak{g}}(z) < \infty$ and $M_f^{\mathfrak{g}}(z) < c + m_f^{\mathfrak{g}}(z)$, so there is $A \in \mathfrak{g}(z)$ such that $\sup f(A) < c + m_f^{\mathfrak{g}}(z)$. Further there is $B \in \mathfrak{g}(z)$ such that $\inf f(B) > \sup f(A) - c$. Since $\mathfrak{g}$ is a quasi-topology, $A \cap B \in \mathfrak{g}$ and so $A \cap B \in \mathfrak{g}(z)$. Let $u, v \in A \cap B$ be arbitrary. Assume, e.g., $f(v) \ge f(u)$. Then $|f(u) - f(v)| = f(v) - f(u) \le \sup f(A) - \inf f(B) < \sup f(A) + c - \sup f(A) = c$. This yields $\operatorname{diam} f(A \cap B) \le c$ and $k_f^{\mathfrak{g}}(z) \le c$.

Now, assume that $\mathfrak{g}$ is not a quasi-topology. Then there are $A, B \in \mathfrak{g}$ with $A \cap B \notin \mathfrak{g}$. There is $z \in A \cap B$ such that $H \not\subset A \cap B$ for each $H \in \mathfrak{g}(z)$. (If, namely for each $x \in A \cap B$, there is $H_x \in \mathfrak{g}(x)$ such that $H_x \subset A \cap B$, then $A \cap B = \bigcup_{x \in A\cap B} H_x \in \mathfrak{g}$, a contradiction.)

Define a function $f\colon X \to \mathbb{R}$ as $f(x) = 0$ for $x \in A \setminus B$, $f(x) = 1$ for $x \in A \cap B$ and $f(x) = 2$ otherwise.

Let $H \in \mathfrak{g}(z)$ be arbitrary. Then there is $y \in H \setminus (A \cap B)$. We have $|f(z) - f(y)| = 1$ and hence $k_f^{\mathfrak{g}}(z) \ge \operatorname{diam} f(A \cap B) \ge 1$. Further, $A, B \in \mathfrak{g}(z)$, $\sup f(A) = 1 = \inf f(B)$ and hence $M_f^{\mathfrak{g}}(z) \le 1$, $m_f^{\mathfrak{g}}(z) \ge 1$ and $\tilde{k}_f^{\mathfrak{g}}(z) = M_f^{\mathfrak{g}}(z) - m_f^{\mathfrak{g}}(z) \le 0 < 1 = k_f^{\mathfrak{g}}(z)$. □

From Proposition 6 we have

Proposition 9. *A function f is simultaneously upper and lower $\mathfrak{g}$-continuous at x if and only if $\tilde{k}_f^{\mathfrak{g}}(x) = 0$.*

Proposition 10. *We have $uC_{\mathfrak{g}}(f) \cap lC_{\mathfrak{g}}(f) = \bigcap_{n\in\mathbb{N}} M_n$, where $M_n \in \mathfrak{g}$.*

P r o o f. We have

$$\Big\{x : \tilde{k}_f^{\mathfrak{g}}(x) > 0\Big\} = \bigcup_{r,s\in\mathbb{Q}:r<s} \Big(\big\{x : m_f^{\mathfrak{g}}(x) \le r\big\} \cap \big\{x : M_f^{\mathfrak{g}}(x) \ge s\big\}\Big)$$

and hence

$$uC_{\mathfrak{g}}(f) \cap lC_{\mathfrak{g}}(f) = \Big\{x : \tilde{k}_f^{\mathfrak{g}}(x) = 0\Big\} = \bigcap_{r,s\in\mathbb{Q}:r<s} \Big(m_f^{-1}((r,\infty)) \cup M_f^{-1}((-\infty,s))\Big).$$

By Proposition 3 and 5, $uC_{\mathfrak{g}}(f) \cap lC_{\mathfrak{g}}(f)$ is the countable intersection of sets from $\mathfrak{g}$. □

Remark 1. The sequence $(M_n)_n$ in Proposition 10 need not be decreasing. Let $X = \{a,b,c\}$, $\mathfrak{g} = \big\{\emptyset, \{a,c\}, \{b,c\}, \{a,b,c\}\big\}$ and $f\colon X \to \mathbb{R}$ be defined as $f(a) = -1$, $f(b) = 1$, $f(c) = 0$. Then $uC_{\mathfrak{g}}(f) \cap lC_{\mathfrak{g}}(f) = \{c\}$, however the set $\{c\}$ cannot be the intersection of decreasing sequence of sets from $\mathfrak{g}$.

As corollary we obtain that the function $\tilde{k}_f^{\mathfrak{g}}$ need not be upper $\mathfrak{g}$-continuous, although it is the sum of two upper $\mathfrak{g}$-continuous functions. (If namely $\tilde{k}_f^{\mathfrak{g}}$ would be upper $\mathfrak{g}$-continuous, then $uC_{\mathfrak{g}}(f) \cap lC_{\mathfrak{g}}(f) = \bigcap_{n\in\mathbb{N}} M_n$, where $M_n = (\tilde{k}_f^{\mathfrak{g}})^{-1}((-\infty, 1/n)) \in \mathfrak{g}$ and $M_{n+1} \subset M_n$.)

Proposition 11. *The $\mathfrak{g}$-oscillation $k_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous.*

P r o o f. Let $k_f^{\mathfrak{g}}(x) < a$. Then there is $A \in \mathfrak{g}(x)$ such that $\operatorname{diam} f(A) < a$. For each $y \in A$ we have $A \in \mathfrak{g}(y)$ and $k_f^{\mathfrak{g}}(y) < a$, thus $k_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous at x. □

Theorem 1. *Let $\mathfrak{g}$ be a GT on X and let Y be a metric space. The set $C_{\mathfrak{g}}(f)$ is the countable intersection of sets M_n, where $M_n \in \mathfrak{g}$ and $M_{n+1} \subset M_n$.*

P r o o f. Put $M_n = (k_f^{\mathfrak{g}})^{-1}((-\infty, 1/n))$. Evidently, $M_{n+1} \subset M_n$ and, according to Proposition 3 and 11, $M_n \in \mathfrak{g}$. By Proposition 2 we have $C_{\mathfrak{g}}(f) = \bigcap_{n\in\mathbb{N}} M_n$. □

By [8], $\mathcal{H} \subset \mathcal{P}(X)$ is a hereditary class, if $B \subset A \in \mathcal{H}$ implies $B \in \mathcal{H}$.

Proposition 12. *Let $\mathfrak{g}$ be a GT on X and let Y be a metric space. If there is a function $f\colon X \to Y$ such that $C_{\mathfrak{g}}(f) = \emptyset$ and the set $f(X)$ is countable, then there is a hereditary class $\mathcal{M} \subset \mathcal{P}(X)$ such that $\mathcal{M} \cap \mathfrak{g} = \{\emptyset\}$ and $X = \bigcup_{n\in\mathbb{N}} X_n$, where $X_n \in \mathcal{M}$.*

P r o o f. Let $f(X) \subset \{r_1, r_2, \dots\}$ ($r_i \ne r_j$ for $i \ne j$) and put $X_n = f^{-1}(r_n)$. Then $X = \bigcup_{n\in\mathbb{N}} X_n$ and the sets X_n are mutually disjoint. Put

$$\mathcal{M} = \big\{A \subset X : (B \in \mathfrak{g} \;\&\; B \subset A) \Rightarrow B = \emptyset\big\}.$$

Evidently, $\mathcal{M}$ is a hereditary class and $\mathcal{M} \cap \mathfrak{g} = \{\emptyset\}$. If $X_n \notin \mathcal{M}$ for some $n \in \mathbb{N}$, then there is $B \in \mathfrak{g}$ such that $B \ne \emptyset$ and $B \subset X_n$. For each $z \in B$ we have $B \in \mathfrak{g}(z)$ and $f(z) = r_n$. Therefore for each neighbourhood V of r_n we have $f(B) \subset V$, i.e., $z \in C_{\mathfrak{g}}(f)$, a contradiction. □

Theorem 2. *Let $\mathfrak{g}$ be a GT on X and let (Y,d) be a metric space with at least one accumulation point. Let $\mathcal{M} \subset \mathcal{P}(X)$ be a hereditary class such that $\mathcal{M} \cap \mathfrak{g} = \{\emptyset\}$ and $X = \bigcup_{i\in\mathbb{N}} X_i$, where $X_i \in \mathcal{M}$. Let $M \subset X$. Then $M = C_{\mathfrak{g}}(f)$ for some $f\colon X \to Y$ if and only if $M = \bigcap_{n\in\mathbb{N}} M_n$, where $M_n \in \mathfrak{g}$ and $M_{n+1} \subset M_n$ for $n \in \mathbb{N}$.*

Proof. Necessity follows from Theorem 1. Sufficiency. Let $X = \bigcup_{i\in\mathbb{N}} Y_i$, where $Y_i \in \mathcal{M}$. Put $X_1 = Y_1$ and $X_{n+1} = Y_{n+1} \setminus \bigcup_{i=1}^{n} Y_i$. Then $X = \bigcup_{i\in\mathbb{N}} X_i$, $X_i \in \mathcal{M}$ and X_i are mutually disjoint.

Let y_0 be an accumulation point of Y and let $(y_n)_n$ be a sequence in Y converging to y_0 such that $y_i \neq y_j$ for $i \neq j$. We can assume that $d(y_0, y_j) < 1/j$.

Let $\phi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $\phi(j,i) \geq j$. Put $M_0 = X$ and define a function $f\colon X \to Y$ as: $f(x) = y_0$ for $x \in M$ and $f(x) = y_{\phi(n,i)}$ for $x \in X_i \cap (M_{n-1} \setminus M_n)$. We will show that $M = C_{\mathfrak{g}}(f)$.

Let $x \in M$ and $\varepsilon > 0$. Let n be such that $1/n < \varepsilon$. Then $x \in M_n$, so $M_n \in \mathfrak{g}(x)$. For each $y \in M_n$ we have $y \notin X \setminus M_j$ for $j \leq n$. So, $f(y) = y_{\phi(j,i)}$ with $j > n$. We have $d\big(f(x), f(y)\big) = d\big(y_0, y_{\phi(j,i)}\big) < 1/\phi(j,i) \leq 1/j < 1/n < \varepsilon$, i.e., $x \in C_{\mathfrak{g}}(f)$.

Now, let $x \notin M$. Then $f(x) = y_{\phi(n,i)}$ for some $i, n \in \mathbb{N}$. Let V be a neighbourhood of $y_{\phi(n,i)}$ such that $y_{\phi(m,j)} \notin V$ for $(m,j) \neq (n,i)$. Let $A \in \mathfrak{g}(x)$. Then $A \cap X_k \neq \emptyset$ for some $k \neq i$, otherwise $A \subset X_i$ and so A is a nonempty member of $\mathfrak{g} \cap \mathcal{M}$, a contradiction. For $z \in A \cap X_k$ we have $f(z) = y_{\phi(m,k)} \neq y_{\phi(n,i)}$ and hence $f(z) \notin V$. Therefore $x \notin C_{\mathfrak{g}}(f)$. □

If $\mathcal{M} = \big\{A \subset X : i_{\mathfrak{g}}(A) = \emptyset\big\}$ we obtain the characterization of the $\mathfrak{g}$-continuity, $\sigma(\mathfrak{g})$-continuity and $\alpha(\mathfrak{g})$-continuity points. If $\mathcal{M} = \big\{A \subset X : i_{\mathfrak{g}}(c_{\mathfrak{g}}(A)) = \emptyset\big\}$ we obtain the characterization of the $\pi(\mathfrak{g})$-continuity and $\beta(\mathfrak{g})$-continuity points. Especially, if $\mathcal{M}$ is the family of nowhere dense sets, we obtain the characterization of pre-continuity and β-continuity points on spaces of first category.

A topological space is almost resolvable (resolvable) if it is the union of countably many (two) sets with empty interiors. Every first countable toplogical space wihtout isolated points, locally compact Hausdorff topological space without isolated points or linear topological space is resolvable. Every separable topological space without isolated points is almost resolvable [1]. If $\mathcal{M}$ is the family of sets with empty interiors, we obtain

Corollary 1. *Let X be an almost resolvable topological space and let (Y,d) be a metric space with at least one accumulation point. Let $\mathfrak{g}$ be a GT on X such that the interior of A is nonempty for each nonempty $A \in \mathfrak{g}$. Let $M \subset X$. Then $M = C_{\mathfrak{g}}(f)$ for some $f\colon X \to Y$ if and only if $M = \bigcap_{n\in\mathbb{N}} M_n$, where $M_n \in \mathfrak{g}$ and $M_{n+1} \subset M_n$.*

Especially, if $\mathfrak{g}$ is the family of all open sets in X, we obtain the characterization of continuity points [1]; if $\mathfrak{g}$ is the family of all semi-open sets, we obtain the characterization of quasicontinuity points [4]; if $\mathfrak{g}$ is the family of all α-sets, we obtain the characterization of α-continuity points.

Proposition 13. *Let $f, g\colon X \to Y$ be functions. Then*

$$\sup\{|k_f^{\mathfrak{g}}(x) - k_g^{\mathfrak{g}}(x)| : x \in X\} \le 2\sup\Big\{d\big(f(x), g(x)\big) : x \in X\Big\}$$

(we assume that $\infty - \infty = 0$).

P r o o f. Suppose that

$$\sup\{|k_f^{\mathfrak{g}}(x) - k_g^{\mathfrak{g}}(x)| : x \in X\} > 2\sup\Big\{d\big(f(x), g(x)\big) : x \in X\Big\} = s.$$

Then there is $x \in X$ with $|k_f^{\mathfrak{g}}(x) - k_g^{\mathfrak{g}}(x)| > s$. Let, e.g., $k_g^{\mathfrak{g}}(x) - k_f^{\mathfrak{g}}(x) > s$. Then there is c such that $k_g^{\mathfrak{g}}(x) > c > k_f^{\mathfrak{g}}(x) + s$. Hence there is $A \in \mathfrak{g}$ such that $\operatorname{diam} f(A) < c - s$. Since $\operatorname{diam} g(A) > c$ there are $u, v \in A$ such that $d\big(g(u), g(v)\big) > c$. This yields

$$\begin{aligned} d\big(f(u), f(v)\big) &< c - s < d\big(g(u), g(v)\big) - s \\ &\le d\big(g(u), f(u)\big) + d\big(f(u), f(v)\big) + d\big(f(v), g(v)\big) - s \\ &\le d\big(f(u), f(v)\big) + 2\sup\Big\{d\big(g(x), f(x)\big) : x \in X\Big\} - s \\ &= d\big(f(u), f(v)\big), \end{aligned}$$

a contradiction. □

Corollary 2. *If a net $\{f_s : s \in S\}$ of functions $f_s\colon X \to Y$ uniformly converges to a function $f\colon X \to Y$ then the net $\big\{k_{f_s}^{\mathfrak{g}} : s \in S\big\}$ uniformly converges to $k_f^{\mathfrak{g}}$.*

Corollary 3. *If a net $\{f_s : s \in S\}$ of functions $f_s\colon X \to Y$ uniformly converges to a function $f\colon X \to Y$ and f_s are $\mathfrak{g}$-continuous (at x) then f is $\mathfrak{g}$-continuous (at x), too.*

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