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A NOTE ON MEASURABILITY OF MULTIFUNCTIONS APPROXIMATELY CONTINUOUS IN SECOND VARIABLE

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ABSTRACT. Let $I \subset \mathbb{R}$ be an interval, $(X, \mathcal{M}(X))$ a measure space, and $(Z, \|\cdot\|)$ a reflexive Banach space. We prove that a multifunction F from $X \times I$ to Z is measurable whenever it is $\mathcal{M}(X)$-measurable in the first and approximately continuous and almost everywhere continuous in the second variable.

1. Introduction

It is well known that Lebesgue measurability of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ implies Lebesgue measurability of the functions $f(x,\cdot)$ and $f(\cdot,y)$ for almost every $x \in \mathbb{R}$ and almost every $y \in \mathbb{R}$. The converse is, however, not true even if all these functions are Lebesgue measurable. There are various sufficient conditions on the functions $f(x,\cdot)$ and $f(\cdot,y)$ ensuring that f is measurable (see, e.g., [2], [3], [8], [11]). It is known that if $(X, \mathcal{M}(X))$ is a measurable space, (Y, ρ) a separable metric space, and (Z, d) a metric space, then a function $f\colon X \times Y \to Z$, $\mathcal{M}(X)$-measurable in the first and continuous in the second variable is measurable with respect to the product of the σ-field $\mathcal{M}(X)$ and the Borel σ-field of Y. This result was also proved in the case of a multifunction (see [12, Theorem 2]). Unfortunately, without additional hypotheses, this result cannot be extended to multifunctions with a weaker assumption in place of the continuity. Among others, the continuum hypothesis implies that this result fails if the continuity is replaced with the approximate continuity, as can be seen from the proof of [2, Theorem 2]. Our purpose is to show that if $Y = I \subset \mathbb{R}$ is an interval and (Z, d) is a reflexive Banach space (with a metric d generated by

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the norm in Z), then the reinforcement of the approximate continuity of a multifunction F from $X \times Y$ to Z in the second variable with continuity almost everywhere ensures its measurability.

2. Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. Let $I \subset \mathbb{R}$ be an interval and let $\mathcal{L}(\mathbb{R})$ be the σ-field of Lebesgue measurable subsets of $\mathbb{R}$ (in case of need, the Lebesgue measure m in $\mathcal{L}(\mathbb{R})$ is understood).

Let S and Z be two nonempty sets. We assume that for every point $s \in S$ a non-empty subset $\Phi(s)$ of Z is given. In this case, Φ is a multifunction from S to Z and we will write $\Phi : S \mapsto Z$.

If $\Phi : S \mapsto Z$ and $G \subset Z$ are given, then we define

$$\Phi^-(G) = \big\{s \in S : \Phi(s) \cap G \neq \emptyset\big\}.$$

If (Z, d) is a metric space, $z_0 \in Z$ and $M \subset Z$, then $B(z_0, r)$ will denote an open ball centered in z_0 with radius $r > 0$, $B(M, r) = \bigcup\big\{B(z, r) : z \in M\big\}$, and $\mathcal{B}(Z)$ the σ-field of Borel subsets of Z.

Let $\mathcal{P}(Z)$ be the power set of (Z, d) and let $\mathcal{P}_0(Z) = \mathcal{P}(Z) \setminus \emptyset$. We put

$$\mathcal{C}_b(Z) = \big\{A \in \mathcal{P}_0(Z) : A \text{ is closed and bounded}\big\}.$$

In the sequel, convergence in the space $\mathcal{C}_b(Z)$ will denote the convergence in the Hausdorff metric denoted by h.

Now, we collect some material which can be useful for the next section: terminology, known facts from the literature, and some properties which are new for multifunctions of one variable.

Let $\big(S, \mathcal{M}(S)\big)$ be a measurable space and (Z, d) a metric one.

We will say a multifunction $\Phi : S \mapsto Z$ is $\mathcal{M}(S)$-measurable if for each open set $G \subset Z$, $\Phi^-(G) \in \mathcal{M}(S)$.

Let $\Phi_n : S \mapsto Z$ for $n \in \mathbb{N}$ and $\Phi : S \mapsto Z$ be multifunctions with values in $\mathcal{C}_b(Z)$. To say that $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ means that the sequence $\big(\Phi_n(s)\big)_{n\in\mathbb{N}}$ converges to $\Phi(s)$ for each $s \in S$. One can prove that (see [7, (1.21)]):

(1) If (Z, d) is separable and the sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ, then Φ is $\mathcal{M}(S)$-measurable whenever Φ_n is $\mathcal{M}(S)$-measurable for each $n \in \mathbb{N}$.

Let $\big(S, \mathcal{T}(S)\big)$ be a topological space and (Z, d) a metric one. Let $\Phi : S \mapsto Z$ be a multifunction. Unless otherwise stated, we assume that $\Phi(s) \in \mathcal{C}_b(Z)$.

The statement that Φ is *h-continuous* will mean that Φ treated as a function from S to the space $\big(\mathcal{C}_b(Z), h\big)$ is continuous.

We will say a multifunction $\Phi : I \mapsto Z$ is *approximately h-continuous* at $s \in I$ if there exists a set $A \in \mathcal{L}(\mathbb{R})$, including s, such that s is a density point of A and the restriction $F|_A$ is h-continuous at s; Φ is *approximately h-continuous* if it is approximately h-continuous at any point $s \in I$.

It is known that a multifunction $\Phi : I \mapsto Z$ almost everywhere approximately h-continuous is $\mathcal{L}(\mathbb{R})$-measurable ([5, Theorem 2]).

From now on, let $(Z, ||\cdot||)$ be a reflexive Banach space with a metric d generated by the norm; θ will denote the origin of Z, $||K|| = h\big(K, \{\theta\}\big)$ when $K \in \mathcal{C}_b(Z)$, and $\mathrm{co}(K)$ the convex hull of K.

If $A \subset Z$, $B \subset Z$, and $\lambda \in \mathbb{R}$ then, as usual,

$$A + B = \{a + b : a \in A \wedge b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a : a \in A\}.$$

It is known that ([1, Lemma 2.2 (ii)])

(2) If $A_i, B_i \in \mathcal{C}_b(Z)$ for $i = 1, 2$, then

$$h(A_1 + A_2, B_1 + B_2) \le h(A_1, B_1) + h(A_2, B_2).$$

We put

$$\mathcal{C}_{bc}(Z) = \big\{A \in \mathcal{C}_b(Z) : A \text{ is convex}\big\}.$$

By reflexivity of $(Z, ||\cdot||)$, the space $\mathcal{C}_{bc}(Z)$ with the addition defined above is a commutative semigroup which satisfies the cancelation low (see [10]). Note that the reflexivity of $(Z, ||\cdot||)$ is used to show that

(3) $A + B \in \mathcal{C}_{bc}(Z)$ whenever $A, B \in \mathcal{C}_{bc}(Z)$.

The completeness of (Z, d) implies that $\big(\mathcal{C}_b(Z), h\big)$ is complete. Therefore, Price's inequality (see [9, (2.9), p. 4])

$$h\big(\mathrm{co}(A), \mathrm{co}(B)\big) \le h(A, B)$$

implies the following.

(4) A Cauchy sequence in $\mathcal{C}_{bc}(Z)$ must converge to an element of $\mathcal{C}_{bc}(Z)$.

Now, let $T \in \mathcal{L}(\mathbb{R})$ and let $\Phi : T \mapsto Z$ be a measurable multifunction with values in $\mathcal{C}_{bc}(Z)$. Suppose that Φ is bounded, i.e., all its values are contained in a fixed totally bounded set $K \subset Z$.

We define an integral of Φ as follows (in the case $Z = \mathbb{R}^k$, cf. [4, p. 218]).

If Φ takes only a finite number of values $B_1, B_2, \ldots, B_n$, then we put

$$\int_E \Phi(t)\, dt = \sum_{i=1}^{n} m(D_i) \cdot B_i,$$

where $E \subset T$ is a bounded measurable set and $D_i = \big\{t \in E : \Phi(t) = B_i\big\}$ for $i = 1, 2, \ldots, n$. By (3), $\int_E \Phi(t)\, dt \in \mathcal{C}_{bc}(Z)$.

It is easy to see that:

(5) If $A, B \in \mathcal{L}(\mathbb{R})$ are non-overlapping and $E = A \cup B$, then

$$\int_E \Phi(t)\,dt = \int_A \Phi(t)\,dt + \int_B \Phi(t)\,dt.$$

Let $\Psi : T \mapsto Z$ be a measurable and bounded multifunction with values in $\mathcal{C}_{bc}(Z)$. Using (2), one obtains

$$h\left(\int_E \Phi(t)\,dt, \int_E \Psi(t)\,dt\right) \leq \int_E h\big(\Phi(t), \Psi(t)\big)\,dt \tag{6}$$

whenever Φ and Ψ take a finite number of values.

For a general case of a measurable and bounded multifunction, the definition of its integral is based on the following lemma ([6, Lemma 1]).

Lemma 1. *Let a totally bounded convex set $K \subset Z$ and a number $\delta > 0$ be given. Then there exists a finite family $\mathcal{F}_\delta \subset \mathcal{C}_{bc}(Z)$ such that if $D \in \mathcal{C}_{bc}(K)$, then there exists a smallest set $B \in \mathcal{F}_\delta$ such that $D \subset B \subset B(D, \delta)$.*

Now, take K in the lemma to be the totally bounded convex set containing all the values of Φ. Suppose $t \in T$. Let $\mathcal{F}_\delta(t)$ be the family corresponding to $\delta > 0$, and let $\Phi_\delta(t)$ be the smallest member of $\mathcal{F}_\delta(t)$ containing $\Phi(t)$.

Then $h\big(\Phi(t), \Phi_\delta(t)\big) < \delta$ and $\Phi_\delta : T \mapsto Z$ takes only a finite number of values. Moreover, if $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers and $\lim_{n\to\infty} \delta_n = 0$, then, by (6),

$$\left(\int_E \Phi_{\delta_n}(t)\,dt\right)_{n\in\mathbb{N}}$$

is a Cauchy sequence in $\mathcal{C}_{bc}(Z)$. Thus, by (4), the limit

$$h - \lim_{\delta\to 0} \int_E \Phi_\delta(t)\,dt$$

exists in $\mathcal{C}_{bc}(Z)$ and we take this limit to be the integral of Φ on E, i.e.,

$$\int_E \Phi(t)\,dt =: h - \lim_{\delta\to 0} \int_E \Phi_\delta(t)\,dt \in \mathcal{C}_{bc}(Z).$$

Note that by a passage to a limit in (5) and (6), we can see the following.

(7) The properties (5) and (6) are true for all measurable and bounded multifunctions from T to Z with values in $\mathcal{C}_{bc}(Z)$.

The following result is known [6, Theorem 6].

Theorem 1. *If a bounded multifunction $\Phi : I \mapsto Z$ with closed and convex values is approximately h-continuous, then it is a derivative, i.e.,*

$$\Phi(t) = h - \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \Phi(s)\, ds \qquad \text{for} \quad t \in I.$$

Now, we present a different approach of defining integrability for multifunctions. It is based on the definition of Riemann integral. Moving from Hukuhara's idea of integrability (in the case $Z = \mathbb{R}^k$, cf. [4]), we define an R-integrability of multifunctions in a more general case.

Let $I = [a, b]$ and let $\Phi : I \mapsto Z$ be a multifunction with values in $\mathcal{C}_{bc}(Z)$. Let $\Delta = \{a_0, a_1, \dots a_n\}$ be a partition of I and let $\lambda(\Delta) = \max\{a_{i+1} - a_i\}$ be the diameter of the partition. Let $\mathcal{P}$ denote the family of all pairs (Δ, τ), where $\tau = (t_0, t_1, \dots t_{n-1})$ is a sequence of points such that $t_i \in [a_i, a_{i+1}]$ for $i = 0, \dots, n-1$. We put

$$C_\Phi(\Delta, \tau) = \sum_{i=0}^{n-1} (a_{i+1} - a_i) \Phi(t_i)$$

for $(\Delta, \tau) \in \mathcal{P}$. Note that (3) implies $C_\Phi(\Delta, \tau) \in \mathcal{C}_{bc}(Z)$. Moreover, by (2),

$$h\big(C_\Phi(\Delta, \tau), C_\Psi(\Delta, \tau)\big) \le \sum_{i=0}^{n-1} (a_{i+1} - a_i)\, h\big(\Phi(t_i), \Psi(t_i)\big) \tag{8}$$

whenever $\Psi : I \mapsto Z$ is a multifunction with values in $\mathcal{C}_{bc}(Z)$.

We say that a multifunction $\Phi : I \mapsto Z$ is R-integrable (on I) if there exists $B \in \mathcal{C}_{bc}(Z)$ such that

$$\forall \varepsilon > 0\, \exists \eta > 0\, \forall (\Delta, \tau) \in \mathcal{P} \left[\lambda(\Delta) < \eta \Rightarrow h\big(C_\Phi(\Delta, \tau), B\big) < \varepsilon \right],$$

and we define $(R) \int_I \Phi(t)\, dt$ to be the set B.

Note that by (8), we have

$$h\left(\int_I \Phi(t)\, dt, \int_I \Psi(t)\, dt \right) \le \int_I h\big(\Phi(t), \Psi(t)\big)\, dt \le (b - a)\, \varepsilon,$$

provided that $h\big(\Phi(t_i), \Psi(t_i)\big) \le \varepsilon$ for each $t \in I$. Therefore, similarly to the case of real functions,

(9) if $\Phi : I \mapsto Z$ is h-continuous, then Φ is R-integrable.

Proposition 1. *If a multifunction $\Phi : I \mapsto Z$ is bounded and almost everywhere h-continuous, then Φ is R-integrable.*

P r o o f. Let $K \in \mathcal{C}_{bc}(Z)$ be such that $\Phi(t) \subset K$ for $t \in I$. Let D_Φ denote a set of discontinuity points of Φ. By assumption, $m(D_\Phi) = 0$. Fix $\varepsilon > 0$. Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of open intervals such that $D_\Phi \subset \bigcup_{n\in\mathbb{N}} I_n$ and $\Sigma_{n\in\mathbb{N}} m(I_n) < \varepsilon$. Without loss of generality, we can assume that $I_n \cap I_m = \emptyset$ for $n \neq m$. Let $I_n = (\alpha_n, \beta_n)$ for $n \in \mathbb{N}$ and $A_\varepsilon = [a,b] \setminus \bigcup_{n\in\mathbb{N}} I_n$. Then, $m(A_\varepsilon) > b - a - \varepsilon$. We define a multifunction $\Phi_\varepsilon : I \mapsto Z$ by

$$\Phi_\varepsilon(t) = \begin{cases} \Phi(t) & \text{if } t \in A_\varepsilon, \\ \frac{\beta_n - t}{\beta_n - \alpha_n}\Phi(\alpha_n) + \frac{t-\alpha_n}{\beta_n-\alpha_n}\Phi(\beta_n) & \text{if } t \in (\alpha_n, \beta_n) \cap I, n \in \mathbb{N}. \end{cases}$$

Note that $\Phi_\varepsilon(t) \in \mathcal{C}_{bc}(Z)$. Moreover, Φ_ε is h-continuous and, by (9), also R-integrable. Let $B \in \mathcal{C}_{bc}(Z)$ be such that $\int_I \Phi_\varepsilon(t)\,dt = B$. Let $(\Delta, \tau) \in \mathcal{P}$ and $\eta > 0$ be such that $\lambda(\Delta) < \eta$ and $h\big(C_{\Phi_\varepsilon}(\Delta,\tau), B\big) < \varepsilon$.

Then,

$$\begin{aligned} h\big(C_\Phi(\Delta,\tau), B\big) &\le h\big(C_\Phi(\Delta,\tau), C_{\Phi_\varepsilon}(\Delta,\tau)\big) \\ &\quad + h\big(C_{\Phi_\varepsilon}(\Delta,\tau), B\big) \\ &= h\big(\Sigma_{i=0}^{n-1}(a_{i+1}-a_i)\,\Phi(t_i), \Sigma_{i=0}^{n-1}(a_{i+1}-a_i)\Phi_\varepsilon(t_i)\big) \\ &\quad + h\big(C_{\Phi_\varepsilon}(\Delta,\tau), B\big), \end{aligned}$$

and then, by (2),

$$h\big(C_\Phi(\Delta,\tau), B\big) \le \Sigma_{i=0}^{n-1}(a_{i+1}-a_i)\,h\big(\Phi(t_i), \Phi_\varepsilon(t_i)\big) + h\big(C_{\Phi_\varepsilon}(\Delta,\tau), B\big).$$

For that reason

$$h\big(C_\Phi(\Delta,\tau), B\big) \le 2\,\varepsilon\,||K|| + \varepsilon,$$

since $\Phi(t_i) = \Phi_\varepsilon(t_i)$ for $t_i \in [a_{i-1}, a_i] \cap A_\varepsilon$ and $h\big(\Phi(t_i), \Phi_\varepsilon(t_i)\big) \le 2\,||K||$ for $t_i \in [a_{i-1}, a_i] \setminus A_\varepsilon$. This finishes the proof of Proposition 1. □

Following H u k u h a r a [4], one can prove that R-integrability of a measurable and bounded multifunction $\Phi : I \mapsto Z$ implies its integrability. Moreover, $(R)\int_I \Phi(t)\,dt = \int_I \Phi(t)\,dt$.

3. Main results

Now, we pass on to multifunctions of two variables. Let $\big(X, \mathcal{M}(X), \mu\big)$ be a measure space with a σ-finite complete measure μ defined on $\mathcal{M}(X)$ and let $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$ be the σ-field generated by the family of sets $A \times B$, where $A \in \mathcal{M}(X)$ and $B \in \mathcal{B}(\mathbb{R})$.

Let $(Z||\cdot||)$ be still a reflexive Banach space with the metric d generated by the norm, and we will still consider the multifunctions $F : X \times I \mapsto Z$ with values in $\mathcal{C}_{bc}(Z)$.

Let $(x, y) \in X \times I$ be a fixed point. Then the multifunction $F_x : I \mapsto Z$ defined by $F_x(y) = F(x, y)$ will be called the x-section of F, and the multifunction $F^y : X \mapsto Z$ defined by $F^y(y) = F(x, y)$, the y-section of F.

Similarly, if $E \subset X \times I$ and $(x, y) \in X \times I$, then $E_x = \{y \in I : (x, y) \in E\}$ will be called the x-section of E, and $E^y = \{x \in X : (x, y) \in E\}$, the y-section of E.

Our purpose is to prove the measurability of a multifunction $F : X \times I \mapsto Z$ whose all x-sections are approximately h-continuous and almost everywhere h-continuous and all y-sections are $\mathcal{M}(X)$-measurable. In order to attain, we will need the notion of (J) property.

The (J) property for real functions of two real variables was introduced by L i p i ń s k i [8] and considered intensively by G r a n d e [3]. It is also generalized into the case of multifunction [7].

Definition 1. A bounded multifunction $F : X \times I \mapsto Z$ has the (J) property if for each $y \in I$, F^y is $\mathcal{M}(X)$-measurable, for each $x \in X$, F_x is $\mathcal{L}(\mathbb{R})$-measurable, and for each interval $P \subset I$, a multifunction $\Phi_P : X \mapsto Z$ given by

$$\Phi_P(x) = \int_P F(x, y)\, dy$$

is $\mathcal{M}(X)$-measurable.

A multifunction with the (J) property need not be measurable.

Example 1. Let $E \subset \mathbb{R}^2$ be Sierpiński's set, i.e., $E \notin \mathcal{L}(\mathbb{R}^2)$ and its x-sections and y-sections have at most two elements for any $(x, y) \in \mathbb{R}^2$. Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be given by

$$F(x, y) = \begin{cases} [0, 1] & \text{if} \quad (x, y) \notin E, \\ \{0\} & \text{if} \quad (x, y) \in E. \end{cases}$$

Then F is not $\mathcal{L}(\mathbb{R}^2)$-measurable, but F has the (J) property.

It is known that ([7, Prop. 2.24]):

Proposition 2. *If the space (Z, d) is separable and $F : X \times I \mapsto Z$ is a bounded multifunction such that F_x is R-integrable for each $x \in X$, and F^y is $\mathcal{M}(X)$--measurable for each $y \in I$, then F has the (J) property.*

We can now prove an important (for the final result) theorem on measurability of multifunctions with the (J) property.

Theorem 2. *Let us suppose that (Z, d) is separable. If a bounded multifunction $F : X \times I \mapsto Z$ has the (J) property and for each $x \in X$, F_x is a derivative, i.e.,*

$$F_x(y) = h - \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_y^{y+\Delta y} F_x(t)\, dt \qquad \text{for} \quad y \in I,$$

then F is measurable with respect to the $\mu \times m$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$.

P r o o f. Let $n \in \mathbb{N}$ be fixed and let $\Delta = \{y_{0,n}, y_{1,n}, \ldots, y_{n,n}\}$ be a partition of I into n equal intervals. Let us put

$$F_n(x,y) = \begin{cases} \frac{1}{y_{i,n}-y_{i-1,n}} \int_{y_{i-1,n}}^{y_{i,n}} F(x,y)\,dy & \text{if } x \in X \text{ and } y \in (y_{i-1,n}, y_{i,n}), \\ \{\theta\}, & \text{if } x \in X \text{ and } y = y_{i,n},\ i = 0,1,\ldots n. \end{cases}$$

Next, let $\Phi_{i,n} : X \mapsto Z$, for $i = 1,2,\ldots n$, be a multifunction given by

$$\Phi_{i,n}(x) = \int_{y_{i-1,n}}^{y_{i,n}} F(x,y)\,dy.$$

By the (J) property of F, we can see that

(10) the multifunction $\Phi_{i,n}$ is $\mathcal{M}(X)$-measurable for each $i = 1,2,\ldots n$.

Now, define a multifunction $\Phi_n : X \times \bigcup_{i=1}^{n} (y_{i-1,n}, y_{i,n}) \mapsto Z$ by

$$\Phi_n(x,y) = \Phi_{i,n}(x) \qquad \text{for} \quad y \in (y_{i-1,n}, y_{i,n}).$$

Then, (10) shows that

$$\Phi_n^{-}(V) = \bigcup_{i=1}^{n} \Phi_{i,n}^{-}(V) \times (y_{i-1,n}, y_{i,n}) \in \mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$$

whenever V is an open subset of Z. Consequently, F_n is $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$-measurable and, by (1), we only need to show that

(11) $h - \lim_{n\to\infty} F_n(x,y) = F(x,y)$ for every $x \in X$ and for almost every $y \in I$.

Let us fix $(x_0, y_0) \in X \times I$ such that $y_0 \neq y_{i,n}$ for $n \in \mathbb{N}$ and $i = 1,2,\ldots,n$, and choose a sequence $(y_{i_n,n})$ such that $y_{i_n-1} < y_0 < y_{i_n}$. Since F_{x_0} is the derivative at y_0, it follows that

$$F(x_0,y_0) = h - \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{y_0}^{y_0+\Delta y} F(x_0,y)\,dy.$$

Let us put

$$A_n = \frac{1}{y_0 - y_{i_n-1,n}} \int_{y_{i_n-1,n}}^{y_0} F(x_0,y)\,dy \quad \text{and} \quad B_n = \frac{1}{y_{i_n,n} - y_0} \int_{y_0}^{y_{i_n,n}} F(x_0,y)\,dy.$$

Then

$$\lim_{n\to\infty} h(A_n, F_0) = 0 \quad \text{and} \quad \lim_{n\to\infty} h(B_n, F_0) = 0, \tag{12}$$

where $F_0 = F(x_0, y_0)$.

Let us put $z_n = h\big(F_n(x_0, y_0), F_0\big)$. Note that

$$\begin{aligned} z_n &= h\left(\frac{1}{y_{i_n,n} - y_{i_n-1,n}} \int\limits_{y_{i_n-1,n}}^{y_{i_n,n}} F(x_0,y)\,dy, \frac{1}{y_{i_n,n} - y_{i_n-1,n}} \int\limits_{y_{i_n-1,n}}^{y_{i_n,n}} F_0\,dy\right) \\ &= \frac{1}{y_{i_n,n} - y_{i_n-1,n}} h\left(\int\limits_{y_{i_n-1,n}}^{y_{i_n,n}} F(x_0,y)\,dy, \int\limits_{y_{i_n-1,n}}^{y_{i_n,n}} F_0\,dy\right). \end{aligned}$$

By (7), we have

$$\int\limits_{y_{i_n-1,n}}^{y_{i_n,n}} F(x_0,y)\,dy = \int\limits_{y_{i_n-1,n}}^{y_0} F(x_0,y)\,dy + \int\limits_{y_0}^{y_{i_n,n}} F(x_0,y)\,dy$$

and

$$\int\limits_{y_{i_n-1,n}}^{y_{i_n,n}} F_0\,dy = \int\limits_{y_{i_n-1,n}}^{y_0} F_0\,dy + \int\limits_{y_0}^{y_{i_n,n}} F_0\,dy.$$

Next, (2) shows that

$$\begin{aligned} &h\left(\int\limits_{y_{i_n-1,n}}^{y_0} F(x_0,y)\,dy + \int\limits_{y_0}^{y_{i_n,n}} F(x_0,y)\,dy, \int\limits_{y_{i_n-1,n}}^{y_0} F_0\,dy + \int\limits_{y_0}^{y_{i_n,n}} F_0\,dy\right) \\ &\le h\left(\int\limits_{y_{i_n-1,n}}^{y_0} F(x_0,y)\,dy, \int\limits_{y_{i_n-1,n}}^{y_0} F_0\,dy\right) + h\left(\int\limits_{y_0}^{y_{i_n,n}} F(x_0,y)\,dy, \int\limits_{y_0}^{y_{i_n,n}} F_0\,dy\right). \end{aligned}$$

Moreover,

$$\frac{1}{y_{i_n,n} - y_{i_n-1,n}} < \frac{1}{y_0 - y_{i_n-1,n}} \quad \text{and} \quad \frac{1}{y_{i_n,n} - y_{i_n-1,n}} < \frac{1}{y_{i_n,n} - y_0}.$$

Therefore,

$$\begin{aligned} z_n < {} & \frac{1}{y_0 - y_{i_n-1,n}} h\left(\int\limits_{y_{i_n-1,n}}^{y_0} F(x_0,y)\,dy, \int\limits_{y_{i_n-1,n}}^{y_0} F_0\,dy\right) \\ &+ \frac{1}{y_{i_n,n} - y_0} h\left(\int\limits_{y_0}^{y_{i_n,n}} F(x_0,y)\,dy, \int\limits_{y_0}^{y_{i_n,n}} F_0\,dy\right), \end{aligned}$$

and finally,

$$h\big(F_n(x_0,y_0), F_0\big) < h(A_n, F_0) + h(B_n, F_0).$$

Thus, (12) shows that (11) is true, which finishes the proof of Theorem 2. □

As a straightforward consequence of Propositions 1 and 2, Theorems 1 and 2, we get the following corollary.

Corollary 1. *If the space (Z,d) is separable and a bounded multifunction $F\colon X \times I \mapsto Z$ has F_x approximately h-continuous and almost everywhere h-continuous for each $x \in X$ and F^y is $\mathcal{M}(X)$-measurable for each $y \in I$, then F is measurable with respect to the $\mu \times m$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$.*

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