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ON POINTWISE $\mathcal{M}$-CONTINUITY OF MAPPINGS

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ABSTRACT. Classical Levine's theorem [N. Levine: *Semi-open sets and semi--continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41] asserts that for a semi-continuous mapping on a second countable topological space, the discontinuity points form a 1st category set. There are two directions in literature in which this result is generalized: by considering either multi-valued mappings or mappings on some second noncountable spaces (for the latter, see for instance [T. Neubrunn: *Quasi-continuity (topical survey)*, Real Anal. Exchange **14** (1988/89), 259–306]). In this paper, we offer another path, namely, the path of so-called $\mathcal{M}$-spaces, essentially weaker than the topological ones. Pointwise $\mathcal{M}$-continuity of a mapping between two $\mathcal{M}$-spaces is defined and characterized. These characterizations are the basic tool for our generalization.

1. Preliminaries

A subfamily $\mathfrak{m}_X$ of the power set 2^X of a nonempty set X is called a *minimal structure* on X [3], [4], [6], if $\varnothing \in \mathfrak{m}_X$ and $X \in \mathfrak{m}_X$. A pair $(X, \mathfrak{m}_X)$ is then called an $\mathcal{M}$-space. For each subset S of the space X, the $\mathfrak{m}_X$-*closure* and the $\mathfrak{m}_X$-*interior* are defined, as in [3], in a usual manner:

1. $\mathfrak{m}_X\text{-cl}\,(S) = \bigcap\{F;\, S \subset F,\, X \setminus F \in \mathfrak{m}_X\}$,
2. $\mathfrak{m}_X\text{-int}\,(S) = \bigcup\{U;\, S \supset U,\, U \in \mathfrak{m}_X\}$.

A minimal structure $\mathfrak{m}_X$ being closed under any unions, is called a *supratopology* while the couple $(X, \mathfrak{m}_X)$ is called a *supratopological space* [1]. For an $\mathcal{M}$-space $(X, \mathfrak{m}_X)$, we write $\mathrm{c}(\mathfrak{m}_X) = \{F : X \setminus F \in \mathfrak{m}_X\}$. It is obvious that each topological space is supratopological and each supratopological space is an $\mathcal{M}$-space. The following two lemmas are necessary for the rest of this paper.

Lemma 1 ([3]). *Let A and B be any subsets of an $\mathcal{M}$-space $(X, \mathfrak{m}_X)$. The following properties hold:*

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1. (a) $\mathfrak{m}_X\text{-cl}\,(X \setminus A) = X \setminus \mathfrak{m}_X\text{-int}\,(A)$,
   (b) $\mathfrak{m}_X\text{-int}\,(X \setminus A) = X \setminus \mathfrak{m}_X\text{-cl}\,(A)$,
2. (a) *if* $X \setminus A \in \mathfrak{m}_X$, *then* $\mathfrak{m}_X\text{-cl}\,(A) = A$,
   (b) *if* $A \in \mathfrak{m}_X$ *then* $\mathfrak{m}_X\text{-int}\,(A) = A$,
3. (a) $\mathfrak{m}_X\text{-cl}\,(\varnothing) = \varnothing$, $\mathfrak{m}_X\text{-cl}(X) = X$,
   (b) $\mathfrak{m}_X\text{-int}\,(\varnothing) = \varnothing$, $\mathfrak{m}_X\text{-int}(X) = X$,
4. (a) *if* $A \subset B$, *then* $\mathfrak{m}_X\text{-cl}\,(A) \subset \mathfrak{m}_X\text{-cl}\,(B)$,
   (b) *if* $A \subset B$, *then* $\mathfrak{m}_X\text{-int}\,(A) \subset \mathfrak{m}_X\text{-int}\,(B)$,
5. (a) $A \subset \mathfrak{m}_X\text{-cl}\,(A)$,
   (b) $A \supset \mathfrak{m}_X\text{-int}\,(A)$,
6. (a) $\mathfrak{m}_X\text{-cl}\big(\mathfrak{m}_X\text{-cl}\,(A)\big) = \mathfrak{m}_X\text{-cl}\,(A)$,
   (b) $\mathfrak{m}_X\text{-int}\big(\mathfrak{m}_X\text{-int}\,(A)\big) = \mathfrak{m}_X\text{-int}\,(A)$.

Lemma 2 ([6])**.** *Consider an* $\mathcal{M}$*-space* $(X, \mathfrak{m}_X)$. *Let* $A \subset X$. *Then,* $x \in \mathfrak{m}_X\text{-cl}\,(A)$ *if and only if* $U \cap A \neq \varnothing$ *for every* $U \in \mathfrak{m}_X$ *containing* x.

2. $\mathcal{M}$-continuity at a point

Definition 1. Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be two $\mathcal{M}$-spaces. A mapping $f : (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ is said to be $\mathcal{M}$*-continuous at* $x \in X$ if, for every set $V \in \mathfrak{m}_Y$ with $f(x) \in V$, there exists $U \in \mathfrak{m}_X$ such that $x \in U$ and $U \subset f^{-1}(V)$ (equiv. $f(U) \subset V$). If x is not an $\mathcal{M}$-continuity point of f, then it is an $\mathcal{M}$*-discontinuity point* of f.

Theorem 1. *Let* $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ *be two supratopological spaces and let* $x_0 \in X$. *For any mapping* $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$, *the following statements are equivalent:*

(a) f *is* $\mathcal{M}$*-continuous at* x_0*;*

(b) *if* $x_0 \in f^{-1}(V)$, *then* $x_0 \in \mathfrak{m}_X\text{-int}\big(f^{-1}(V)\big)$ *for every* $V \in \mathfrak{m}_Y$*;*

(c) *if* $x_0 \in \mathfrak{m}_X\text{-cl}\,(S)$, *then* $f(x_0) \in \mathfrak{m}_Y\text{-cl}\big(f(S)\big)$ *for every* $S \subset X$*;*

(d) *if* $x_0 \in \mathfrak{m}_X\text{-cl}\big(f^{-1}(F)\big)$, *then* $x_0 \in f^{-1}(F)$ *for every* $F \in \mathrm{c}\,(\mathfrak{m}_Y)$.

For the equivalence (b)$\Leftrightarrow$(d), *it is enough to assume that* $(X, \mathfrak{m}_X)$ *and* $(Y, \mathfrak{m}_Y)$ *are* $\mathcal{M}$*-spaces.*

Proof. (a) $\Rightarrow$ (b). Let $V \in \mathfrak{m}_Y$ be an arbitrarily chosen set such that $x_0 \in f^{-1}(V)$. Then, $f(x_0) \in V$, and by hypothesis, there exists $U \in \mathfrak{m}_X$ such that $x_0 \in U \subset f^{-1}(V)$. Thus $x_0 \in \mathfrak{m}_X\text{-int}\big(f^{-1}(V)\big)$.

(b) $\Rightarrow$ (c). Suppose there exists a set $S \subset X$ such that $x_0 \in \mathfrak{m}_X\text{-cl}\,(S)$ and $f(x_0) \notin \mathfrak{m}_Y\text{-cl}\big(f(S)\big)$. Consider the set $V = Y \setminus \mathfrak{m}_Y\text{-cl}\big(f(S)\big) = \mathfrak{m}_Y\text{-int}\big(Y \setminus f(S)\big)$.

Since $\mathfrak{m}_Y$ is a supratopology on Y, $V \in \mathfrak{m}_Y$ (as it is a union of some family of sets from $\mathfrak{m}_Y$ containing $Y \setminus f(S)$). Moreover, $f(x_0) \in V$. Hence, $x_0 \in f^{-1}(V)$, and by hypothesis, $x_0 \in \mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right)$. Put $U = \mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right)$. Since $\mathfrak{m}_X$ is a supratopology $U \in \mathfrak{m}_X$, we have what follows:

$$\begin{aligned} U \cap S &= \mathfrak{m}_X\text{-int}\left(f^{-1}\left(Y \setminus \mathfrak{m}_Y\text{-cl}\left(f(S)\right)\right)\right) \cap S \subset f^{-1}\left(Y \setminus f(S)\right) \cap S \\ &= \left(X \setminus f^{-1}\left(f(S)\right)\right) \cap S = \varnothing. \end{aligned}$$

Thus, by Lemma 2, $x_0 \notin \mathfrak{m}_X\text{-cl}\,(S)$. It is in contrary to the assumption.

(c) $\Rightarrow$ (a). Suppose f is not $\mathcal{M}$-continuous at x_0, that is, there exists a set $V \in \mathfrak{m}_Y$ with $f(x_0) \in V$ such that, for any $U \in \mathfrak{m}_X$ with $x_0 \in U$, one has $U \not\subset f^{-1}(V)$. Set $S = X \setminus f^{-1}(V)$. We shall show that $x_0 \in \mathfrak{m}_X\text{-cl}\,(S)$ and $f(x_0) \notin \mathfrak{m}_Y\text{-cl}\left(f(S)\right)$. Namely, observe that $U \cap S = U \cap \left(X \setminus f^{-1}(V)\right) \neq \varnothing$ and by Lemma 2, $x_0 \in \mathfrak{m}_X\text{-cl}\,(S)$. On the other hand, $V \cap f(S) = f\left(f^{-1}(V) \cap S\right) = \varnothing$. So, by Lemma 2 once again, we obtain $f(x_0) \notin \mathfrak{m}_Y\text{-cl}\left(f(S)\right)$.

(b) $\Rightarrow$ (d). Let $F \in \mathrm{c}\,(\mathfrak{m}_X)$ be such a set that $x_0 \in \mathfrak{m}_X\text{-cl}\left(f^{-1}(F)\right)$ but $x_0 \notin f^{-1}(F)$. We have $x_0 \notin X \setminus \mathfrak{m}_X\text{-cl}\left(f^{-1}(F)\right) = \mathfrak{m}_X\text{-int}\left(X \setminus f^{-1}(F)\right) = \mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right)$, where $V = Y \setminus F \in \mathfrak{m}_Y$. Since $x_0 \in f^{-1}(V)$, we get (b) does not hold.

(d) $\Rightarrow$ (b). Suppose there exists $V \in \mathfrak{m}_Y$ such that $x_0 \notin \mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right)$ and $x_0 \in f^{-1}(V)$. Then $x_0 \in X \setminus \mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right) = \mathfrak{m}_X\text{-cl}\left(X \setminus f^{-1}(V)\right) = \mathfrak{m}_X\text{-cl}\left(f^{-1}(F)\right)$, where $F = Y \setminus V \in \mathrm{c}\,(\mathfrak{m}_Y)$. Observe that $f^{-1}(F) \cap f^{-1}(V) = \varnothing$, and since $x_0 \in f^{-1}(V)$, one gets $x_0 \notin f^{-1}(F)$. Therefore the condition (d) does not hold. This contradiction completes the proof. □

The following corollary clearly follows from the proof of Theorem 1.

Theorem 2. *Let $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ be two $\mathcal{M}$-spaces. If a mapping $f : (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ is $\mathcal{M}$-continuous at a point $x_0 \in X$, then the following hold:*

1. *if $x_0 \in f^{-1}(V)$, then $x_0 \in \mathfrak{m}_X\text{-int}\left(f^{-1}(V)\right)$ for every $V \in \mathfrak{m}_Y$,*
2. *if $x_0 \in \mathfrak{m}_X\text{-cl}\left(f^{-1}(F)\right)$, then $x_0 \in f^{-1}(F)$ for every $F \in \mathrm{c}\,(\mathfrak{m}_Y)$.*

Proof. It is enough to analyse the proof of implication (a)$\Rightarrow$(b) and to use the fact that (b)$\Leftrightarrow$(d) holds for $\mathcal{M}$-spaces. □

Definition 2. A subfamily $\beta_{\mathfrak{m}_X}$ of the minimal structure $\mathfrak{m}_X$ on X is said to be an $\mathfrak{m}$-*base for* $\mathfrak{m}_X$ if for every set $S \in \mathfrak{m}_X$ there exists $\beta_S \subset \beta_{\mathfrak{m}_X}$ with $S = \bigcup \beta_S$. If $\beta_{\mathfrak{m}_X}$ can be found at most countable, then the $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ is called $\mathfrak{m}$-*2nd countable.*

Lemma 3. *Let* $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ *be two* $\mathcal{M}$*-spaces,* $x_0 \in X$, $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$. *If* $\beta_{\mathfrak{m}_Y}$ *is an* $\mathfrak{m}$*-base for* $\mathfrak{m}_Y$, *then the condition* (b) *of Theorem 1 is equivalent to the following one:*

(b') *if* $x_0 \in f^{-1}(V)$, *then* $x_0 \in \mathfrak{m}_X\text{-int}\big(f^{-1}(V)\big)$ *for every* $V \in \beta_{\mathfrak{m}_Y}$.

Proof. (b)⇒(b') is obvious. (b')⇒(b). Let $x_0 \in f^{-1}(V)$, where V is any member of $\mathfrak{m}_Y$. There is a subfamily $\beta_V \subset \beta_{\mathfrak{m}_Y}$ such that $V = \bigcup \beta_V$. So, $x_0 \in f^{-1}\big(\bigcup \beta_V\big) = \bigcup_{U \in \beta_V} f^{-1}(U)$ and thus $x_0 \in f^{-1}(U)$ for some $U \in \beta_V$. By hypothesis and by 4.(b) of Lemma 1, $x_0 \in \mathfrak{m}_X\text{-int}\big(f^{-1}(U)\big) \subset \mathfrak{m}_X\text{-int}\big(f^{-1}(V)\big)$. □

A mapping $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$, where $(X, \mathfrak{m}_X)$ and $(Y, \mathfrak{m}_Y)$ are two $\mathcal{M}$-spaces, is *$\mathcal{M}$-continuous on* $(X, \mathfrak{m}_X)$ [6, Definition 3.3] if it is $\mathcal{M}$-continuous at each point $x \in X$. The set of all $\mathcal{M}$-discontinuity points of f will be denoted by $\mathcal{D}_f$. The following result is a consequence of conditions (b), (d) (resp.) of Theorem 1, and condition (b') of Lemma 3.

Theorem 3. *Let* $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ *be two supratopological spaces,* $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$. *We have*

1. $\mathcal{D}_f = \bigcup_{V \in \mathfrak{m}_Y} \big(f^{-1}(V) \setminus \mathfrak{m}_X\text{-int}(f^{-1}(V))\big)$,
2. $\mathcal{D}_f = \bigcup_{F \in \mathrm{c}\,(\mathfrak{m}_Y)} \big(\mathfrak{m}_X\text{-cl}\,(f^{-1}(F)) \setminus f^{-1}(F)\big)$,
3. $\mathcal{D}_f = \bigcup_{V \in \beta_{\mathfrak{m}_Y}} \big(f^{-1}(V) \setminus \mathfrak{m}_X\text{-int}\,(f^{-1}(V))\big)$, *where* $\beta_{\mathfrak{m}_Y}$ *is any* $\mathfrak{m}$*-base for* $\mathfrak{m}_Y$.

3. $\mathcal{M}$-semi-open sets, $\mathcal{M}$-semi-continuity and $\mathcal{M}$-discontinuity points

We start this section with some properties for subsets of any $\mathcal{M}$-space.

Lemma 4. *Consider an* $\mathcal{M}$*-space* $(X, \mathfrak{m}_X)$. *Let* $\{S_i\}_{i \in \mathcal{I}} \subset 2^X$. *Then,*

1. $\mathfrak{m}_X\text{-int}\big(\bigcap_{i \in \mathcal{I}} S_i\big) \subset \bigcap_{i \in \mathcal{I}} \mathfrak{m}_X\text{-int}\,(S_i)$,
2. $\mathfrak{m}_X\text{-cl}\big(\bigcap_{i \in \mathcal{I}} S_i\big) \subset \bigcap_{i \in \mathcal{I}} \mathfrak{m}_X\text{-cl}\,(S_i)$,
3. $\mathfrak{m}_X\text{-int}\big(\bigcup_{i \in \mathcal{I}} S_i\big) \supset \bigcup_{i \in \mathcal{I}} \mathfrak{m}_X\text{-int}\,(S_i)$,
4. $\mathfrak{m}_X\text{-cl}\big(\bigcup_{i \in \mathcal{I}} S_i\big) \supset \bigcup_{i \in \mathcal{I}} \mathfrak{m}_X\text{-cl}\,(S_i)$.

Proof. 1. Let $x \in \mathfrak{m}_X\text{-int}\big(\bigcap_{i \in \mathcal{I}} S_i\big)$. Then, there is a $U \in \mathfrak{m}_X$ such that $x \in U \subset \bigcap_{i \in \mathcal{I}} S_i$. Obviously, we get $U \subset S_i$ for each $i \in \mathcal{I}$, that is, $x \in \mathfrak{m}_X\text{-int}\,(S_i)$ for each $i \in \mathcal{I}$.
2. Let $x \in \mathfrak{m}_X\text{-cl}\big(\bigcap_{i \in \mathcal{I}} S_i\big)$. By Lemma 2, one has $U \cap \bigcap_{i \in \mathcal{I}} S_i \neq \varnothing$ for each $U \in \mathfrak{m}_X$ with $x \in U$. Thus, $U \cap S_i \neq \varnothing$, for each $i \in \mathcal{I}$, and consequently,

$x \in \mathfrak{m}_X\text{-cl}\,(S_i)$. Statement 3. follows by 2. and 1.(b) of Lemma 1. Statement 4. follows by 1. and 1.(a) of Lemma 1. $\square$

Definition 3 (see [3, Definition 2.4(b)] or [4, Definition 2.4(b)]). Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. A set $A \subset X$ is said to be $\mathfrak{m}_X$*-semi-open* (or $\mathcal{M}$*-semi-open*) if there exists a set $U \in \mathfrak{m}_X$ such that

$$U \subset A \subset \mathfrak{m}_X\text{-cl}\,(U).$$

It is known (see [3, Theorem 2.7(i)] or [4, Theorem 2.7(i)]) that if A is $\mathfrak{m}_X$-semi-open, then $A \subset \mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(A)\big)$. Lemmas 5 and 6 slightly reformulate some known results.

Lemma 5 ([4, Lemma 3.3]). *For an* $\mathcal{M}$*-space* $(X, \mathfrak{m}_X)$*, the following two conditions are equivalent:*

1. $(X, \mathfrak{m}_X)$ *is a supratopological space;*
2. *if* $\mathfrak{m}_X\text{-cl}\,(F) = F$ *then* $X \setminus F \in \mathfrak{m}_X$ *for each* $F \subset X$*.*

It is necessary to prove Lemma 5.

Lemma 6 ([6, Theorem 2.7(ii)]). *Let* $(X, \mathfrak{m}_X)$ *be a supratopological space. Then, a subset* A *of the space* X *is* $\mathfrak{m}_X$*-semi-open if and only if* $A \subset \mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(A)\big)$.

The family of all $\mathfrak{m}_X$-semi-open subsets of an $\mathcal{M}$-space $(X, \mathfrak{m}_X)$ will be denoted by $\mathrm{SO}\,(X, \mathfrak{m}_X)$. According to Lemma 4 (3. and 4.), and Lemma 6, the following result may be easily observed.

Lemma 7. *If* $(X, \mathfrak{m}_X)$ *is a supratopological space, then so is* $\big(X, \mathrm{SO}\,(X, \mathfrak{m}_X)\big)$.

P r o o f. It is sufficient to use Lemma 4 (3. and 4.) and Lemma 6. Namely, for any family $\{S_i\}_{i\in\mathfrak{I}} \subset \mathrm{SO}\,(X, \mathfrak{m}_X)$ we have what follows:

$$\begin{aligned} \bigcup_{i\in\mathfrak{I}} S_i &\subset \bigcup_{i\in\mathfrak{I}} \mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(S_i)\big) \\ &\subset \mathfrak{m}_X\text{-cl}\left(\bigcup_{i\in\mathfrak{I}} \mathfrak{m}_X\text{-int}\,(S_i)\right) \\ &\subset \mathfrak{m}_X\text{-cl}\left(\mathfrak{m}_X\text{-int}\left(\bigcup_{i\in\mathfrak{I}} S_i\right)\right). \end{aligned}$$

Thus, $\bigcup_{i\in\mathfrak{I}} S_i \in \mathrm{SO}\,(X, \mathfrak{m}_X)$. $\square$

Lemma 8. (a) *In an* $\mathcal{M}$*-space* $(X, \mathfrak{m}_X)$*,* $A \in \mathrm{SO}\,(X, \mathfrak{m}_X)$ *implies*

$$\mathfrak{m}_X\text{-cl}\,(A) = \mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(A)\big).$$

(b) *If* $(X, \mathfrak{m}_X)$ *is a supratopological space, then, moreover, the above identity implies* $A \in \mathrm{SO}\,(X, \mathfrak{m}_X)$.

P r o o f. (a) Let $A \in \mathrm{SO}(X, \mathfrak{m}_X)$. Then, $A \subset \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(A))$. Using 4.(a) of Lemma 1, we have $\mathfrak{m}_X\text{-cl}\,(A) \subset \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(A)))$ and so, by 6.(a) of the same lemma, $\mathfrak{m}_X\text{-cl}\,(A) \subset \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(A))$. On the other hand, $A \supset \mathfrak{m}_X\text{-int}\,(A)$ obviously implies $\mathfrak{m}_X\text{-cl}\,(A) \supset \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(A))$. Thus, $\mathfrak{m}_X\text{-cl}\,(A) = \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(A))$.

(b) By 5.(b) of Lemma 1 and our hypothesis, one gets $\mathfrak{m}_X\text{-int}\,(A) \subset A \subset \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(A))$. Then, since the space $(X, \mathfrak{m}_X)$ is supratopological, by Definition 3, we directly obtain that $A \in \mathrm{SO}(X, \mathfrak{m}_X)$. □

Corollary 1. *If $(X, \mathfrak{m}_X)$ is a supratopological space, $A \in \mathrm{SO}(X, \mathfrak{m}_X)$ if and only if $\mathfrak{m}_X\text{-cl}\,(A) = \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(A))$.*

Lemma 9 ([6, Corollary 3.1])**.** *Let $(X, \mathfrak{m}_X)$ be a supratopological space and $(Y, \mathfrak{m}_Y)$ be an $\mathcal{M}$-space. Then, for any mapping $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$, the following two conditions are equivalent:*

1. *f is $\mathcal{M}$-continuous;*
2. *$f^{-1}(V) \in \mathfrak{m}_X$ for each $V \in \mathfrak{m}_Y$.*

Definition 4. Let $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ be $\mathcal{M}$-spaces. A mapping $f\colon X \to Y$ is said to be *$\mathcal{M}$-semi-continuous* if $f^{-1}(V) \in \mathrm{SO}(X, \mathfrak{m}_X)$ for each $V \in \mathfrak{m}_Y$.

It is worth noticing the following.

Theorem 4. *Let $(X, \mathfrak{m}_X)$ be a supratopological space and $(Y, \mathfrak{m}_Y)$ be an $\mathcal{M}$-space. A mapping $f\colon X \to Y$ is $\mathcal{M}$-semi-continuous if and only if f is $\mathcal{M}$-continuous if defined on $(X, \mathrm{SO}(X, \mathfrak{m}_X))$ instead of $(X, \mathfrak{m}_X)$.*

P r o o f. As $(X, \mathrm{SO}(X, \mathfrak{m}_X))$ is a supratopological space (Lemma 7), the conclusion follows from Lemma 9. □

Definition 5. Let $(X, \mathfrak{m}_X)$ be an $\mathcal{M}$-space. A subset A of the space X is said to be *$\mathfrak{m}_X$-nowhere dense* if $\mathfrak{m}_X\text{-int}\,(\mathfrak{m}_X\text{-cl}\,(A)) = \varnothing$. A is said to be *of $\mathfrak{m}_X$-first category* if A can be written as an at most countable union of $\mathfrak{m}_X$-nowhere dense sets.

Lemma 10. *For any subset S of an $\mathcal{M}$-space $(X, \mathfrak{m}_X)$, the set*

$$\mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(S)) \setminus \mathfrak{m}_X\text{-int}\,(S)$$

is $\mathfrak{m}_X$-nowhere dense.

P r o o f. Set $A = \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(S)) \setminus \mathfrak{m}_X\text{-int}\,(S)$. Since $\mathfrak{m}_X\text{-int}\,(S) \subset \mathfrak{m}_X\text{-cl} \times (\mathfrak{m}_X\text{-int}\,(S))$ (5.(a) of Lemma 1), we have

$$A = (X \setminus \mathfrak{m}_X\text{-int}\,(S)) \cap \mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(S)).$$

Using 2. of Lemma 4, we get

$$\mathfrak{m}_X\text{-cl}\,(A) \subset \mathfrak{m}_X\text{-cl}\,\big(X \setminus \mathfrak{m}_X\text{-int}\,(S)\big) \cap \mathfrak{m}_X\text{-cl}\,\Big(\mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(S)\big)\Big),$$

and by 1.(a) and 6. of Lemma 1,

$$\mathfrak{m}_X\text{-cl}\,(A) \subset X \setminus \mathfrak{m}_X\text{-int}\,(S) \cap \mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(S)\big),$$

respectively. Then by 4.(b) of Lemma 1, 1. of Lemma 4, and 1.(b) of Lemma 1, we obtain

$$\begin{aligned}\mathfrak{m}_X\text{-int}\,\big(\mathfrak{m}_X\text{-cl}\,(A)\big) &\subset \mathfrak{m}_X\text{-int}\,\big(X \setminus \mathfrak{m}_X\text{-int}\,(S)\big) \cap \mathfrak{m}_X\text{-int}\,\Big(\mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(S)\big)\Big)\\ &= \Big(X \setminus \mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(S)\big)\Big) \cap \mathfrak{m}_X\text{-int}\,\Big(\mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(S)\big)\Big).\end{aligned}$$

However, $\mathfrak{m}_X\text{-int}\,\big(\mathfrak{m}_X\text{-cl}\,(\mathfrak{m}_X\text{-int}\,(S))\big) \subset \mathfrak{m}_X\text{-cl}\,\big(\mathfrak{m}_X\text{-int}\,(S)\big)$ (5.(b) of Lemma 1), so $\mathfrak{m}_X\text{-int}\,\big(\mathfrak{m}_X\text{-cl}\,(A)\big) = \varnothing$. The proof is complete. □

Now, we are ready to prove the main result of this section.

Theorem 5. *Let $(X, \mathfrak{m}_X)$, $(Y, \mathfrak{m}_Y)$ be supratopological spaces with $(Y, \mathfrak{m}_Y)$ being $\mathfrak{m}_X$-2nd countable, and let $f\colon (X, \mathfrak{m}_X) \to (Y, \mathfrak{m}_Y)$ be $\mathcal{M}$-semi-continuous. Then, the set $\mathcal{D}_f$ is of $\mathfrak{m}_X$-first category.*

P r o o f. Let $\beta_{\mathfrak{m}_Y}$ be a countable $\mathfrak{m}$-base for $\mathfrak{m}_Y$. By 3. of Theorem 3, we have

$$\mathcal{D}_f \subset \bigcup_{V \in \beta_{\mathfrak{m}_Y}} \Big[\mathfrak{m}_X\text{-cl}\,\Big(\mathfrak{m}_X\text{-int}\,\big(f^{-1}(V)\big)\Big) \setminus \mathfrak{m}_X\text{-int}\,\big(f^{-1}(V)\big)\Big].$$

Thus, by Lemma 10, the desired result follows. □

Theorem 5 generalizes the well-known Levine's result. Indeed, putting in the above theorem $\mathfrak{m}_X = \tau$ and $\mathfrak{m}_Y = \sigma$, where τ and σ are topologies on X and Y, respectively, one has

Corollary 2 ([2, Theorem 13])**.** *Let $f\colon (X, \tau) \to (Y, \sigma)$ be a semi-continuous mapping and let (Y, σ) be a 2nd axiom space. Then, the set of all discontinuity points of f is of first category.*

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