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NULL SETS WITH RESPECT TO A CONTINUOUS FUNCTION

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Dedicated to Władysław Wilczyński on the occasion of his 65th birthday

ABSTRACT. This short paper concerns "peso nullo" subsets of the real line defined in [Caccioppoli, R.: *L'integrazione e la ricerca delle primitive rispetto ad una funzione continua qualunque*, Ann. Mat. Pura Appl. **40** (1955), 15–34]. The framework is that of integration with respect to a function g which is continuous but not necessarily of bounded variation. Here we shall call these sets g-null. Since the family of g-null sets is a σ-ideal, the natural question is whether it is a family of null sets with respect to a Borel measure on the real line. The paper gives a negative answer to this question.

In the sequel, λ is the Lebesgue measure on the line $\mathbb{R}$.

In [5, p. 308], H. Lebesgue asks about the possibility of developing a theory of (Stieltjes) integration with respect to a function g which is continuous but not of bounded variation. Of course, the question of developing a theory of differentiation with respect to such functions is also an interesting problem. See the book of S. Saks [8, p. 272], and the paper by J. Liberman [6].

Obviously, the notion of a measure with respect to g can be avoided when dealing with this subject, but the notion of a "negligible set" is necessary, and Caccioppoli suggests the notion of a set of *"peso nullo" with respect to g* that we shall call a g-null set. For an exact description, see [4].

To give an idea of the definition, consider a continuous function g on an interval $I = [a, b]$. Take now a sequence (g_n) of continuous functions on I and a sequence (H_n) of subsets of I with the following properties:

- g_n is piecewise monotone,
- g_n coincides with g on H_n,
- $I \setminus H_n$ is a disjoint union of open intervals and g_n is constant on each of them.

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Following Caccioppoli [4], if the sequence (H_n) is increasing and the sequence (g_n) uniformly converges to g, we call (g_n) a *generating sequence* for g. Caccioppoli proves that generating sequences exist for every continuous g and calls a subset X of I g-null if and only if for every generating sequence (g_n) of g, $\lambda\big(g_n(X)\big) = 0$ for every n. Obviously, if $\lambda\big(g(X)\big) = 0$, then X is g-null. He conjectured that even the converse holds, but in [1] it is proved that this is not true.

The following proposition gives a characterization of g-null sets.

Proposition 1. *Let g be a continuous function, with no interval of constancy, defined on an interval $[a, b]$.*

A set $X \subset [a, b]$ is g-null if and only if $\lambda\big(g(A)\big) = 0$ for any set $A \subseteq X$ on which g is monotone.

Proof. Let X be g-null, $A \subseteq X$, and g increasing on A.

Denote by C a maximal set containing A such that g is increasing on C. Because of the maximality of C, we have that C is closed. Moreover, if $[a_n, b_n]$ is an interval of the complement of C, then $g(a_n) = g(b_n)$. For every n, let us extend $g_{/C}$ defining it constant on $[a_n, b_n]$; the new function will be called $\overline{g}$. Since A is g-null, for every generating sequence (g_n) of g, we have $\lambda\big(g_n(A)\big) = 0$ for every n. But $\overline{g}$ can be included in a generating sequence of g, so $\lambda\big(\overline{g}(A)\big) = 0$ and then $\lambda\big(g(A)\big) = 0$.

For the converse, suppose that $\lambda\big(g(A)\big) = 0$ for every $A \subseteq X$ such that g is monotone on A. Let (g_n) be a generating sequence for g. Then, for each $n \in \mathbb{N}$,

$$\lambda\big(g_n(X)\big) = \lambda\big(g_n(X \cap H_n)\big) + \lambda\big(g_n(X \setminus H_n)\big) = \lambda\big(g(X \cap H_n)\big) + \lambda\big(g_n(X \setminus H_n)\big) = 0.$$

□

For every g, the family of g-null sets is a σ-ideal.

Following the so called method I and method II of M. E. Munroe [7], a Borel measure can be defined in a canonical way from g: if X is a subset of the line, for every $p \in \mathbb{N}$ define

$$\mu_{g,p}(X) = \inf \sum_{k \in \mathbb{N}} |g(x'_k) - g(x''_k)|,$$

where $\big([x'_k, x''_k]\big)$ is a sequence of intervals covering X each having a length less than $\frac{1}{p}$; take then

$$\mu_g(X) = \lim_p \mu_{g,p}(X).$$

There is a long list of authors whose papers concern the measures generated by different types of functions. References can be found in the papers referred here and obviously on the web. For example, in [3], the problem of signed measures generated by continuous functions is studied.

In [1, Propositions (2.1) and (2.2)], it is proved that

Proposition 2. *A g-null set is μ_g-null.*

Proposition 3. *If every μ_g-null set has a λ-null image by g, then every g-null set has a λ-null image.*

As observed, the converse of Proposition 2 is not true. Moreover, a μ_g-null set has not necessarily a λ-null image. On the other hand, it is easy to prove that:

Proposition 4. *For every Borel measure μ in $[a,b]$ there exists a continuous function g such that the family of g-null sets coincides with the family of μ-null sets.*

Proof. It suffices to choose g as the function

$$g(x) = \mu\big([a,x[\big)$$

which generates the measure μ. □

The next proposition proves a bit more.

Proposition 5. *Let $[a,b]$ be an interval on the line. There exists a continuous function g on $[a,b]$ such that for every Borel measure μ there exists a μ-null set which is not g-null.*

Proof. Since we use a recursive procedure, we start by the description of a general construction.

Let $[c,d]$ be an interval. Take two numbers u and v, with $u < v$, and denote by w the middle point $w = (u+v)/2$. Take four intervals in $[c,d]$: $I_1 = [x_1, y_1] \prec I_2 = [x_2, y_2] \prec I_3 = [x_3, y_3] \prec I_4 = [x_4, y_4]$ $(y_i < x_{i+1})$ with $x_1 = c$ and $y_4 = d$. Now, define a function f on $[c,d]$ by

$$\begin{aligned} f(c) &= f(x_2) = u, \\ f(y_1) &= f(y_2) = f(x_3) = f(x_4) = w, \\ f(y_3) &= f(d) = v \end{aligned}$$

and further, extend f to $[c,d]$ connecting the points of the graph $\big(x_i, f(x_i)\big)$ and $\big(y_i, f(y_i)\big)$ linearly. To start, we observe that there is no loss of generality if we focus on $[0,1]$.

We apply the procedure to the interval $[0,1]$, choosing $u = 0$ and $v = 1$. We obtain a function from $[0,1]$ onto $[0,1]$ that we call g_1. Denote by I_i^1, $i = 1, \cdots, 4$ the intervals obtained in this first step and note that

$$g_1\big(I_1^1\big) = g_1\big(I_2^1\big) = [0, 1/2], \qquad g_1\big(I_3^1\big) = g_1\big(I_4^1\big) = [1/2, 1].$$

g_1 is increasing on $I_1 \cup I_3$ and on $I_2 \cup I_4$.

In step 2, apply the procedure to each interval I_i^1, choosing $u = \inf g_1(I_i^1)$ and $v = \sup g_1(I_i^1)$. By the procedure, we obtain g_2 on each I_i^1 and we linearly connect the points of the graph to define g_2 on the remaining points of $[0,1]$. If I_i^2, $i = 1, 2, \dots, 16$ are now the intervals in this step, $g_2(I_i^2)$ is of the type $\big[k/4, (k+1)/4\big]$.

In step $n-1$, we have $2^{2(n-1)}$ intervals I_i^{n-1} and a function g_{n-1}. In step n, we apply the procedure to intervals I_i^{n-1} choosing $u = \inf g_{n-1}(I_i^{n-1})$ and $v = \sup g_{n-1}(I_i^{n-1})$. We obtain 2^{2n} intervals I_i^n and a function g_n that maps I_i^n onto an interval of the type $\big[k/2^n, (k+1)/2^n\big]$.

Now we have a sequence of continuous functions (g_n) defined on $[0,1]$ and, for each x in $[0,1]$, put

$$g(x) = \lim_n g_n(x).$$

Cauchy criterion ensures that the convergence is uniform.
Let σ be any sequence with $\sigma_i \in \{0,1\}$. For each n, define

$$D_\sigma^n = \bigcap_{i=1}^{n} \bigcup_{p=1}^{2^{2i-1}} I^i_{2p-\sigma_i}, \qquad D_\sigma = \bigcap_n D_\sigma^n.$$

We easily verify that if σ' and σ'' are two different sequences, then

$$D_{\sigma'} \cap D_{\sigma''} = \emptyset.$$

The set D_σ^n is the disjoint union of 2^n closed intervals of the type I_i^n and g_n is increasing on D_σ^n. This implies that g is increasing on D_σ. It is not difficult to prove that for each k and for each n, $k/2^n \in g(D_\sigma)$[1]. Hence, $g(D_\sigma)$ is dense. It is also closed, so $g(D_\sigma) = [0,1]$.

Proposition 1 implies that D_σ is not a $g-$null set. But $(D_\sigma)_\sigma$ is a non denumerable disjoint family of subsets of $[0,1]$ and so, for every Borel measure μ, at least one of the sets of the family must be μ-null because of the Countable Chain Condition. This completes the proof. □

Let $\mathcal{P}_g$ be a σ-ideal of g-null sets. It is easy to prove that

Proposition 6. *Let g be a continuous function defined on $[0,1]$ with no intervals of constancy. Then:*

- *$\mathcal{P}_g$ does not coincide with the family $\mathcal{H}$ of σ-porous sets.*
- *$\mathcal{P}_g$ does not coincide with the family $\mathcal{I}$ of sets of first category.*

[1] $x \in \cap_n I_1^n \Rightarrow g_n(x) \in g_n(\cap_n I_1^n) \subseteq \cap_n g_n(I_1^n) = \cap_n [0, 1/2^n] = \{0\}$. This argument shows that 0 is a value of the function g on the set D_σ. To prove that $k/2^n$ is a value of g, start with the interval I_r^n of the nth step such that $g_n(I_r^n) = \big[k/2^n, (k+1)/2^n\big]$. For $h \geq n$, g_h maps the first interval of each succeeding decomposition of I_r^n onto $\big[k/2^n, k/2^n + 1/2^{n+h}\big]$.

P r o o f. Divide $g\big([0,1]\big)$ in a set A of first category and a λ-null set B. Since g is continuous and not constant in any interval, the inverse image of a nowhere dense set is nowhere dense [2], hence $g^{-1}(A)$ is of first category. According to a theorem by J. T k a d l e c [9, Th. 2], $g^{-1}(B)$ contains a non-σ-porous set Z. Z is g-null since its image is λ-null. So, $\mathcal{P}_g \neq \mathcal{H}$.

To prove that $\mathcal{P}_g \neq \mathcal{I}$, let C be a subset of $[0,1]$ with $C \notin \mathcal{P}_g$. We have that $g^{-1}\big(B \cap g(C)\big)$ is g-null and $g^{-1}\big(A \cap g(C)\big) \in \mathcal{I} \setminus \mathcal{P}_g$. □

An interesting open problem related to the results of this paper is: suppose that g is a continuous function such that $\mathcal{P}_g$ is affine invariant. Does $\mathcal{P}_g$ coincide with the λ-null sets?

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REFERENCES

[1] AVERSA, V.: *Sugli insiemi di peso nullo rispetto ad una funzione continua*, Ricerche Mat. **26** (1977), 79–84.

[2] AVERSA, V.—PREISS, D.: *Lusin's theorem for derivatives with respect to a continuous function*, Proc. Amer. Math. Soc. **127** (1999), 3229–3235.

[3] BROWN, B. H.: *A measure on the real line constructed from an arbitrary point function*, Proc. London Math. Soc. (3) **27** (1973), 1–21.

[4] CACCIOPPOLI, R.: *L'integrazione e la ricerca delle primitive rispetto ad una funzione continua qualunque*, Ann. Mat. Pura Appl. **40** (1955), 15–34.

[5] LEBESGUE, H. L.: *Leçons sur L'intégration et la Recherche des Fonctions Primitives.* AMS Chelsea Publ., Providence, RI, 1973.

[6] LIBERMAN, J.: *Théorème de Denjoy sur la dérivée d'une fonction arbitraire par rapport à une fonction continue*, Rec. Math. Moscow, n. Ser. 9, **51** (1941), 221–236.

[7] MUNROE, M. E.: *Measure and Integration* (2nd ed.). Addison-Wesley Publ. Comp., Reading, Mass., 1971.

[8] SAKS, S.: *Theory of the Integral* (2nd ed.). Dover Publ., Mineola, NY, 1964.

[9] TKADLEC, J.: *Construction of some non-σ-porous set on the real line*, Real Anal. Exchange **9** (1983–1984), 473–482.

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[2] If $]a,b[\subseteq \overline{g^{-1}(A)}$, then $g\big(]a,b[\big) \subseteq g\big(\overline{g^{-1}(A)}\big) \subseteq g\big(g^{-1}(\overline{A})\big) \subseteq \overline{A}$.