VERSITA

Tatra Mountains
Mathematical Publications
DOI: 10.2478/v10127-012-0032-x
Tatra Mt. Math. Publ. **52** (2012), 115–131

MODES OF CONTINUITY INVOLVING ALMOST AND IDEAL CONVERGENCE

Antonio Boccuto — Xenofon Dimitriou — Nicolas Papanastassiou

ABSTRACT. We investigate some properties of almost convergence and compare it with other kinds of convergence. Moreover, we present different modes of continuity considering almost, Single and ideal convergence, giving some relations between them and some examples. Finally, we pose some open problems.

1. Introduction

Almost convergence was deeply investigated by G. G. Lorentz in [12], who studied several of its features. Some properties of continuous functions with respect to almost convergence and some relations between continuous and linear functions were investigated in [2], [3], [6]. In particular, in [6] it was shown that every function $f\colon \mathbb{R} \to \mathbb{R}$, sequentially continuous with respect to almost convergence, is linear. Similar studies were continued in [7], in which abstract methods of convergence, and in particular some types of matrix methods, were considered (see also [15]). Some other investigations about statistical convergence and (strong) summability in the context of matrix methods were done in [10].

In [11, Proposition 3.3] a characterization of the classical continuity was given by means of ideal convergent sequences. Some relations between different types of ideal continuity were examined in [5]. A particular case of ideal convergence, which can be viewed as generated both by an ideal of $\mathbb{N}$ and by a summability matrix method associated with a positive regular matrix is the statistical convergence, introduced in [8], [17] (see also [7], [9], [11]).

2010 Mathematics Subject Classification: Primary: 26A03, 26A09, 26A15, 40A35, 54A20, 54C05, 54C08; Secondary: 40C05, 40G15.
Keywords: almost convergence, Single convergence, ideal convergence, Pringsheim-Single convergence, (Lorentz) uniform equal convergence, modes of continuity.
Supported by Universities of Perugia and Athens.

In Section 2 we give a characterization of almost convergence, and we compare it with the Single convergence. Moreover, we investigate some differences between almost and ideal convergence.

In Section 3 we present some different kinds of sequential continuity in terms of almost and ideal convergence, and give some comparison results. Finally, we pose some open problems.

Our thanks to the referee for his/her helpful suggestions.

2. Almost and ideal convergence

In this section we give a characterization of almost convergence and we show that it is in general different from ideal and Single convergence.

Definitions 2.1.

(a) An *ideal* of $\mathbb{N}$ is a subset $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ with $\mathbb{N} \notin \mathcal{I}$, $A \cup B \in \mathcal{I}$ whenever A, $B \in \mathcal{I}$, and with the property that, if $C \in \mathcal{I}$ and $D \subset C$, then $D \in \mathcal{I}$.

(b) A *filter* of $\mathbb{N}$ is a subset $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ with $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, and with $F \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $F \supset E$.

(c) An ideal $\mathcal{I}$ is called *admissible* if and only if it contains the ideal $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$.

(d) A filter $\mathcal{F}$ is said to be *free* if and only if it contains the filter $\mathcal{F}_{\text{cofin}}$ of the cofinite subsets of $\mathbb{N}$.

(e) Given an ideal $\mathcal{I}$ of $\mathbb{N}$, we call the set

$$\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$$

the dual filter associated with $\mathcal{I}$. In this case we say that $\mathcal{I}$ is the *dual ideal* associated with $\mathcal{F}$.

Observe that an ideal is admissible if and only if its dual filter is free. From now on, we always deal with admissible ideals or free filters.

(f) We say that $\mathcal{I}$ is a *P-ideal* if and only if for any sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subset \mathbb{N}$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$ (see also [4]).

$\mathcal{I}_{\text{fin}}$ and $\mathcal{I}_{\text{d}}$ (the ideal of all subsets of $\mathbb{N}$ with zero asymptotic density (see also [11])) are some examples of P-ideals.

The following result will be useful in the sequel.

Proposition 2.2. *If $\mathcal{F}$ is a free filter of $\mathbb{N}$, $P \in \mathcal{F}$ and $A \subset \mathbb{N}$, $A \notin \mathcal{F}$ and $\mathbb{N} \setminus A \notin \mathcal{F}$, then both $P \cap A$ and $P \setminus A$ are infinite.*

Proof. Suppose by contradiction that $P \cap A$ is finite. Note that $\mathbb{N} \setminus (P \cap A)$ belongs to $\mathcal{F}$, since it is cofinite and $\mathcal{F}$ is free. Then

$$P \setminus A = P \cap \big(\mathbb{N} \setminus (P \cap A)\big) \in \mathcal{F}.$$

Hence $\mathbb{N} \setminus A \in \mathcal{F}$, which contradicts the hypothesis. Thus, $P \cap A$ is infinite. Analogously it is possible to prove that $P \setminus A$ is infinite. This finishes the proof. $\square$

We now recall the ideal, Single and almost convergence, and the Pringsheim convergence for double sequences (see also [12], [13]). From now on, let $(X, \|\cdot\|)$ be a normed linear space, $X \neq \{0\}$, and $x_0 \in X$ be a fixed point.

Definitions 2.3.

(a) A sequence $(x_n)_n$ in X is said to be *$\mathcal{I}$-convergent* to x_0, and we write $(\mathcal{I}) \lim_n x_n = x_0$, if and only if $\{n \in \mathbb{N} : \|x_n - x_0\| > \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$.

(b) Let $y_{m,n}$, $m \geq 0$, $n \geq 1$ be a double sequence in X. We say that x_0 is the *Pringsheim limit* of $(y_{m,n})_{m,n}$ if and only if for all $\varepsilon > 0$ there is $\overline{n} \in \mathbb{N}$ with $\|y_{m,n} - x_0\| \leq \varepsilon$ for all $m, n \geq \overline{n}$.

(c) Given a sequence $(x_n)_n$ in X, we say that $(x_n)_n$ *almost converges* (*(A)-converges*) to x_0, and we write $(A) \lim_n x_n = x_0$, if and only if $\lim_n y_{m,n} = x_0$ uniformly with respect to m, where

$$y_{m,n} := \frac{1}{n} \sum_{i=1}^{n} x_{m+i}, \qquad m \geq 0, \quad n \geq 1. \tag{1}$$

(d) A sequence $(x_n)_n$ in X *Pringsheim-Singly converges* (*(PS)-converges*) to x_0, and we write shortly $(PS) \lim_n x_n = x_0$, if and only if x_0 is the Pringsheim limit of $(y_{m,n})_{m,n}$, where $(y_{m,n})_{m,n}$ is as in (1).

(e) A sequence $(x_n)_n$ in X *Singly converges* (*(S)-converges*) to x_0, and we write in brief $(S) \lim_n x_n = x_0$, if and only if $\lim_n y_{m,n} = x_0$ for all $m \geq 0$, where $(y_{m,n})_{m,n}$ is as in (1).

(f) Fix $x_0 \in X$. Given $\varepsilon > 0$, we say that an integer $m \geq 0$ is *ε-stable* with respect to a double sequence $(y_{m,n})_{m,n}$ if and only if there is an integer $\overline{n} = \overline{n}(\varepsilon, m)$, with the property that $\|y_{m,n} - x_0\| \leq \varepsilon$ for all $n \geq \overline{n}$.

Note that a sequence $(x_n)_n$ Singly converges to x_0 if and only if, for each $\varepsilon > 0$, every integer $m \geq 0$ is ε-stable with respect to the double sequence $(y_{m,n})_{m,n}$ defined in (1).

We will prove the equivalence between (A)- and (PS)-convergence. First, we compare almost and Singly convergences. Of course, (A)-convergence implies (S)-convergence. We now give an example of sequence, Singly but not almost convergent.

Example 2.4. Let $X = \mathbb{R}, x_0 = \frac{1}{2}$, and let us construct a sequence $(x_k)_k$ as follows. Set $x_1 = 0$, and for all $n \in \mathbb{N}$ put $x_k = 1$ if $n^2 + 1 \le k \le n^2 + n$ and $x_k = 0$ if $n^2 + n + 1 \le k \le (n+1)^2$. Observe that

$$\frac{1}{k}\sum_{i=1}^{k} x_i \le \frac{1}{2} \qquad \text{for all} \quad k \in \mathbb{N}. \tag{2}$$

Moreover, for every $n \in \mathbb{N}$ and $n^2 + 1 \le k \le (n+1)^2$, we get

$$\frac{1}{k}\sum_{i=1}^{k} x_i \ge \frac{1}{(n+1)^2}\sum_{i=1}^{n} i = \frac{n}{2(n+1)}. \tag{3}$$

Since $\lim_n \frac{n}{2(n+1)} = \frac{1}{2}$, from (2) and (3) it follows that

$$\lim_k y_{0,k} = \lim_k \frac{1}{k}\sum_{i=1}^{k} x_i = \frac{1}{2}. \tag{4}$$

Set now

$$S_{m,n} := n\, y_{m,n} = \sum_{i=1}^{n} x_{m+i}, \qquad m \ge 0, \quad n \ge 1. \tag{5}$$

Let us proceed by induction on m. We have proved in (4) that $\lim_k y_{0,k} = \frac{1}{2}$. Now suppose that $\lim_n y_{m-1,n} = \frac{1}{2}$, and we claim that $\lim_n y_{m,n} = \frac{1}{2}$. In this way we will show that $(x_k)_k$ Singly converges to $\frac{1}{2}$.

For all $m \ge 0$, $n \ge 1$ we get

$$S_{m-1,n+1} - S_{m,n} = \sum_{i=1}^{n+1} x_{m-1+i} - \sum_{i=1}^{n} x_{m+i} = \sum_{i=0}^{n} x_{m+i} - \sum_{i=1}^{n} x_{m+i} = x_m, \tag{6}$$

and hence

$$\begin{aligned}\lim_n y_{m,n} = \lim_n \frac{1}{n} S_{m,n} = \lim_n \frac{1}{n-1} S_{m,n-1} \\ = \lim_n \frac{1}{n-1} S_{m-1,n} - \lim_n \frac{1}{n-1} x_m \\ = \lim_n \frac{1}{n} S_{m-1,n} \cdot \lim_n \frac{n}{n-1} = \lim_n y_{m-1,n} = \frac{1}{2}\end{aligned}$$

by the inductive hypothesis, which gives the claim.

We now prove that $(x_k)_k$ does not almost converge to $\frac{1}{2}$. Indeed, let $\varepsilon_0 = \frac{1}{2}$, and for every $n \in \mathbb{N}$ and $n^2 + 1 \le k \le n^2 + n$, set $m = m(k) = n^2$. We have

$$\left|\frac{x_{m+1} + \cdots + x_{m+n}}{n} - \frac{1}{2}\right| = \frac{1}{2} = \varepsilon_0.$$

Thus we get the assertion. $\square$

We now prove that (PS)-convergence implies almost convergence. This will give the equivalence between (PS)- and almost convergence, since the converse implication is obvious.

Theorem 2.5. *Let $(x_n)_n$ be a sequence in X, (PS)-convergent to $x_0 \in X$. Then $(x_n)_n$ almost converges to x_0 (thus (PS)-convergence is not generated by any matrix summability method—see* [12], [15]*).*

Proof. Observe that, in order to obtain this result, it will be sufficient to show that (PS)-convergence implies Single convergence, that is

$$\lim_n y_{m,n} = x_0 \qquad \text{for every} \quad m \geq 0, \tag{7}$$

where $(y_{m,n})_{m,n}$ is as in (1). Almost convergence will follow from (PS)-convergence and (7).

Indeed, for each $\varepsilon > 0$, let $\overline{n}$ be such that

$$\|y_{m,n} - x_0\| \leq \varepsilon \qquad \text{for all} \quad m, n \geq \overline{n}\,.$$

From (7) it follows that in correspondence with $m = 0, 1, \ldots, \overline{n} - 1$ there is $n_m \in \mathbb{N}$ with

$$\|y_{m,n} - x_0\| \leq \varepsilon \qquad \text{for all} \quad n \geq n_m.$$

Let $n^* := \max\{\overline{n}, n_0, n_1, \ldots n_{\overline{n}-1}\}$. For all $m \geq 0$ and $n \geq n^*$, we get

$$\|y_{m,n} - x_0\| \leq \varepsilon, \qquad \text{that is almost convergence.}$$

Now, to prove (7), it will be sufficient to show that every $m \geq 0$ is 2ε-stable for each $\varepsilon > 0$.

Fix arbitrarily $\varepsilon > 0$, and set $m = \overline{n} = \overline{n}(\varepsilon)$, where $\overline{n}$ is as in the definition of (PS)-convergence. So, every integer $q \geq m$ is ε-stable, and hence a fortiori even 2ε-stable.

We now claim that $m - 1$ is σ_1-stable for each $\sigma_1 > \varepsilon$. To this aim, let $S_{m,n}$, $m \geq 0$, $n \geq 1$, be as in (5). By (6) we get

$$\begin{aligned} y_{m-1,n+1} &= \frac{S_{m-1,n+1}}{n+1} = \frac{x_m}{n+1} + \frac{S_{m,n}}{n+1} \\ &= \frac{x_m}{n+1} - \frac{S_{m,n}}{n(n+1)} + \frac{S_{m,n}}{n} = \frac{x_m}{n+1} - \frac{y_{m,n}}{n+1} + y_{m,n}. \end{aligned}$$

Therefore, taking into account the ε-stability of m, there is an integer $\widetilde{n} = \widetilde{n}(m)$ with the property that for all $n \geq \widetilde{n}$ we have

$$\|y_{m-1,n+1} - y_{m,n}\| \leq \frac{\|x_m\|}{n+1} + \frac{\|y_{m,n}\|}{n+1} \leq \frac{\|x_m\| + \|x_0\| + \varepsilon}{n+1}.$$

Observe that for every $\sigma_1 > \varepsilon$ there is an integer $\widehat{n}$ large enough, with

$$\frac{\|x_m\| + \|x_0\| + \varepsilon}{n+1} \leq \sigma_1 - \varepsilon \qquad \text{whenever} \quad n \geq \widehat{n}.$$

For each $n \geq \max\{\widetilde{n}, \widehat{n}\}$, we get

$$\|y_{m-1,n+1} - x_0\| \leq \|y_{m-1,n+1} - y_{m,n}\| + \|y_{m,n} - x_0\| \leq \sigma_1 - \varepsilon + \varepsilon = \sigma_1.$$

Hence, $m-1$ is σ_1-stable.

Proceeding by a similar argument, it is possible to check that $m-2$ is σ_2-stable for each $\sigma_2 > \sigma_1$. By induction, we obtain that $m-j$ is σ_j-stable for all $\sigma_j > \sigma_{j-1}$, $j = 3, \dots, m$. If we take $\sigma_j = \varepsilon\big(1 + \frac{j}{m}\big)$, $j = 1, \dots, m$, then $\sigma_m = 2\varepsilon$, and so we get that every $m \geq 0$ is 2ε-stable.

Thus we obtain (7), and hence almost convergence of the sequence $(x_n)_n$. □

We now see that, in general, (A)- and (S)-convergences do not coincide with $\mathcal{I}$-convergence. We begin by giving an example of sequence, (A)-convergent to 0 (and a fortiori (S)-convergent to 0) but not $\mathcal{I}$-convergent to 0 for any admissible ideal $\mathcal{I}$ of $\mathbb{N}$. Let $X = \mathbb{R}$, and consider the sequence $n \mapsto (-1)^n$. Observe that, if m is even, then we get

$$\sum_{k=1}^{n} (-1)^{k+m} = \sum_{k=1}^{n} (-1)^k = \begin{cases} -1 & \text{if } n \text{ is odd,} \\ \;\;\,0 & \text{if } n \text{ is even.} \end{cases}$$

If m is odd, then we have

$$\sum_{k=1}^{n} (-1)^{k+m} = -\sum_{k=1}^{n} (-1)^k = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

So for all $m \geq 0$, $n \geq 1$ we get

$$\left| \sum_{k=1}^{n} (-1)^{k+m} \right| \leq 1,$$

and hence the sums $\sum_{k=1}^{n} (-1)^{k+m}$ are bounded uniformly with respect to m. From this it follows that the sequence $n \mapsto (-1)^n$ is almost convergent to 0.

On the other hand, we claim that this sequence is not $\mathcal{I}$-convergent to 0 for any ideal $\mathcal{I}$ of $\mathbb{N}$. Indeed, let $\varepsilon_0 = \frac{1}{2}$; we get that $\big\{n \in \mathbb{N} : |(-1)^n| > \varepsilon_0\big\} = \mathbb{N} \notin \mathcal{I}$, and thus the claim follows. □

In a similar way, given a normed linear space $(X, \|\cdot\|)$ and a fixed element $x_0 \in X$, $x_0 \neq 0$, it is possible to show that the sequence $n \mapsto (-1)^n \cdot x_0$ is almost convergent to 0, but is not $\mathcal{I}$-convergent to 0.

In the following proposition we see that for each ideal $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ there are sequences $\mathcal{I}$-convergent to a point x_0, but not (S)-convergent to x_0 (and a fortiori not (A)-convergent to x_0).

Proposition 2.6. *Let $\mathcal{I} \neq \mathcal{I}_{\mathit{fin}}$ be an ideal of $\mathbb{N}$, $(X, \|\cdot\|)$ be a normed linear space, $X \neq \{0\}$, and $x_0 \in X$. Then there is a sequence $(x_n)_n$ in X with $(\mathcal{I}) \lim_n x_n = x_0$ but not (S)-convergent to x_0.*

Proof. Let $A \in \mathcal{I}$ be an infinite set, $A = \{k_1 < \cdots < k_n < \cdots\}$. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, then A does exist. Let us define $f\colon A \to \mathbb{N}$ by setting $f(k_n) = \min\{m \in \mathbb{N} : m > k_n,\ m \notin A\}$, $n \in \mathbb{N}$, and let $\varepsilon_0 > \|x_0\|$. In the first step, pick $z_1 \in X$ with $\|z_1 - x_0\| > f(k_1)(\varepsilon_0 + \|x_0\|)$. Note that such an element z_1 does exist, since $\sup\{\|x - y\| : x, y \in X\} = +\infty$. In the second step, choose $z_2 \in X$ with

$$\|z_2 - x_0\| > \|z_1 - x_0\| + f(k_1)(\varepsilon_0 + \|x_0\|).$$

By proceeding analogously, in the nth step we can find an element $z_n \in X$ with

$$\|z_n - x_0\| > \|z_1 - x_0\| + \|z_2 - x_0\| + \cdots + \|z_{n-1} - x_0\| + f(k_n)(\varepsilon_0 + \|x_0\|). \tag{8}$$

Now, let us define $x_m = z_n$ if $m = k_n$, $n \in \mathbb{N}$, and $x_m = x_0$ if $m \notin A$. By construction, it is easy to see that $(\mathcal{I}) \lim_m x_m = x_0$.

We now prove that the sequence $(x_m)_m$ does not (S)-converge to x_0. Choose $m = 0$, fix arbitrarily $n \in \mathbb{N}$ and let $p = k_n$; we have $f(k_n) > k_n = p$. Moreover, we get

$$\begin{aligned} x_{m+1} + \cdots + x_{m+p} &= x_1 + \cdots + x_{k_n} \\ &= x_{k_1} + \cdots + x_{k_n} + x_{l_1} + \cdots + x_{l_{n'}} \\ &= z_1 + \cdots + z_n + n' x_0, \end{aligned}$$

(where $l_1, \ldots, l_{n'}$ are suitable elements of $\mathbb{N} \setminus A$). Taking (8) into account, we have

$$\begin{aligned} &\left\| \frac{x_1 + \cdots + x_p}{p} - x_0 \right\| \\ &= \left\| \frac{x_{k_1} + \cdots + x_{k_n} + (n' - p)x_0}{p} \right\| \\ &= \frac{1}{p} \|z_1 + \cdots + z_n - n x_0\| \\ &= \frac{1}{p} \|(z_1 - x_0) + \cdots + (z_n - x_0)\| \\ &\geq \frac{1}{p} \Big(\|z_n - x_0\| - \|z_1 - x_0\| - \cdots - \|z_{n-1} - x_0\| \Big) \\ &\geq \frac{1}{p} \Big(f(k_n)(\varepsilon_0 + \|x_0\|) + \|z_1 - x_0\| + \|z_2 - x_0\| + \cdots + \|z_{n-1} - x_0\| \\ &\quad - \|z_1 - x_0\| - \|z_2 - x_0\| - \cdots - \|z_{n-1} - x_0\| \Big) \\ &> \frac{1}{p} \cdot p \cdot (\varepsilon_0 + \|x_0\|) = \varepsilon_0 + \|x_0\| \geq \varepsilon_0. \end{aligned}$$

Thus the sequence $(x_n)_n$ is not (S)-convergent to 0. This completes the proof. □

3. Modes of continuity

In this section we deal with modes of continuity using almost and ideal convergence. Let $\mathcal{I}$ be a fixed admissible ideal of $\mathbb{N}$.

Definitions 3.1. Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two normed linear spaces, X, $Y \neq \{0\}$, $f\colon X \to Y$ be a function and $x_0 \in X$. We say that f is:

- *A_1-continuous at x_0,* if and only if $\lim_n f(x_n) = f(x_0)$ (in the usual sense) whenever $(A) \lim_n x_n = x_0$;
- *A_1-continuous on X,* if and only if f is A_1-continuous at each $x_0 \in X$;
- *A_2-continuous at x_0,* if and only if $(A) \lim_n f(x_n) = f(x_0)$ whenever $\lim_n x_n = x_0$;
- *A_2-continuous on X,* if and only if f is A_2-continuous at each $x_0 \in X$;
- *A_3-continuous at x_0,* if and only if $(A) \lim_n f(x_n) = f(x_0)$ whenever $(A) \lim_n x_n = x_0$;
- *A_3-continuous on X,* if and only if f is A_3-continuous at each $x_0 \in X$;
- *$\mathcal{I}$-continuous at x_0,* if and only if $(\mathcal{I}) \lim_n f(x_n) = f(x_0)$ whenever $(\mathcal{I}) \lim_n x_n = x_0$;
- *$\mathcal{I}$-continuous on X,* if and only if f is $\mathcal{I}$-continuous at each $x_0 \in X$;
- *$A\mathcal{I}$-continuous at x_0,* if and only if $(\mathcal{I}) \lim_n f(x_n) = f(x_0)$ whenever $(A) \lim_n x_n = x_0$;
- *$A\mathcal{I}$-continuous on X,* if and only if f is $A\mathcal{I}$-continuous at each $x_0 \in X$;
- *$\mathcal{I}A$-continuous at x_0,* if and only if $(A) \lim_n f(x_n) = f(x_0)$ whenever $(\mathcal{I}) \lim_n x_n = x_0$;
- *$\mathcal{I}A$-continuous on X,* if and only if f is $\mathcal{I}A$-continuous at each $x_0 \in X$.

Remark 3.2. Observe that $\mathcal{I}$-continuity is equivalent to the usual continuity (see [11, Proposition 3.3]), and that $A\mathcal{I}_{\text{fin}}$- and $\mathcal{I}_{\text{fin}}A$-continuity coincide with A_1- and A_2-continuity, respectively.

Moreover, since usual convergence implies (A)-convergence (see also [12]), then clearly both ordinary and A_3-continuity at a point $x_0 \in X$ imply A_2-continuity at x_0, and A_1-continuity at x_0 implies usual continuity at x_0.

We now prove the following.

Proposition 3.3. *Let $f\colon X \to Y$ be a function, and $\mathcal{I}$ be a P-ideal of $\mathbb{N}$. If f is $A\mathcal{I}$-continuous at a point $x_0 \in X$, then f is constant.*

Proof. First of all observe that, in order to prove the proposition, it is enough to check that $f(z + x_0) = f(x_0)$ for all $z \in X$.

Let $\mathbb{E}$ be the set of all even natural numbers. We first consider the case in which $\mathbb{E} \notin \mathcal{I}$ and $\mathbb{N} \setminus \mathbb{E} \notin \mathcal{I}$. Let $z \in X$ be arbitrarily fixed, set $z_n = z + x_0$ if and only if $n \in \mathbb{E}$ and $z_n = -z + x_0$ if and only if $n \in \mathbb{N} \setminus \mathbb{E}$. By construction, it is not difficult to see that $(A) \lim_n z_n = x_0$. Since f is $A\mathcal{I}$-continuous at x_0, we get $(\mathcal{I}) \lim_n f(z_n) = f(x_0)$. Since $\mathcal{I}$ is a P-ideal, there exists a set $P \in \mathcal{F}$, where $\mathcal{F}$ is the dual filter associated with $\mathcal{I}$, $P = \{k_1 < \cdots < k_n < \cdots\}$, with the property that $\lim_n f(z_{k_n}) = f(x_0)$ in the usual sense (see also [11]). By Proposition 2.2 we get that both $P \cap \mathbb{E}$ and $P \setminus \mathbb{E}$ are infinite. So there exists a strictly increasing sequence $(l_n)_n$ in $\mathbb{N}$ with $k_{l_n} \in \mathbb{E}$ for all $n \in \mathbb{N}$. Thus $f(x_0) = \lim_n f(z_{k_{l_n}}) = f(z + x_0)$, and hence f is constant on X, by arbitrariness of z.

If $\mathbb{N} \setminus \mathbb{E} \in \mathcal{I}$, then $\mathbb{E} \in \mathcal{F}$. Fix arbitrarily $z \in X$ and let us proceed analogously as above. In this case we obtain that $P \cap \mathbb{E} \in \mathcal{F}$, and hence it is infinite. So we get $f(x_0) = f(z + x_0)$, and thus f is constant, since z was chosen arbitrarily.

If $\mathbb{E} \in \mathcal{I}$, then $\mathbb{N} \setminus \mathbb{E} \in \mathcal{F}$. By exchanging the roles of $\mathbb{E}$ and $\mathbb{N} \setminus \mathbb{E}$, we get that $P \setminus \mathbb{E} \in \mathcal{F}$, and so it is infinite. Thus we obtain the existence of a strictly increasing sequence $(l_n)_n$ in $\mathbb{N}$ with $k_{l_n} \in \mathbb{N} \setminus \mathbb{E}$ for all $n \in \mathbb{N}$. Hence, $f(x_0) = f(-z + x_0)$, and so again we get that f is constant on X, thanks to arbitrariness of z. This finishes the proof. □

As a consequence of Proposition 3.3, we have

Corollary 3.4. *If $f \colon X \to Y$ is A_1-continuous at a point $x_0 \in X$, then f is constant. Thus a continuous and not constant function is not A_1-continuous.*

Proposition 3.5. *Let X, Y be two normed linear spaces, $\mathcal{I} \neq \mathcal{I}_{fin}$ be an admissible ideal of $\mathbb{N}$ and $f \colon X \to Y$ be an $\mathcal{I}A$-continuous function on X (that is, f is $\mathcal{I}A$-continuous at each $x \in X$). Then f is bounded (that is, $f(X)$ is a bounded set in Y).*

P r o o f. Let $f(X)$ be not bounded. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ there is an infinite set $K = \{k_1 < k_2 < \ldots\}$ such that $K \in \mathcal{I}$. Let $x_0 \in X$. We proceed as in the proof of Proposition 2.6. Then we can construct a sequence $(y_n)_n \subseteq f(X)$ as follows

$$y_n = \begin{cases} f(x_0), & n \notin K, \\ \quad z_m, & n \in K, \quad n = k_m, \quad m \in \mathbb{N} \end{cases}$$

and such that $y_n \overset{A}{\not\to} f(x_0)$. But $y_n \overset{\mathcal{I}}{\to} f(x_0)$ and since $y_n \in f(X)$ for all $n \in \mathbb{N}$, it follows that there exists a sequence $(x'_n)_n \subseteq X$ such that $y_n = f(x'_n)$ for every $n \in \mathbb{N}$. We set $(x_n)_n \subseteq X$ as follows

$$x_n = \begin{cases} x_0, & n \notin K, \\ x'_n, & n \in K, \quad n = k_m, \quad m \in \mathbb{N}. \end{cases}$$

So, if $n \in \mathbb{N}$, then we have the following cases:

(i) $n \in K = \{k_m : m \in \mathbb{N}\} \Rightarrow x_n = x'_n \Rightarrow f(x_n) = f(x'_n) = y_n$;

(ii) $n \notin K = \{k_m : m \in \mathbb{N}\} \Rightarrow x_n = x_0 \Rightarrow f(x_n) = f(x_0) = y_n$.

Hence $y_n = f(x_n)$ for any $n \in \mathbb{N}$. Moreover, by construction we have that

$$x_n \xrightarrow{\mathcal{I}} x_0 \quad \text{but} \quad f(x_n) = y_n \overset{A}{\not\to} f(x_0).$$

This contradicts the hypothesis of $\mathcal{I}A$-continuity of f at x_0. Since x_0 was chosen arbitrarily, it follows that $f(X)$ is a bounded set. Thus the proof is completed. $\square$

Corollary 3.6. *Every continuous and not bounded function is not $\mathcal{I}A$-continuous for $\mathcal{I} \neq \mathcal{I}_{fin}$.*

The following notion will be useful to prove some properties of $\mathcal{I}A$-continuous functions.

Definition 3.7. A function $f\colon X \to Y$ is *locally bounded* at a point $x_0 \in X$, if and only if there are two positive real numbers δ, M with $\|f(x)\| \le M$ whenever $x \in X$, $\|x - x_0\| < \delta$.

Proposition 3.8. *Every function $f\colon X \to Y$, A_2-continuous at a point $x_0 \in X$, is locally bounded at x_0.*

Proof. If we deny the thesis, then for each δ and $M > 0$ there exists $\overline{x} \in X$ with $\|\overline{x} - x_0\| < \delta$ and $\|f(\overline{x})\| > M$. So, in correspondence with $\delta = 1$ and $M = 1 + \|f(x_0)\|$ there is $x_1 \in X$ with $\|x_1 - x_0\| < 1$ and $\|f(x_1)\| > 1 + \|f(x_0)\|$. To $\delta = 1/2$ and $M = 2 + \|f(x_1)\| + 2\|f(x_0)\|$ there corresponds an element $x_2 \in X$ with $\|x_2 - x_0\| < 1/2$ and $\|f(x_2)\| > 2 + \|f(x_1)\| + 2\|f(x_0)\|$. So, in the $m+n$th step an element $x_{m+n} \in X$ can be found, with $\|x_{m+n} - x_0\| < \frac{1}{m+n}$ and

$$\begin{aligned}\|f(x_{m+n})\| &> (m+n) + \|f(x_1)\| + \cdots + \|f(x_m)\| \\ &\qquad + \|f(x_{m+1})\| + \cdots + \|f(x_{m+n-1})\| + (m+n)\|f(x_0)\| \\ &> n + \|f(x_{m+1})\| + \cdots + \|f(x_{m+n-1})\| + n\|f(x_0)\|. \end{aligned} \tag{9}$$

Note that $\|x_n - x_0\| < 1/n$ for all $n \in \mathbb{N}$, and hence $\lim_n x_n = x_0$ in the usual sense. Taking (9) into account, for all m, $n \in \mathbb{N}$ we have:

$$\begin{aligned}&\left\| \frac{f(x_{m+1}) + \cdots + f(x_{m+n})}{n} - f(x_0) \right\| \\ &= \left\| \frac{[f(x_{m+1}) - f(x_0)] + \cdots + [f(x_{m+n}) - f(x_0)]}{n} \right\| \\ &\ge \frac{1}{n}\big(\|f(x_{m+n}) - f(x_0)\| - \|f(x_{m+1}) - f(x_0)\| - \cdots - \|f(x_{m+n-1}) - f(x_0)\|\big)\end{aligned}$$

$$\geq \frac{1}{n}\big(\|f(x_{m+n})\| - \|f(x_{m+1})\| - \cdots - \|f(x_{m+n+1})\| - n\|f(x_0)\|\big)$$

$$\geq \frac{1}{n}\big(n + \|f(x_{m+1})\| + \cdots + \|f(x_{m+n+1})\| + n\|f(x_0)\|$$

$$- \|f(x_{m+1})\| - \cdots - \|f(x_{m+n+1})\| - n\|f(x_0)\|\big) = \frac{1}{n} \cdot n = 1.$$

Thus we obtain that the sequence $\big(f(x_n)\big)_n$ does not (A)-converge to $f(x_0)$. Hence, f is not A_2-continuous at x_0, a contradiction. □

We now give a condition under which $\mathcal{I}A$-continuity implies usual continuity. Let (σ) be a mode of convergence, with the property that a sequence $(x_n)_n$ in X (σ)-converges to $x_0 \in X$ whenever $\lim_n x_n = x_0$ in the usual sense. Almost, Single and ideal convergence are examples of (σ)-convergences.

Theorem 3.9. *Let X, Y be two normed linear spaces with $X, Y \neq \{0\}$ and $\dim Y < \infty$, and $x_0 \in X$. Assume that $f\colon X \to Y$ is locally bounded at x_0. Moreover, suppose that $\big(f(x_n)\big)_n$ (σ)-converges to $f(x_0)$ whenever $(x_n)_n$ is a sequence in X, convergent to x_0 in the ordinary sense and with $x_l \neq x_s$ for all $l \neq s$.*

Then f is continuous at x_0 in the usual sense.

Proof. By contradiction, suppose that f is not continuous at x_0. There exists a positive real number ε_0 with the property that for every $\tau > 0$ there is $x_\tau \in X$ with $\|x_\tau - x_0\| < \tau$ and $\|f(x_\tau) - f(x_0)\| \geq \varepsilon_0$. We construct a sequence $(x_n)_n$ in X as follows. In the first step, pick an $x_1 \in X$ with

$$\|x_1 - x_0\| < 1 \quad \text{and} \quad \|f(x_1) - f(x_0)\| \geq \varepsilon_0.$$

Note that $x_1 \neq x_0$. In the second step, choose an $x_2 \in X$ with

$$\|x_2 - x_0\| < \min\left(\frac{\|x_1 - x_0\|}{2}, \frac{1}{2}\right) \quad \text{and} \quad \|f(x_2) - f(x_0)\| \geq \varepsilon_0,$$

and observe that $x_2 \neq x_1$ and $x_2 \neq x_0$. Proceeding by induction, in the nth step, pick an element $x_n \in X$ with

$$\|x_n - x_0\| < \min\left(\frac{\|x_{n-1} - x_0\|}{2}, \frac{1}{n}\right) \quad \text{and} \quad \|f(x_n) - f(x_0)\| \geq \varepsilon_0.$$

It is not difficult to check that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_l \neq x_s$ whenever $l \neq s$. Note that the sequence $(x_n)_n$ converges to x_0 in the usual sense. By the assumption, f is locally bounded at x_0, and hence there exist $\delta > 0$, $M > 0$ with $\|f(x)\| \leq M$ whenever $x \in X$, $\|x - x_0\| < \delta$. Since $\lim_n x_n = x_0$ in the ordinary sense, in correspondence with this positive real number δ there is $n_0 \in \mathbb{N}$ with $\|x_n - x_0\| < \delta$ for all $n \geq n_0$. For each $n \in \mathbb{N}$, set $y_n = x_{n_0+n}$. Note that $\lim_n y_n = x_0$ in the usual sense, $\|f(y_n) - f(x_0)\| \geq \varepsilon_0$ for all $n \in \mathbb{N}$ and $\|f(y_n)\| \leq M$ for every $n \in \mathbb{N}$. Let $B_M \subset Y$ be the closed ball of center 0 and radius M. Since $\dim Y < \infty$, B_M is compact, and so there are a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and an element $y_0 \in B_M$ with $\lim_k f(y_{n_k}) = y_0$. Since $\|f(y_n) - f(x_0)\| \geq \varepsilon_0$ for each $n \in \mathbb{N}$, then $y_0 \neq f(x_0)$. Note that $\lim_n y_n = x_0$ and hence $\lim_k y_{n_k} = x_0$. By hypothesis, we get

$$(\sigma) \lim_k f(y_{n_k}) = f(x_0).$$

On the other hand, since $\lim_k f(y_{n_k}) = y_0$, we also have

$$(\sigma) \lim_k f(y_{n_k}) = y_0.$$

Thus $y_0 = f(x_0)$, a contradiction. □

A consequence of Proposition 3.8 and Theorem 3.9 is the following.

Corollary 3.10. *Let X, Y be two normed linear spaces as in Theorem 3.9. If $f\colon X \to Y$ is A_2-continuous or $\mathcal{I}A$-continuous at x_0, then f is continuous at x_0 in the ordinary sense.*

Remark 3.11. Observe that, if $X \neq \{0\}$ is any normed linear space, then the identity function $\mathrm{id}\colon X \to X$ is continuous in the usual sense, A_2 and A_3-‑continuous, but not $\mathcal{I}A$-continuous at any point $x_0 \in X$ for every $\mathcal{I} \neq \mathcal{I}_{\mathrm{fin}}$, thanks to Proposition 2.6.

The following example shows that, in general, the hypothesis $\dim Y < \infty$ in Theorem 3.9 cannot be dropped.

Example 3.12. Let $X = \mathbb{R}$, $Y = l^2$ be the space of all real sequences $(x_n)_n$ with the property that

$$\sum_{n=1}^{\infty} x_n^2 < +\infty.$$

Let $\mathbb{Q}$ be the set of all rational numbers, $\mathbb{Q} := \{r_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let e_n be the element of l^2 whose nth coordinate is 1 and the other coordinates equal to 0, and let us define $f\colon \mathbb{R} \to l^2$ as follows:

$$\begin{cases} f(0) = 0, & \\ f(x) = e_n & \text{if } x = r_n, x \neq 0, \\ f(x) = 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Note that f is not continuous at 0 in the usual sense. Indeed, let $(x_n)_n$ be any sequence of rational numbers, convergent to 0 and with $x_n \neq 0$ for every $n \in \mathbb{N}$. There is a sequence $(i_n)_n$ in $\mathbb{N}$ with $x_n = r_{i_n}$ for all $n \in \mathbb{N}$, and hence

$$\|f(x_n) - f(0)\|_2 = \|f(x_n)\|_2 = \|e_{i_n}\|_2 = 1 \qquad \text{for all} \quad n \in \mathbb{N}.$$

So f is not continuous at 0 in the ordinary sense.

We now prove that the sequence $\bigl(f(x_n)\bigr)_n$ (A)-converges to 0 for *any* sequence $(x_n)_n$ in $\mathbb{R}$ with $x_l \neq x_s$ for all $l \neq s$.

Fix arbitrarily $\varepsilon > 0$, and let $p_0 \in \mathbb{N}$ be with $p_0 > \frac{1}{\varepsilon^2}$. Choose arbitrarily $p \geq p_0$ and $n \in \mathbb{N}$. In correspondence with n and p, denote by $r_{i_1}, \dots, r_{i_q}$ those elements of the type x_{n+j}, with $x_{n+j} \in \mathbb{Q} \setminus \{0\}$, $j = 1, \dots, p$, provided that they exist. Since the x_n's are distinct, we get

$$\begin{aligned}\left\| \frac{f(x_{n+1}) + \cdots + f(x_{n+p})}{p} \right\|_2 &= \frac{1}{p} \left\| e_{i_1} + \cdots + e_{i_q} \right\|_2 \\ &= \frac{1}{p} q^{1/2} \leq \frac{1}{\sqrt{p}} \leq \frac{1}{\sqrt{p_0}} < \varepsilon.\end{aligned}$$

If $x_{n+j} \notin \mathbb{Q} \setminus \{0\}$ for all $j = 1, \dots, p$, then we have

$$\left\| \frac{f(x_{n+1}) + \cdots + f(x_{n+p})}{p} \right\|_2 = 0.$$

In both cases, we obtain almost convergence to 0 of the sequence $\big(f(x_n)\big)_n$. □

Remarks 3.13. (a) Observe that in the argument of Example 3.12, in general, the condition that the elements x_n of the involved sequence are distinct, cannot be dropped.

Indeed, for each $n \in \mathbb{N}$ and $2^{n-1} \leq k < 2^n - 1$, set $x_k = \frac{1}{n}$, and choose $m = 0$. Of course, $(x_k)_k$ converges to 0 in the usual sense. Let $\frac{1}{n} = r_{i_n}$, $n \in \mathbb{N}$. We get

$$\begin{aligned}\| f(x_1) + \cdots + f(x_{2^n - 1}) \|_2 &= \left\| \sum_{k=0}^{n-1} 2^k e_{i_k} \right\|_2 \\ &= \sqrt{\sum_{k=0}^{n-1} (2^k)^2} = \sqrt{\sum_{k=0}^{n-1} 4^k} = \sqrt{\frac{4^n - 1}{3}},\end{aligned}$$

and hence

$$\begin{aligned}\lim_n \left\| \frac{f(x_1) + \cdots + f(x_{2^n - 1})}{2^n - 1} \right\|_2 &= \lim_n \frac{1}{2^n - 1} \cdot \sqrt{\frac{4^n - 1}{3}} \\ &= \lim_n \sqrt{\frac{4^n - 1}{3(4^n - 2^{n+1} + 1)}} = \sqrt{\frac{1}{3}}.\end{aligned}$$

Thus the sequence $\big(f(x_k)\big)_k$ does not Singly converge to 0, and a fortiori it does not almost converge to 0.

(b) All above results of Section 3 hold, even if in Definitions 3.1 almost convergence is replaced with Single convergence, and the techniques of the proofs are the same.

Let now $\mathcal{I}_u$ be the ideal of all subsets of $\mathbb{N}$ having Banach (or uniform) density zero. We have the following.

Proposition 3.14. *Let X be a normed linear space, $x_0 \in X$ and $x = (x_n)_n$ be a bounded sequence in X, such that $(\mathcal{I}_u) \lim_n x_n = x_0$. Then, $(A) \lim_n x_n = x_0$.*

Proof. Put $M = \|x\|_\infty + \|x_0\|$, where $\|x\|_\infty$ denotes the supremum norm of $(x_n)_n$. For each $\varepsilon > 0$ and $m, n \in \mathbb{N}$ set

$$B^{(\varepsilon)}_{m,n} = \left\{ m+1 \le k \le m+n : \|x_k - x_0\| \ge \frac{\varepsilon}{2} \right\}.$$

By hypothesis, we get that $\lim_n \frac{\sharp(B^{(\varepsilon)}_{m,n})}{n} = 0$ uniformly with respect to $m \ge 0$, where $\sharp$ denotes the cardinality of the set into brackets, and thus there exists an $N = N(\varepsilon) \in \mathbb{N}$ with

$$\frac{\sharp(B^{(\varepsilon)}_{m,n})}{n} \le \frac{\varepsilon}{2M} \tag{10}$$

for all $n \ge N(\varepsilon)$ and $m \ge 0$.

By (10), for each $n \ge N(\varepsilon)$ and $m \ge 0$ we get

$$\begin{aligned} 0 &\le \left\| \frac{\sum_{k=m+1}^{m+n} x_k}{n} - x_0 \right\| = \left\| \frac{\sum_{k=m+1}^{m+n} x_k - n\, x_0}{n} \right\| = \left\| \frac{\sum_{k=m+1}^{m+n} (x_k - x_0)}{n} \right\| \\ &\le \frac{\sum_{k=m+1}^{m+n} \|x_k - x_0\|}{n} = \frac{1}{n} \left(\sum_{k \in B^{(\varepsilon)}_{m,n}} \|x_k - x_0\| + \sum_{\substack{k \notin B^{(\varepsilon)}_{m,n} \\ m+1 \le k \le m+n}} \|x_k - x_0\| \right) \\ &\le \frac{M\, \sharp(B^{(\varepsilon)}_{m,n})}{n} + \frac{1}{n} \frac{n\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

taking into account that

$$\|x_k - x_0\| \le \|x_k\| + \|x_0\| \le \|x\|_\infty + \|x_0\| = M$$

for all $k \in \mathbb{N}$, and consequently $(A) \lim_n x_n = x_0$. So the proof is complete. □

A consequence of Proposition 3.14 is the following.

Corollary 3.15. *Let $f \colon X \to Y$ be a bounded continuous function. Then f is $\mathcal{I}_u A$-continuous on X.*

Proof. Let $x_0 \in X$ and $(x_n)_n$ be a sequence in X, $(\mathcal{I}_u)$-convergent to $x_0 \in X$. Since f is continuous, the sequence $\big(f(x_n)\big)_n$ $(\mathcal{I}_u)$-converges to $f(x_0)$ (see Remark 3.2). But f is bounded, and thus $\big(f(x_n)\big)_n$ is a bounded sequence in Y. By Proposition 3.14 we get that $(A) \lim_n x_n = x_0$. This concludes the proof. □

Remark 3.16. By a similar argument, we can prove that for any normed linear spaces X, Y and any bounded sequence $(x_n)_n$ in X with $(\mathcal{I}_d)\lim_n x_n = x_0$ (where $\mathcal{I}_d$ is the ideal of all subsets of $\mathbb{N}$ of asymptotic density zero) we get that $(S)\lim_n x_n = x_0$, and consequently any bounded continuous function $f\colon X \to Y$ is $\mathcal{I}_d S$-continuous on X (compare with Proposition 2.6).

We now recall the following (see also [14]).

Definition 3.17. Let $X \neq \emptyset$ be a set and f, f_n $(n \in \mathbb{N})$, be real-valued functions defined on X. We say that the sequence $(f_n)_n$ *converges uniformly equally to f on X*, and we write $f_n \overset{u.e.}{\to} f$, if and only if there exist $(\varepsilon_n)_n \in c_0^+$ (= the positive elements of the space of all sequences convergent to zero) and a natural number $k = k[(\varepsilon_n)_n]$ such that

$$\sharp\big(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}\big) \leq k \qquad \text{for all} \quad x \in X,$$

where $\sharp$ denotes the cardinality of the set into brackets.

Remark 3.18. A similar definition can be stated when $(\mathbb{R}, |\cdot|)$ is replaced with an arbitrary normed linear space $(Y, \|\cdot\|)$.

Thus we can formulate the following.

Definitions 3.19. Let X, Y be two normed linear spaces, $x_0 \in X$ and $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. Then:

(i) If $(x_n)_n$ is a sequence in X and $\phi_n, \phi\colon \mathbb{N} \to X$,

$$\phi_n(m) = \frac{x_{m+1} + \cdots + x_{m+n}}{n},$$

$\phi(m) = x_0$ for every $n, m \in \mathbb{N}$, then we say that $(x_n)_n$ *converges Lorentz uniformly equally to x_0*, and we write $x_n \overset{Au.e.}{\longrightarrow} x_0$, if and only if $\phi_n \overset{u.e.}{\to} \phi$.

(ii) If $f\colon X \to Y$ is a function, then we say that f is:

- *B_1-continuous at x_0*, if and only if $\lim_n f(x_n) = f(x_0)$ whenever $(Au.e.)\lim_n x_n = x_0$;
- *B_1-continuous on X*, if and only if f is B_1-continuous at each $x_0 \in X$;
- *B_2-continuous at x_0*, if and only if $(Au.e.)\lim_n f(x_n) = f(x_0)$ whenever $\lim_n x_n = x_0$;
- *B_2-continuous on X*, if and only if f is B_2-continuous at each $x_0 \in X$;
- *B_3-continuous at x_0*, if and only if $(Au.e.)\lim_n f(x_n) = f(x_0)$ whenever $(Au.e.)\lim_n x_n = x_0$;
- *B_3-continuous on X*, if and only if f is B_3-continuous at each $x_0 \in X$;
- *$B\mathcal{I}$-continuous at x_0*, if and only if $(\mathcal{I})\lim_n f(x_n) = f(x_0)$ whenever $(Au.e.)\lim_n x_n = x_0$;

- *$B\mathcal{I}$-continuous on X*, if and only if f is $B\mathcal{I}$-continuous at each $x_0 \in X$;
- *$\mathcal{I}B$-continuous at x_0*, if and only if $(Au.e.)\lim_n f(x_n) = f(x_0)$ whenever $(\mathcal{I})\lim_n x_n = x_0$;
- *$\mathcal{I}B$-continuous on X*, if and only if f is $\mathcal{I}B$-continuous at each $x_0 \in X$.

Remark 3.20. Observe that $B\mathcal{I}_{\text{fin}}$- and $\mathcal{I}_{\text{fin}}B$-continuity coincide with B_1- and B_2-continuity, respectively.

Open problems:

(a) For which admissible ideals of $\mathbb{N}$ does the usual continuity coincide with $\mathcal{I}A$-continuity?

(b) For which normed linear spaces X is it true that every A_3-continuous function $f\colon X \to X$ is linear? A positive answer was given in [6, Theorem 1] when $X = \mathbb{R}$ (see also [1], [16]).

(c) Study similar problems for continuities defined by any sequential matrix summability method (see also [10]).

(d) Study the fundamental properties of the modes of continuities of Definitions 3.19 and their relations to the modes of continuities formulated in Definitions 3.1.

REFERENCES

[1] ACZEL, J.—DHOMBRES, J.: *Functional Equations in Several Variables*. Cambridge Univ. Press, Cambridge, 1987.

[2] ANTONI, J.: *On the A-continuity of real functions II*, Math. Slovaca **36** (1986), 283–288.

[3] ANTONI, J.—ŠALÁT, T.: *On the A-continuity of real functions*, Acta Math. Univ. Comenian. **39** (1980), 159–164.

[4] BALCERZAK, M.—DEMS, K.—KOMISARSKI, A.: *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328** (2007), 715–729.

[5] BOCCUTO, A.—DIMITRIOU, X.—PAPANASTASSIOU, N.—WILCZYŃSKI, W.: *On some different types of ideal continuity*, 2012 (preprint).

[6] BORSÍK, J.—ŠALÁT, T.: *On F-continuity of real functions*, Tatra Mt. Math. Publ. **2** (1993), 37–42.

[7] CONNOR, J.—GROSSE-ERDMANN, K. G.: *Sequential definitions of continuity for real functions*, Rocky Mountain J. Math. **33** (2003), 93–121.

[8] FAST, H.: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 41–44.

[9] FREEDMAN, A. R.—SEMBER, J. J.: *Densities and summability*, Pacific J. Math. **95** (1981), 293–305.

[10] KOLK, E.: *Inclusion relations between the statistical convergence and strong summability*, Acta Comment. Univ. Tartu. Math. **2** (1998), 39–54.

[11] KOSTYRKO, P.—ŠALÁT, T.—WILCZYŃSKI, W.: *$\mathcal{I}$-convergence*, Real Anal. Exchange **26** (2000/2001), 669–685.

[12] LORENTZ, G. G.: *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[13] MÓRICZ, F.—STADTMŰLLER, U.: *Summability of double sequences by weighted mean methods and Tauberian conditions for convergence in Pringsheim's sense*, Int. J. Math. Math. Sci. **65** (2004), 3499–3511.
[14] PAPANASTASSIOU, N.: *On a new type of convergence of sequences of functions*, Atti Semin. Mat. Fis. Univ. Modena **50** (2002), 493–506.
[15] PETERSEN, G. M.: *Regular Matrix Transformations.* McGraw-Hill, London, 1966.
[16] SMÍTAL, J.: *On Functions and Functional Equations.* Adam Hilger, Bristol, UK, 1988.
[17] STEINHAUS, H.: *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–74.

Received April 20, 2012

Antonio Boccuto
Department of Mathematics and Computer Sciences
University of Perugia
via Vanvitelli 1
I–06123 Perugia
ITALY
E-mail: boccuto@yahoo.it
boccuto@dmi.unipg.it

Xenofon Dimitriou
Nicolas Papanastassiou
Department of Mathematics
University of Athens
Panepistimiopolis
Athens 15784
GREECE
E-mail: xenofon11@gmail.com
dxenof@math.uoa.gr
npapanas@math.uoa.gr