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FUNCTIONS WITH VALUES IN LOCALLY CONVEX SPACES WITH WEAKLY COMPACT SEMIVARIATION

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ABSTRACT. The present paper is concerned with some properties of functions with values in locally convex vector spaces, especially functions having weakly compact semivariation and generalizations of some theorems for functions with values in locally convex vector spaces, namely: If X is a sequentially complete locally convex vector space, then the function $x(\cdot)\colon [a,b] \to X$ having a weakly compact semivariation on the interval $[a,b]$ defines a vector valued measure m on Borel subsets of $[a,b]$ with values in X and the range of this measure is a weakly relatively compact subset in X. This theorem is an extension of the result of Sirvint and of Edwards from Banach spaces to locally convex spaces.

1. Introduction

The set $\mathcal{WCS}$ of functions $x(\cdot)$ of weakly compact semivariation on the real line R to a complete complex locally convex vector space X derives its importance from the fact that each function $x(\cdot)$ in $\mathcal{WCS}$ is associated in a natural way with a vector valued measure on the ring $\mathcal{B}$ of bounded Borel sets, and conversely. The relationship resembles that between ordinary real and complex functions of bounded variation and real and signed measures defined on $\mathcal{B}$. A precise statement will be given later on.

The extension to locally convex vector spaces is very useful because it makes possible to include also many important spaces, which are not normable, e.g., nuclear locally convex vector spaces often appearing in applications.

The paper is divided into four sections. The most propositions and theorems are proved under the assumption that LCS X is Hausdorff sequentially complete (further said only s-complete) locally convex topological vector space.

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2. Functions with bounded semivariation

In this section we recall the definition of functions $x(\cdot)\colon [a,b] \to X$ with bounded semivariation, where X is a LCS.

Definition 1. A function $x(\cdot)\colon [a,b] \to X$ is of bounded semivariation on $[a,b]$, if for an arbitrary semi-norm $p \in P$ there exists a real number $K_p(a,b)$, $0 \le K_p(a,b) < \infty$ such that for any $k \ge 1$ and for an arbitrary division of the interval $[a,b]$ with $-\infty < a \le t_0 < t_1 < \cdots < t_k \le b < \infty$, the set

$$V_{x(\cdot)} = \left\{ \sum_{j=1}^{k} \alpha_j \big(x(t_j) - x(t_{j-1})\big) \,\Big|\, k \in \mathbf{N},\ a = t_0 < t_1 < \cdots < t_k = b, |\alpha_j| \le 1 \right\}$$

is bounded in X. Equivalently,

Definition 2. $x(\cdot)$ has *bounded semi-variation* if and only if $p\big(V_{x(\cdot)}\big)$ is a bounded set of real numbers for each $p \in Q$, where Q is the set of seminorms determining the topology of X.

We shall recall the proof of the following property of functions with bounded semi-variation.

Theorem 3. *A function $x(\cdot)$ has bounded semi-variation if and only if elements of the set $\{x' \circ x \mid x' \in X'\}$ have bounded variations.*

Proof. a) Let us suppose that $x(\cdot)$ has bounded semi-variation and let $p'_X(x') \le 1$. For any division $a = t_0 < t_1 < \cdots < t_k = b$ of the interval $[a,b]$ and for any j, there exists α_j such that $|\alpha_j| = 1$ and $\big|x'\big(x(t_j - x(t_{j-1}))\big|=\alpha_j x'\big(x(t_j - x(t_{j-1})\big)$. It follows that

$$\begin{aligned}
&\sum_{j=1}^{k} \big|x'\big(x(t_j - x(t_{j-1}))\big| \\
&= x' \left(\sum_{j=1}^{k} \alpha_j \big(x(t_j) - x(t_{j-1})\big) \right) \\
&\le p'_X(x') K_{p'} \le K_{p'},
\end{aligned}$$

where $K_{p'}$ is a bound for the semi-norm of elements of $V_{x(\cdot)}$.

b) We now suppose that the elements of the set $\big\{x' \circ x(\cdot) \mid x' \in X',\ p'_X(x') \le 1\big\}$ have uniformly bounded variations. We have

$$p\left(\sum_{j=1}^{k} \alpha_j \big(x(t_j) - x(t_{j-1})\big) \right) = \sup_{p'_X(x') \le 1} \left| x' \left(\sum_{j=1}^{k} \alpha_j \big(x(t_j) - x(t_{j-1})\big) \right) \right|$$

and

$$\left| x'\left(\sum_{j=1}^{k} \alpha_j \big(x(t_j) - x(t_{j-1})\big)\right) \right| = \left| \sum_{j=1}^{k} \alpha_j x'\big(x(t_j) - x(t_{j-1})\big) \right| \le \sum_{j=1}^{k} \left| x'\big(x(t_j) - x(t_{j-1})\big) \right|$$

and the last sum is bounded by a constant independent of x'. □

3. Functions with weakly compact semivariation

Let us consider a function $x(\cdot)\colon R \to X$. Denote, for $[a, b]$,

$$E(a,b) = \left\{ \sum_{i=1}^{n} \Big[x(t_{2i}) - x(t_{2i-1}) \Big],\ n \ge 1,\ -\infty < a \le t_1 < t_2 < \cdots < t_{2n} \le b < \infty \right\},$$

We shall say that the function $x(\cdot)\colon R \to X$ has a weakly compact semivariation if the set $E(a,b)$ is relatively weakly compact.

Using weak compactness we can prove

Theorem 4. *If X is a s-complete LCS and $x(\cdot) : R \to X$ has on every interval $[a,b]$ weakly compact semivariation, then the limits of the function $x(\cdot)$ from the right and from the left, $x(\pm 0)$, exist at every point t and if X is also metrisable, then the function $x(\cdot)$ is continuous for all t except of a set at most countable.*

Theorem 5. *Let X be an s-complete LCS and (x_n) be a sequence of elements from X. Then a necessary and sufficient condition for the unconditional convergence of the series $\sum_{n=1}^{\infty} x_n$ is that the set*

$$E_0 = \left\{ \sum_{m=1}^{M} x_{n_m},\ M \ge 1,\ 1 \le n_1 < n_2 < \cdots < n_M < \infty \right\}$$

is relatively weakly compact.

Recall the following definitions.

Definition 6. The series $\sum x_i$, $x_i \in X$, unconditionally (permutally) converges, if for any permutation $p(i)$ of natural numbers the series $\sum x_{p(i)}$ converges in X.

Definition 7. The series $\sum x_i$, $x_i \in X$, subseries converges, if for any increasing sequence (n_i) of natural numbers the series $\sum x_{n_i}$ converges to some element in X.

Note that in an s-complete LCS the unconditional and subseries convergences coincide.

Remark. As can be shown by an example, every function with weakly compact semivariation is a function with bounded semivariation but not conversely.

4. Connection with vector measures

In this section we investigat the existence of the unique correspondence between the vector valued functions with weakly compact semivariation and continuous from the right on $[a, b]$ and the vector valued measures on Borel sets in $[a, b]$.

Denote by $\mathcal{B}[a, b]$ the σ-algebra of Borel subsets of the interval $[a, b]$. Let $\mathcal{B}$ be the union of all σ-algebras $\mathcal{B}[a, b]$, $a, b \in R$. Then $\mathcal{B}$ is a δ-ring. Recall that a δ-ring is a ring S having the property: for every sequence (A_i) of sets of S there holds $\bigcap_{i=1}^{\infty} A_i \in S$.

Let us recall the definition of vector valued measure with values in a LCS.

Definition 8. A function $m\colon \mathcal{B} \to X$, where X is a LCS and $\mathcal{B}$ is the δ-ring of Borel subsets of the interval $[a, b]$, is called vector valued measure, if

$$m(\emptyset) = 0, \quad m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n),$$

if the sets E_n are mutually disjoint in $\mathcal{B}$, with the union in $\mathcal{B}$, the series in the right being unconditionally convergent.

Now we shall prove that the vector valued measure induces a function with weakly compact semivariation.

Theorem 9. *If X is a s-complete LCS and $m\colon \mathcal{B} \to X$ is a vector valued measure, then the function $x(\cdot)$ defined by the relations*

$$x(0) = 0, \quad x(d) - x(c) = m\big((c, d]\big), \quad c, d \in R, \quad -\infty < c < d < \infty,$$

has weakly compact semivariation on every compact interval $[a, b]$ and is continuous from the right.

Proof. Since m is a vector valued measure, for any compact interval $[a, b]$ the set $E(a, b) =$

$$= \left\{ \sum_{i=1}^{M} \big[x(t_{2i}) - x(t_{2i-1})\big],\ n \geq 1,\ -\infty < a \leq t_1 < t_2 < \cdots < t_{2n} \leq b < \infty \right\}$$

$$= \left\{ \sum_{i=1}^{M} m(E_i),\ E_i = (t_{2i-1}, t_{2i}], \quad \text{being mutually disjoint} \right\}$$

is, according to L e w i s [8] and T w e d d l e [12], cf. also [11], relatively weakly compact in LCS X for any interval $[a, b]$, i.e., the function $x(\cdot)$ has weakly compact semivariation for any interval $[a, b]$, and it is easy to prove that is continuous from the right. $\square$

Conversely, we have the following.

Theorem 10. *If X is s-complete LCS and $x(\cdot)\colon R \to X$ has weakly compact semivariation on every interval $[a, b]$ and is continuous from the right, then there exists a unique vector valued measure $m\colon \mathcal{B} \to X$ such that*

$$m(E) = x(d) - x(c), \qquad E = (c, d].$$

P r o o f. Let $x(\cdot)$ be a function with weakly compact semivariation on every compact interval $[a, b]$ and continuous from the right. Let m be defined by the relation

$$m(E) = x(d) - x(c), \qquad E = (c, d],$$

and extended to the weakly σ-additive measure on the σ-ring of Borel sets in $[a, b]$.

Now let C be a ring of Borel subsets of the interval $[a, b]$ and m be a weakly σ-additive measure on C. Consider an arbitrary sequence (E_n), $E_n \in C$, E_n being disjoint, such that there exists $F \in C$ such that $E_n \subset F$, $n = 1, 2, \dots$ The set F can be chosen so that $F \subset [a, b]$, $a, b \in R$. Then

$$\left\{ \sum_{i=1}^{M} \big[x(t_{2i}) - x(t_{2i-1})\big],\ E_i = (t_{2i-1}, t_{2i}],\ a \le t_1 < t_2 < \cdots < t_{2M} \le b < \infty \right\}$$

is relatively weakly compact and therefore the series $\sum_{i=1}^{\infty} m(E_n)$ weakly unconditionally converges.

It follows that the necessary and sufficient condition in order to exist a unique vector valued measure $\tilde{m}$ on the δ-ring $\mathcal{B}$ with values in LCS X coinciding with m on C, $\tilde{m}(E) = x(d) - x(c)$, $E = (c, d]$, is that for any sequence of disjoint sets $E_n \in C$, for which there exists a set $F \in C$ such that $E_n \subset F$, $n = 1, 2, \dots$, the series

$$\sum_{i=1}^{\infty} m(E_n)$$

converges in X, (K l u v á n e k [7, Theorem A]). $\square$

At the end of this section we shall show the existence of the unique correspondence between the space WCS_0 of the vector valued functions with weakly compact semivariation and continuous from the right on $[a, b]$ and the vector valued measures on Borel sets in $[a, b]$.

If we now denote by m_0 the restriction of m to $\mathcal{B} \cap (a, b]$, and $y(t) = m\big([a,t]\big)$, $(a < t \le b)$, $y(a) = 0$, then $y(\cdot) \in WCS_0$ and we have

$$m_0(E) = y(d) - y(c) = x(d) - x(c),$$

whenever $E = (c, d] \subset (a, b]$. So we obtain

Corollary 11. *If $x(\cdot)\colon R \to X$ is a function of class WCS_0, where X in arbitrary quasicomplete LCS, then there exists a unique vector valued measure $m\colon \mathcal{B} \to X$ such that*

$$x(d) - x(c) = m(E), \qquad E = (c, d].$$

P r o o f. Let $x(\cdot)$ be a function with weakly compact semivariation on every compact interval $[a, b]$ and continuous from the right. Let m be defined by the relation

$$m(E) = x(d) - x(c), \qquad E = (c, d],$$

and extended to a σ-additive measure on the σ-ring of Borel sets in $[a, b]$. $\square$

This justifies the assertion at the beginning of the introduction that every function $x(\cdot)$ in $\mathcal{WCS}$ can be "normalized" to give a function of WCS_0 and that each function $x(\cdot)$ in $\mathcal{WCS}$ is associated in a natural way with a vector valued measure on the ring $\mathcal{B}$ of bounded Borel sets, and conversely.

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