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ON A DRYGAS INEQUALITY

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ABSTRACT. The goal of the paper is to give sufficient conditions for which the solution to the Drygas inequality is continuous.

Introduction

Throughout the paper, let $(X, +)$ be an abelian topological group uniquely divided by two satisfying the following two conditions:

1. $\bigcup_{n=0}^{\infty} 2^n U = X$, for every neighbourhood U of zero;
2. for every neighbourhood V of zero, there exists a neighbourhood W of zero such that $W \subset V$, $\frac{1}{2}W \subset W$.

We will investigate the problem of the continuity of solutions to the following Drygas inequality

$$\varphi(x+y) + \varphi(x-y) \leq 2\varphi(x) + \varphi(y) + \varphi(-y), \qquad x, y \in X, \tag{1}$$

where φ is a real function defined on X. This inequality is strictly connected with the so-called Drygas equation of the form

$$\varphi(x+y) + \varphi(x-y) = 2\varphi(x) + \varphi(y) + \varphi(-y), \qquad x, y \in X,$$

which is a generalization of an important quadratic functional equation

$$\varphi(x+y) + \varphi(x-y) = 2\varphi(x) + 2\varphi(y), \qquad x, y \in X \tag{2}$$

introduced in [1] in connection with the characterization of quasi-inner-product spaces. The general solution to the quadratic functional equation is of the form $\varphi(x) = B(x,x)$, $x \in X$, where B is a biadditive and symmetric function defined on X^2, and the general solution for the Drygas equation is of the form $\varphi(x) = B(x,x) + A(x)$, $x \in X$, where B (similarly as in the case of quadratic functional equation) is a biadditive and symmetric function, and A is an additive

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function [2]. Other interesting results connected with the Drygas functional equation may be found in [3], [4], [6] and [7]. In both cases (of quadratic functional equation as well as the Drygas functional equation), polynomial functions are the solutions and sufficient conditions of their continuity are well-known. In [5], we have found some sufficient conditions of the continuity of subquadratic functions that are the functions fulfilling the inequality arising from quadratic functional equation (2) by changing the equality sign by the inequality " $\leq$ " one. We use here similar methods as in [5] to obtain our main results. Throughout the paper, $\mathbb{R}, \mathbb{N}, \mathbb{N}_0$ denote the sets of all reals, the set of all positive integers or the set of all non-negative integers, respectively.

Some lemmas

Putting $x = y = 0$ in (1), we get $\varphi(0) \geq 0$. If $c > 0$ is an arbitrary real, then every function φ having the values in the interval $[c, 2c]$ is a solution to our inequality (1). This easy observation forces to make the following assumption

$$\varphi(0) = 0. \tag{3}$$

We will use the following lemmas.

Lemma 1 ([5]). *Let $\varphi : X \to \mathbb{R}$ be a function. Then for all $u \in X$, $\varepsilon > 0$ and each neighbourhood W_0 of zero there exists a neighbourhood W of zero such that $\frac{1}{2}W \subset W \subset W_0$ and, moreover,*

$$\inf\{f(v); v \in u + W\} + \varepsilon \geq \inf\left\{f(v'); v' \in u + \frac{1}{2}W\right\}.$$

Lemma 2. *Let $\varphi : X \to \mathbb{R}$ be a solution to (1) satisfying condition (3). If φ is locally upper bounded at zero, then it is locally upper bounded at every point $u \in X$.*

P r o o f. Let a neighbourhood V of zero and real number μ be chosen in such a way that

$$\varphi(v) \leq \mu$$

for every $v \in V$. Without loss of generality, we can assume that $V = -V$. It follows from (1) and (3) that

$$\varphi(2x) \leq 3\varphi(x) + \varphi(-x), \qquad x \in X.$$

Fix an arbitrary $u \in X$ and take $n \in \mathbb{N}$ so much that $u \in 2^n V$. It is easy to see that

$$\varphi(x) \leq 4^n \mu$$

for every $x \in U := 2^n V$. This finishes the proof. □

Lemma 3. *Let $\varphi : X \to \mathbb{R}$ be a solution to* (1) *satisfying condition* (3). *If φ is locally bounded (bilaterally) at zero then it is locally bounded (bilaterally) at every point of X.*

P r o o f. Take $m, M \in \mathbb{R}$ and symmetric with respect to zero neighbourhood U of zero in such a way that

$$m \leq \varphi(u) \leq M, \qquad u \in U.$$

By Lemma 2, φ is locally upper bounded at each point of X. We will show that it is also locally bounded below at every point of X. Fix an arbitrary $x_0 \in X$. Let $W_0 \subset U$ be a neighbourhood of zero and $M_{x_0} \in \mathbb{R}$ be chosen so that

$$\varphi(w) \leq M_{x_0}, \qquad w \in (x_0 + W_0) \cup (-x_0 - W_0).$$

Let V_0 be such symmetric with respect to zero neighbourhood of zero that $\frac{1}{2}V_0 \subset V_0 \subset W_0$. If $t \in x_0 + \frac{1}{2}V_0$, then $t = x_0 + \frac{1}{2}v$ with a $v \in V_0$. Putting

$$w := x_0 - \frac{1}{2}v,$$

we obtain

$$\begin{aligned} 2\varphi(t) + \varphi(w) + \varphi(-w) &\geq \varphi(t+w) + \varphi(t-w) \\ &= \varphi(2x_0) + \varphi(v), \end{aligned}$$

and, consequently,

$$2\varphi(t) \geq \varphi(2x_0) + m - 2M_{x_0}, \qquad t \in x_0 + \frac{1}{2}V_0.$$

This completes the proof. □

Lemma 4. *Let $\varphi : X \to \mathbb{R}$ be a solution to* (1) *satisfying condition* (3). *If there exists a point $x_0 \in X$ such that φ is locally bounded below at x_0 and $-x_0$ then it is locally bounded below at zero.*

P r o o f. Let V be a neighbourhood of zero and let $\mu \in \mathbb{R}$ be such that

$$\varphi(x) \geq \mu, \qquad x \in (x_0 + V) \cup (-x_0 + V).$$

For every $v \in V$, we hence get

$$\begin{aligned} 2\varphi(v) &\geq \varphi(v + x_0) + \varphi(v - x_0) - \varphi(x_0) - \varphi(-x_0) \\ &\geq 2\mu - \varphi(x_0) - \varphi(-x_0), \end{aligned}$$

as required. □

Results

Theorem 1. *Let $\varphi : X \to \mathbb{R}$ be a function satisfying inequality* (1) *and condition* (3). *If φ is locally bounded below at each point of X and upper semicontinuous at zero, then it is continuous in X.*

P r o o f. Let $u \in X$ be fixed and take arbitrary $\varepsilon > 0$. It follows from the upper semicontinuity at zero that there exists a neighbourhood U_0 of zero such that

$$\varphi(t) \le \varepsilon, \qquad t \in U_0. \tag{4}$$

Without loss of generality, we may assume that φ is lower bounded on the set $U_0 + u$. Take a symmetric with respect to zero neighbourhood W of zero such that

$$\frac{1}{2}W \subset W \subset U_0,$$

and, simultaneously,

$$\inf\{\varphi(v); v \in u + W\} + \varepsilon \ge \inf\left\{\varphi(v'); v' \in u + \frac{1}{2}W\right\}.$$

For every $v \in u + W$, we get

$$\varphi(u) + \varphi(v) \le 2\varphi\left(\frac{u+v}{2}\right) + \varphi\left(\frac{u-v}{2}\right) + \varphi\left(\frac{v-u}{2}\right) \le 2\varphi\left(\frac{u+v}{2}\right) + 2\varepsilon.$$

Observe that $\frac{v+u}{2} \in u + \frac{1}{2}W$. Therefore,

$$\varphi(u) + \inf\{\varphi(v); v \in u + W\} \le 2\inf\left\{\varphi(v'); v' \in u + \frac{1}{2}W\right\} + 2\varepsilon,$$

and, consequently,

$$\varphi(u) \le \inf\{\varphi(v); v \in u + W\} + 4\varepsilon.$$

This finishes the proof of the lower semicontinuity of φ at the point u. To prove the upper semicontinuity at u, take, as above, an arbitrary $\varepsilon > 0$ and let U_0 be a neighbourhood of zero such that condition (4) is fulfilled. Since φ is upper semicontinuous at u then there exists a neighbourhood $W \subset U_0$ of zero such that

$$\varphi(u) \le \varphi(u + w) + \varepsilon, \qquad w \in W.$$

We may assume that W is symmetric with respect to zero. Thus for each $w \in W$ we have

$$\begin{aligned}\varphi(u - w) &\le 2\ \varphi(u) + \varphi(w) + \varphi(-w) - \varphi(u + w)\\ &\le \varphi(u) + 3\varepsilon.\end{aligned}$$

This finishes the proof of the upper semicontinuity of φ at u and completes the proof of our Theorem 1. $\square$

As an esasy consequence of Theorem 1 and Lemmas 3 and 4, we obtain the following theorems.

Theorem 2. *Let $\varphi : X \to \mathbb{R}$ be a function satisfying inequality* (1) *and condition* (3). *If φ is locally bounded below at the points of $x_0 \neq 0$ and $-x_0$ and upper semicontinuous at zero, then it is continuous in X.*

Theorem 3. *Let $\varphi : X \to \mathbb{R}$ be a function satisfying inequality* (1) *and condition* (3). *If φ is continuous at zero then it is continuous in X.*

Lemma 5. *Let $\varphi : X \to \mathbb{R}$ be a solution to* (1). *If it is locally bounded above at zero and locally bounded below at a point x_0 then it is locally bounded bilaterally at each point of some neighbourhood of zero.*

P r o o f. Let U_0 be a neighbourhood of zero symmetric with respect to zero and let $M \in \mathbb{R}$ be a constant such that

$$\varphi(x) \leq M, \qquad x \in U_0.$$

Since φ is locally bounded below at x_0, we may assume that

$$\varphi(t) \geq m, \qquad t \in x_0 + U_0,$$

with a constant $m \in \mathbb{R}$. By virtue of (1), for every $y \in U_0$, we hence get

$$\varphi(y) \geq \varphi(x_0 + y) + \varphi(x_0 - y) - 2\varphi(x_0) - \varphi(-y) \geq 2\,m - 2\varphi(x_0) - M.$$

This finishes the proof. □

Theorem 4. *Let $\varphi : X \to \mathbb{R}$ be a function satisfying inequality* (1) *and condition* (3). *If φ is upper semicontinuous at zero and locally bounded below at a point x_0, then it is continuous in X.*

P r o o f. It follows from the upper semicontinuity at zero that φ is locally bounded above at zero. According to Lemma 5, there exists a point $x_0 \neq 0$ such that φ is locally bounded below at x_0 and $-x_0$ simultaneously. Our assertion now follows from Theorem 2. □

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