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ON POINTS OF THE REGULAR DENSITY

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ABSTRACT. In this note, we introduce the notion of regular density. Next, we prove that $x \in \mathbb{R}$ is the regular density point of a measurable set A if and only if it is an O'Malley point of A.

1. Notation

In the sequel, we use the following symbols:

χ_A	–	the characteristic function of the set A,
$\mu(A)$	–	the Lebesgue measure of the set A,
$\bigvee_a^b f$	–	the total variation of the function f on $[a,b]$,
$\mathrm{D}(A,I) = \mu(A \cap I)/\lvert I\rvert$	–	the average density of A on the interval I,
$\mathrm{d}(A,x)$	–	the density of the set A at x,
$\mathrm{d}^+(A,x)$	–	the right side density of the set A at x,
$\Phi(A)$	–	the set of all density points of the set A.

2. Definitions

Let $A \subset \mathbb{R}$ be an arbitrary Lebesgue measurable set and $x_0 \in \mathbb{R}$. Put

$$f_{x_0}(h) = \mathrm{D}\big(A, [x_0 - h, x_0 + h]\big) \qquad \text{for} \quad h > 0 \quad \text{and} \quad f_{x_0}(0) = 1.$$

Obviously, the function f_{x_0} is continuous on $\mathbb{R}^+$ if and only if $x_0 \in \Phi(A)$.
Similarly, put

$$f_{x_0}^+(h) = \mathrm{D}\big(A, [x_0, x_0 + h]\big) \qquad \text{for} \quad h > 0 \quad \text{and} \quad f_{x_0}^+(0) = 1.$$

We shall say that x_0 is the *regular density point* of the set A if and only if the following conditions are satisfied: (1) $x_0 \in \Phi(A)$ and (2) $\bigvee_0^1 f_{x_0} < +\infty$. We denote the set of the regular density points of the set A by $\Phi_R(A)$.

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Using the function $f^+_{x_0}$ we obtain the definition of the right hand regular density point. The definitions of Φ^+_R, $f^-_{x_0}$ and Φ^-_R are self-explaining.

For a measurable set $A \subset \mathbb{R}$ and a point $x_0 \in \mathbb{R}$, we will say that x_0 is an *O'Malley point* of the set A if and only if

$$\int_0^1 \frac{1}{h}\big(\chi_{A'}(x_0+h)+\chi_{A'}(x_0-h)\big)\,dh < +\infty.$$

The point x_0 is a *right side O'Malley point of* A if and only if

$$\int_0^1 \frac{1}{h}\chi_{A'}(x_0+h)\,dh < +\infty.$$

A set of all O'Malley (resp. right side O'Malley) points of the set A will be denoted by $\Phi_{OM}(A)$ (resp. $\Phi^+_{OM}(A)$). The operator Φ_{OM} and the topology generated by it were examined in [1]. It was proved that Φ_{OM} has the following properties:

a) if $\mu(A \triangle B) = 0$, A and B are Lebesgue measurable, then $\Phi_{OM}(A) = \Phi_{OM}(B)$,

b) $\Phi_{OM}(\emptyset) = \emptyset$, $\Phi_{OM}(\mathbb{R}) = \mathbb{R}$,

c) $\Phi_{OM}(A \cap B) = \Phi_{OM}(A) \cap \Phi_{OM}(B)$ if A and B are Lebesgue measurable.

However, for O'Malley points, the Lebesgue Density Theorem does not hold. In [1], one can find an example of a measurable set $E \subset [0,1]$ such that $\mu\big(E \setminus \Phi_{OM}(E)\big) > 0$. Simultaneously, $\Phi_{OM}(A) \subset \Phi(A)$ for each measurable $A \subset \mathbb{R}$, so the family

$$\mathcal{T}_{OM} = \big\{A \subset \mathbb{R} : A \text{ is Lebesgue measurable and } A \subset \Phi_{OM}(A)\big\}$$

is a topology strictly stronger than the natural topology and strictly weaker than the density topology.

3. The "right side" case

The following example shows that the condition of regular density is essentially stronger than the condition of the ordinary density.

Example 1. We shall construct the set A such that $0 \in \Phi(A) \setminus \Phi^+_R(A)$.

Let us define the sequence a_n by the following recursion

$$a_1 = 1 \quad \text{and} \quad a_n = a_{n-1}/\big(n(n+1)\big) \qquad \text{for every} \quad n > 1.$$

Simultaneously, for $n > 1$, let $b_n = n \cdot a_{n-1}/(n+1)$. Let $A = (-\infty, 0] \cup \bigcup_{n>1}[a_n, b_n]$. Observe that, for any n, the function f_0 is increasing on the interval $[a_n, b_n]$ and decreasing on the interval $[b_{n+1}, a_n]$.

Let us consider $h \in [b_n, b_{n-1}]$. Then

$$f_0(h) \geq f_0(a_{n-1}) > \frac{\mu(A \cap [0, a_{n-1}])}{a_{n-1}} > \frac{b_n - a_n}{a_{n-1}} = 1 - \frac{1}{n},$$

which implies that $0 \in \Phi(A)$. At the same time,

$$f_0(a_{n-1}) < \frac{b_n}{a_{n-1}} = 1 - \frac{1}{n+1} \quad \text{and} \quad f_0(b_n) > \frac{b_n - a_n}{b_n} = 1 - \frac{1}{n^2},$$

so

$$f_0(b_n) - f_0(a_n) > \frac{1}{n+2} - \frac{1}{n^2}$$

and the series

$$\sum_{n=1}^{\infty} \big(f_0(b_n) - f_0(a_n)\big) \quad \text{is divergent.}$$

In the sequel, we shall express the number $\bigvee_0^1 f_{x_0}^+$ in a more convenient way.

Lemma 1. *Let $x_0 \in \Phi(A)$. Then*

$$\bigvee_0^1 f_{x_0}^+ = \int\limits_{[0,1]\cap(A-x_0)} \frac{1}{h}\big(1 - f_{x_0}^+(h)\big)\, dh + \int\limits_{[0,1]\cap(A'-x_0)} \frac{1}{h} f_{x_0}^+(h)\, dh.$$

P r o o f. Since $x_0 \in \Phi(A)$ the function $f_{x_0}^+$ is continuous. Moreover, for every $\epsilon > 0$ the function $f_{x_0}^+$ restricted to the interval $[\epsilon, 1]$ satisfies the Lipschitz condition with the constant $\frac{1}{\epsilon}$, so it is absolutely continuous. Hence,

$$\bigvee_\epsilon^1 f_{x_0}^+ = \int_\epsilon^1 |(f_{x_0}^+)'(h)|\, dh$$

and finally,

$$\bigvee_0^1 f_{x_0}^+ = \int_0^1 |(f_{x_0}^+)'(h)|\, dh.$$

To simplify the calculations, we can replace the derivative of the function f^+ with the right hand side derivative:

$$\begin{aligned}\bigvee_0^1 f_{x_0}^+ &= \int_0^1 \left| \lim_{h^\star \to h+} \frac{f_{x_0}^+(h^\star) - f_{x_0}^+(h)}{h^\star - h} \right| dh \\ &= \int_0^1 \left| \lim_{h^\star \to h+} \frac{1}{h^\star - h} \left(\frac{\mu(A \cap [x_0, x_0 + h^\star])}{h^\star} - \frac{\mu(A \cap [x_0, x_0 + h])}{h} \right) \right| dh\end{aligned}$$

$$= \int_0^1 \left| \lim_{h^\star \to h+} \frac{1}{h^\star - h} \left(\frac{\mu(A \cap [x_0, x_0+h])}{h^\star} + \frac{\mu(A \cap [x_0+h, x_0+h^\star])}{h^\star} - \frac{\mu(A \cap [x_0, x_0+h])}{h} \right) \right| dh$$

$$= \int_0^1 \left| \lim_{h^\star \to h+} \frac{1}{h^\star - h} \left(\frac{(h-h^\star)\mu(A \cap [x_0, x_0+h])}{h^\star h} + \frac{\mu(A \cap [x_0+h, x_0+h^\star])}{h^\star} \right) \right| dh$$

$$= \int_0^1 \left| \lim_{h^\star \to h+} \left(-\frac{\mu(A \cap [x_0, x_0+h])}{h^\star h} + \frac{1}{h^\star} \frac{\mu(A \cap [x_0+h, x_0+h^\star])}{h^\star - h} \right) \right| dh$$

$$= \int_0^1 \left| \lim_{h^\star \to h+} \left(-\frac{1}{h^\star} f^+_{x_0}(h) + \frac{1}{h^\star} f^+_{(x_0+h)}(h^\star - h) \right) \right| dh$$

$$= \int_0^1 \left| -\frac{1}{h} f^+_{x_0}(h) + \frac{1}{h} \mathrm{d}^+(A, x_0+h) \right| dh.$$

By virtue of the Lebesgue density theorem,

$$\mathrm{d}(A,x) = \mathrm{d}^+(A,x) = \chi_A(x) \quad \text{a.e.},$$

hence

$$\bigvee_0^1 f^+_{x_0} = \int_0^1 \frac{1}{h} \left| -f^+_{x_0}(h) + \chi_A(x_0+h) \right| dh = \int_0^1 \frac{1}{h} \left| -f^+_{x_0}(h) + \chi_{(A-x_0)}(h) \right| dh.$$

For $(x_0+h) \in A$, we have

$$| - f^+_{x_0}(h) + \chi_{(A-x_0)}(h)| = 1 - f^+_{x_0}(h),$$

in the opposite case,

$$| - f^+_{x_0}(h) + \chi_{(A-x_0)}(h)| = f^+_{x_0}(h).$$

Therefore,

$$\bigvee_0^1 f^+_{x_0} = \int_{[0,1]\cap(A-x_0)} \frac{1}{h}\big(1 - f^+_{x_0}(h)\big)\, dh + \int_{[0,1]\cap(A'-x_0)} \frac{1}{h} f^+_{x_0}(h)\, dh.$$

□

The last two numbers will be denoted by $C_1(A,x_0)$ and $C_2(A,x_0)$, respectively.

Proposition 1. *For every measurable $A \subset \mathbb{R}$ and for every $x_0 \in \mathbb{R}$, the numbers $C_1(A,x_0)$, $C_2(A,x_0) \in [0,+\infty]$, moreover, $C_1(A,x_0)$ is finite if and only if $C_2(A,x_0)$ is finite.*

P r o o f. Notice that for every positive ϵ, by virtue of absolute continuity of f_{x_0} on $[\epsilon, 1]$, we have

$$1 \geq |f^+_{x_0}(1) - f^+_{x_0}(\epsilon)| = \left| \int_\epsilon^1 (f^+_{x_0})'(h)\, dh \right|.$$

After the calculations analogous to those from the proof of Lemma 1, the last term can be expressed as

$$\begin{aligned}\left| \int_\epsilon^1 (f^+_{x_0})'(h)\, dh \right| &= \left| \int_\epsilon^1 \frac{1}{h} \left(\chi_{(A-x_0)}(h) - f^+_{x_0}(h) \right) dh \right| \\ &= \left| \int_{[\epsilon,1]\cap(A-x_0)} \frac{1}{h} \left(1 - f^+_{x_0}(h)\right) dh \;-\; \int_{[\epsilon,1]\cap(A'-x_0)} \frac{1}{h} f^+_{x_0}(h)\, dh \right|.\end{aligned}$$

The last expression tends to $|C_1 - C_2|$ when ϵ tends to 0, so when one of the numbers C_1 and C_2 is finite, so is the other. □

Theorem 1. *For every measurable set $A \subset \mathbb{R}$*

$$\Phi^+_{OM}(A) = \Phi^+_R(A).$$

P r o o f. Observe first that if $g(x) : [0,1] \to [0,1]$ and $\epsilon \in (0,1)$, then

$$\int_0^1 \frac{1}{h} g(h)\, dh < +\infty \quad \text{if and only if} \quad \int_0^\epsilon \frac{1}{h} g(h)\, dh < +\infty.$$

Since $x \in \Phi(A)$, we can choose such $\epsilon > 0$ that $f^+_x(h) > \frac{1}{2}$ for $0 < h < \epsilon$. Therefore,

$$\frac{1}{2} \int_{[0,\epsilon]\cap(A'-x)} \frac{1}{h}\, dh \;<\; \int_{[0,\epsilon]\cap(A'-x)} \frac{1}{h} f^+_x(h)\, dh \;\leq\; \int_{[0,\epsilon]\cap(A'-x)} \frac{1}{h}\, dh,$$

so

$$\int_0^\epsilon \frac{1}{h} \chi_{A'}(x+h)\, dh \;=\; \int_{[0,\epsilon]\cap(A'-x)} \frac{1}{h}\, dh < +\infty \;\iff\; \int_{[0,\epsilon]\cap(A'-x)} \frac{1}{h} f^+_x(h)\, dh < +\infty.$$

By virtue of Proposition 1, the last equivalence gives us the thesis. □

4. The bilateral case

Since

$$\int_0^1 \frac{1}{h}\big(\chi_{A'}(x_0+h)+\chi_{A'}(x_0-h)\big)\,dh \;=\int_0^1 \frac{1}{h}\chi_{A'}(x_0+h)\,dh \;+\int_0^1 \frac{1}{h}\chi_{A'}(x_0-h)\,dh,$$

we obtain that for every measurable $A \subset \mathbb{R}$

$$\Phi_{OM}(A) = \Phi_{OM}^{+}(A) \cap \Phi_{OM}^{-}(A).$$

Similarly,

$$\begin{aligned}&\frac{1}{2h}\mu\big(A \cap [x_0-h, x_0+h]\big)\\ &= \frac{1}{2}\left(\frac{1}{h}\mu\big(A \cap [x_0, x_0+h]\big) + \frac{1}{h}\mu\big(A \cap [x_0-h, x_0]\big)\right),\end{aligned}$$

so,

$$f_{x_0}(h) = \frac{1}{2}\big(f_{x_0}^{+}(h) + f_{x_0}^{-}(h)\big).$$

Therefore, if, for a given point x_0, the functions $f_{x_0}^{+}$ and $f_{x_0}^{-}$ are of bounded variation, the function f_{x_0} is of bounded variation, too. This means that

$$\Phi_R(A) \supset \Phi_R^{+}(A) \cap \Phi_R^{-}(A).$$

Now, we shall show

Theorem 2. *For every measurable set $A \subset \mathbb{R}$,*

$$\Phi_R(A) = \Phi_R^{+}(A) \cap \Phi_R^{-}(A).$$

P r o o f. To prove that $\Phi_R(A) \subset \Phi_R^{+}(A) \cap \Phi_R^{-}(A)$, it suffices to show, that $\bigvee_0^1 f_x^{-} = +\infty$ implies $\bigvee_0^1 f_x = +\infty$. We shall prove it for $x = 0$. We will write f, f^{+}, f^{-} instead of f_0, f_0^{+}, f_0^{-}, and A^{+}, A^{-} instead of $A \cap [0,1]$ and $(-A) \cap [0,1]$, respectively. Assume that 0 is the density point of the set A and n is an arbitrary natural number. There exists $H > 0$ such that

$$f^{+}(h) > 1 - \frac{1}{n} \quad \text{and} \quad f^{-}(h) > 1 - \frac{1}{n} \qquad \text{for} \quad h \in (0, H).$$

Let

$$[x', x''] \subset (0, H) \quad \text{with} \quad \mathrm{D}\big(A^{-}, [x', x'']\big) < \frac{1}{n} \quad \text{and} \quad f^{-}(x') = a^{-}.$$

Since
$$a^- > 1 - \frac{1}{n},$$
then
$$\begin{aligned}&f^-(x') - f^-(x'')\\ &= a^- - \frac{a^- x_1 + \mathrm{D}(A^-, [x', x'']) \cdot (x'' - x')}{x''} > a^- - \frac{a^- x' + \frac{1}{n}(x'' - x')}{x''}\\ &= \left(a^- - \frac{1}{n}\right)\frac{x'' - x'}{x''} > \left(1 - \frac{2}{n}\right)\frac{x'' - x'}{x''}.\end{aligned}$$

On the other hand,
$$a^+ = f^+(x') > 1 - \frac{1}{n}$$
and
$$\begin{aligned}f^+(x'') - f^+(x') &= f^+(x'') - a^+\\ &= \frac{a^+ x'}{x''} + \frac{\mu(A^+ \cap (x', x''))}{x''} - a^+ < \frac{a^+ x'}{x''} + \frac{x'' - x'}{x''} - a^+\\ &= \left(1 - a^+\right)\left(\frac{x'' - x'}{x''}\right) < \frac{1}{n}\left(\frac{x'' - x'}{x''}\right).\end{aligned}$$

In particular, for $n = 4$, we obtain
$$f^+(x'') - f^+(x') < \frac{1}{2}\big(f^-(x') - f^-(x'')\big),$$
and consequently,
$$\frac{1}{4}\big(f^-(x') - f^-(x'')\big) < f(x'') - f(x').$$

Assume now that
$$\bigvee_0^1 f^- = +\infty.$$
There exists the decreasing sequence $(x_n)_{n\in\mathbb{N}}$ tending to 0 such that
$$\sum_{n=1}^{\infty} |f^-(x_{n+1}) - f^-(x_n)| = +\infty.$$
We can assume additionaly that for every $n \in \mathbb{N}$,
$$f^-(x_{2n}) - f^-(x_{2n-1}) > 0 \quad \text{and} \quad f^-(x_{2n+1}) - f^-(x_{2n}) < 0$$
and that $x_1 < H$.

We now concentrate our attention on the number
$$f^-(x_2) - f^-(x_1),$$
yet the foregoing argument can be repeated for every positive component.

Since $f^-(x_2) - f^-(x_1) > 0$, we have $\mu\big(A^- \cap (x_2, x_1)\big) < x_1 - x_2$. So,

$$\mu\big((A^-)' \cap (x_2, x_1)\big) > 0.$$

Denote $y_i = f^-(x_i)$. For $y \in (y_1, y_2)$, let $z(y) = \max\{x \in [x_2, x_1] : f^-(x) = y\}$. Let $B = z\big([y_1, y_2]\big) \subset [x_2, x_1]$. Observe that if $x \in \Phi\big(A^- \cap (x_2, x_1)\big)$, then $x \notin B$. Hence, by virtue of the Lebesgue density theorem,

$$\mu\Big(B \setminus \Phi\big((A^-)' \cap (x_2, x_1)\big)\Big) = 0.$$

But $f^-(B) = (y_1, y_2)$ and the function f^- satisfies the Lusin N condition, so

$$\mu\bigg(f^-\Big(B \cap \Phi\big((A^-)' \cap (x_2, x_1)\big)\Big)\bigg) = y_2 - y_1.$$

Let

$$E = f^-\Big(B \cap \Phi\big((A^-)' \cap (x_2, x_1)\big)\Big).$$

If $y \in E$, then there exists exactly one such number $x' \in B \cap \Phi\big((A^-)' \cap (x_2, x_1)\big)$ that $y = f^-(x')$. Let $x'' \in (x', x_1)$. If the interval $[x', x'']$ is short enough, then

(1) $f^-(x') > f^-(x'')$,

(2) $\big[f^-(x''), f^-(x')\big] \subset (y_1, y_2)$,

(3) $\mathrm{D}\big(A^-, [x', x'']\big) < \frac{1}{n}$.

Consequently,

$$f^+(x'') - f^+(x') < \frac{1}{2}\big(f^-(x') - f^-(x'')\big)$$

and

$$\frac{1}{4}\big(f^-(x') - f^-(x'')\big) < f(x'') - f(x').$$

The family $\mathcal{M}$ of intervals $f^-\big([x', x'']\big)$ satisfying the above conditions covers the set E in the Vitali sense. Hence, by virtue of the Vitali covering theorem, there exists a sequence $\big\{[z_k'', z_k']\big\}_{k\in\mathbb{N}}$ of pairwise disjoint intervals from $\mathcal{M}$ such that

$$\mu\left(E \setminus \bigcup_{k\in\mathbb{N}} [z_k, t_k]\right) = 0 \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} [z_k, t_k] \subset (y_2, y_1).$$

Hence,

$$\sum_{k\in\mathbb{N}} (t_k - z_k) = y_1 - y_2.$$

For every $k \in \mathbb{N}$, the interval

$$[z_k, t_k] = f^-\big([x_k', x_k'']\big).$$

From the definition of the set B, it follows that $f^-(x_k') = t_k$. Let $x_k''' \in (x_k', x_k'']$ such that $f(x_k''') = z_k$.

Now, if $N \in \mathbb{N}$ is big enough, we have

$$\bigvee_{x_2}^{x_1} f \geq \sum_{k=1}^{N} |f(x'_k) - f(x'''_k)| \geq \frac{1}{4} \sum_{k=1}^{N} |f^-(x'_k) - f^-(x'''_k)| \geq \frac{1}{8}\big(f^-(x_2) - f^-(x_1)\big).$$

Using the same argument to every component of the type

$$f^-(x_{2n}) - f^-(x_{2n-1}),$$

we obtain that

$$\bigvee_{0}^{1} f = +\infty.$$

This finishes the proof. □

From the last result we obtain

Corollary 1. *For every measurable set $A \subset \mathbb{R}$,*

$$\Phi_{OM}(A) = \Phi_R(A).$$

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