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GENERALIZED DENSITIES ON $\mathbb{R}^n$ AND THEIR APPLICATIONS

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ABSTRACT. We examine some generalized densities (called (ψ, n)-densities) obtained as a result of strengthening the Lebesgue Density Theorem. It turns out that these notions are the generalizations of superdensity, enhanced density and m-density, and have some applications in the theory of sets of finite perimeter and in Sobolev spaces.

1. Introduction

Lebesgue density is an important notion of measure theory and the theory of real functions. The basic theorem connected with this topic is Lebesgue Density Theorem which was proved at the beginning of the 20th century and stated that almost each point of any Lebesgue measurable set is its density point. The notion of density was used by Lebesgue also in his differentiation theorem which was generalized into $\mathbb{R}^n$ by Lebesgue in 1910. The general version of this theorem concerning Radon measures was obtained by Besicovitch [3]. The classical textbooks of geometric measure theory present very general versions of his results (known also as Lebesgue-Besicovitch Differentiation Theorem, see, for instance [8]). The consequence of this theorem (*almost all points of Lebesgue measurable set $E \subset \mathbb{R}^n$ are density points of E and almost all points of $\mathbb{R}^n \setminus E$ are dispersion points of this set*) plays the crucial role in real functions theory.

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In 1936, Stanisław Ulam in The Scottish Book asked about the possibility of strengthening Lebesgue Density Theorem [17], Problem 146. The answer for the real line was presented by S. J. Taylor in 1959 [18]. The notion of superdensity presented in [16] is a special case of Taylor's results. Some generalizations of this notion were the inspiration for [4]. To our surprise, the theory of generalized density on $\mathbb{R}^n$ has wide applications in seemingly distant theories. It shows that our results from [10] and [12] are quite practical.

In Section 2 of this paper, we define two notions of densities on $\mathbb{R}^n$ using them to examine some density-type topologies. We want to draw the reader's attention to the importance of the chosen differentiation basis (in one case, we followed Besicovich). The main aim of Section 3 is to present the mentioned applications in the theory of sets with finite perimeter and in Sobolev spaces.

2. On ψ-density topologies on $\mathbb{R}^n$

Remind that $x \in \mathbb{R}$ is called a density point of a measurable set $A \subset \mathbb{R}$ if $\lim_{\lambda(I)\to 0} \frac{\lambda(A' \cap I)}{\lambda(I)} = 0$, where I is an interval containing x, λ stands for the Lebesgue measure, and A' is the complement of A. In [18], Taylor modified the definition of a density point by introducing a new factor $\psi\big(\lambda(I)\big)$ in denominator of the fraction $\frac{\lambda(A' \cap I)}{\lambda(I)}$, where ψ is a nondecreasing continuous function from $(0,\infty)$ to $(0,\infty)$ such that $\lim_{x\to 0^+} \psi(x) = 0$ (the family of such a function will be denoted by $\mathcal{C}$). He formulated the most important and interesting results in the following theorems.

The First Taylor's Theorem ([18], Theorem 3). *For any Lebesgue measurable set $E \subset \mathbb{R}$ there exists a function $\psi \in \mathcal{C}$ such that for almost all $x \in E$,*

$$\lim_{\lambda(I)\to 0} \frac{\lambda(E' \cap I)}{\lambda(I)\psi(\lambda(I))} = 0,$$

where I runs through all intervals containing x.

The Second Taylor's Theorem ([18], Theorem 4). *For any function $\psi \in \mathcal{C}$ and any real number α, $0 < \alpha < 1$, there exists a perfect set $E \subset [0,1]$ such that $\lambda(E) = \alpha$ and for all $x \in E$*

$$\limsup_{\lambda(I)\to 0} \frac{\lambda(E' \cap I)}{\lambda(I)\psi(\lambda(I))} = \infty,$$

where I runs through all intervals containing x.

Taylor's results gave rise to concepts of ψ-density and the ψ-density topology on $\mathbb{R}$ [19]. The obtained results were extended firstly onto $\mathbb{R}^2$ [10], in Polish and later on $\mathbb{R}^n$ for $n > 2$ [12].

In $\mathbb{R}^n$, $n \geq 2$, we can obtain different notions of density-type topologies depending on differentiation bases at a point. The following differentiation bases S (named after [16]) are the most useful:

- the cube base $\mathcal{S}_{\mathcal{Q}} = \{Q(x,r)\colon x \in \mathbb{R}^n, r > 0\}$, where $Q(x,r)$ is a cube with the center at $x \in \mathbb{R}^n$ and edge length r;
- the symmetric base $\mathcal{S}_{\mathcal{B}} = \{B(x,r)\colon x \in \mathbb{R}^n, r > 0\}$, where $B(x,r)$ is a ball of radius r centered at $x \in \mathbb{R}^n$.

The notion of (ψ, n)-density point of a set $A \subset \mathbb{R}^n$ is a modification of ordinary density. Let us consider n-dimensional cube $Q(x,r)$ with the edges parallel to the coordinate axes. Obviously, a volume of $Q(x,r)$ (denoted by $|Q(x,r)|$) is equal to r^n. By λ_n we denote n-dimensional Lebesgue measure and for $n = 1$ we omit the number. Remind that $x \in \mathbb{R}^n$ is an ordinary density point of a measurable set $A \subset \mathbb{R}^n$ (we will write $A \in \mathcal{L}_n$) if

$$\lim_{r \to 0} \frac{\lambda_n(A' \cap Q(x,r))}{|Q(x,r)|} = 0. \tag{1}$$

Definition 1. Let $\psi \in \mathcal{C}$, $x \in \mathbb{R}^n$, $A \in \mathcal{L}_n$. We will say that x is a (ψ, n)-density point of A with respect to cubes if

$$\lim_{r \to 0} \frac{\lambda_n(A' \cap Q(x,r))}{|Q(x,r)| \cdot \psi(|Q(x,r)|)} = 0. \tag{2}$$

We will say that x is a (ψ, n)-dispersion point of A with respect to cubes if x is a (ψ, n)-density point of A' with respect to cubes.

In other words, x is a (ψ, n)-density point of A with respect to cubes if $\lim_{r\to 0} \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi(r^n)} = 0$. Note that for $n = 1$ the notion of $(\psi, 1)$-density point coincides with ψ-density point on $\mathbb{R}$. If $n = 1$ and $\psi = id$, then we obtain superdensity described by Lukeš, Malý and Zajiček [16]. If $n = 2$, then the notion of $(\psi, 2)$-density point is equivalent to ordinary ψ-density point on the plane described in [10] and [12].

For any $A \in \mathcal{L}_n$ we define the set

$$\Phi_{(\psi,n)}(A) = \{x \in \mathbb{R}^n \colon x \text{ is a } (\psi, n) - \text{density point of } A \text{ with respect to cubes}\}$$

(for short, we will write Φ_ψ in case when $n = 1$). Observe that

$$\Phi_{(\psi,n)}(A) = \left\{x \in \mathbb{R}^n \colon \lim_{r \to 0} \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi(r^n)} = 0\right\} = \bigcap_{k \in \mathbb{N}} \bigcup_{p \in \mathbb{N}} \bigcap_{r \in (0, \frac{1}{p})} A_{kr},$$

where

$$A_{kr} = \left\{x \in \mathbb{R}^n \colon \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi(r^n)} \leq \frac{1}{k}\right\}.$$

Since the function $F(x,r) = \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi(r^n)}$ is continuous with respect to x, and A_{kr} is a closed set for fixed r and k, it follows that $\Phi_{(\psi,n)}(A)$ is a $F_{\sigma\delta}$ set. It means that $\Phi_{(\psi,n)} \colon \mathcal{L}_n \to \mathcal{L}_n$. Obviously, $\Phi_{(\psi,n)}(\emptyset) = \emptyset$, $\Phi_{(\psi,n)}(\mathbb{R}^n) = \mathbb{R}^n$. Moreover, for any measurable $A, B \subset \mathbb{R}^n$ we have $\Phi_{(\psi,n)}(A \cap B) = \Phi_{(\psi,n)}(A) \cap \Phi_{(\psi,n)}(B)$ and if $\lambda_n(A \triangle B) = 0$, then $\Phi_{(\psi,n)}(A) = \Phi_{(\psi,n)}(B)$. Since any (ψ, n)-density point is an ordinary density point, then for any set $A \in \mathcal{L}_n$ the difference $\Phi_{(\psi,n)}(A) \setminus A$ is a set of measure zero. Hence, the operator $\Phi_{(\psi,n)}$ is an almost lower density operator (see [14]). It is not a lower density operator. Indeed, it is sufficient to take a set $A = E \times [0,1]^{n-1}$, where $E \subset [0,1]$ is a set of positive measure constructed in The Second Taylor's Theorem. Then, $A \subset \mathbb{R}^n$ is a perfect set ofpositive measure which has no (ψ, n)-density points.

The pair $(\mathcal{L}_n, \mathcal{N}_n)$, where $\mathcal{N}_n$ is the σ-ideal of null sets on $\mathbb{R}^n$, has the hull property. In consequence, the family

$$\mathcal{T}_{(\psi,n)} = \{A \in \mathcal{L}_n \colon A \subset \Phi_{(\psi,n)}(A)\}$$

is a topology [14]. It will be called (ψ, n)-density topology. It is strictly stronger than the natural topology $\mathcal{T}_{nat}^n$ and strictly weaker than the density topology $\mathcal{T}_d^n$ on $\mathbb{R}^n$ (for $n = 1$ it is the ψ-density topology described in [19]).

To see how different these three topologies are, we recall the definition of similarity.

Definition 2 ([2]). Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two different topologies defined on the same set X. We will say that topological spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are similar (or the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are similar) if they have the same families of sets with nonempty interiors (we will write then $\mathcal{T}_1 \sim \mathcal{T}_2$).

It is easy to observe that $\mathcal{T}_d^n \not\sim \mathcal{T}_{(\psi,n)}$ and $\mathcal{T}_{nat}^n \not\sim \mathcal{T}_{(\psi,n)}$. Take a set E from The First Taylor Theorem. As a set of positive measure, it has a nonempty interior in the density topology on the real line, so $A = E \times \mathbb{R}^{n-1}$ has nonempty interior in $\mathcal{T}_d^n$. Simultaneously, it has the empty interior in $\mathcal{T}_{(\psi,n)}$. Analogously, $B = \mathbb{Q}' \times \mathbb{R}^{n-1}$ has nonempty interior in $\mathcal{T}_{(\psi,n)}$, but its interior in $\mathcal{T}_{nat}^n$ is empty. Among (ψ, n)-density topologies there exist nonsimilar topologies, but the proof is much more difficult. For $n = 1$, it is presented in [13]. In the proof for a certain function $\psi_1 \in \mathcal{C}$ there is constructed the Cantor-like set (perfect and nowhere dense) of positive measure with nonempty interior in $\mathcal{T}_{\psi_1}$ topology and empty interior in $\mathcal{T}_{\psi_2}$ topology for another function $\psi_2 \in \mathcal{C}$.

From the fact that the operator $\Phi_{(\psi,n)}$ is an almost lower density operator, we immediately obtain (for details, see [14]) that $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is neither a first countable, nor a second countable, nor a separable, nor a Lindelöf space. Several other properties of $\mathcal{T}_{(\psi,n)}$ follow from the proofs carried for the topology $\mathcal{T}_\psi$.

It is easy to check that for any set $U \in \mathcal{T}_\psi$ the set $U \times \mathbb{R}^{n-1}$ is open in $\mathcal{T}_{(\psi,n)}$ for the same function $\psi \in \mathcal{C}$. Consequently, if $E \subset \mathbb{R}$ is $\mathcal{T}_\psi$-nowhere dense, then $E \times \mathbb{R}^{n-1}$ is $\mathcal{T}_{(\psi,n)}$-nowhere dense.

To consider the other properties of a space $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$, we will use the following lemma which is a consequence of The Second Taylor's Theorem.

Lemma 3. *For any $n \in \mathbb{N}$ and any $\psi \in \mathcal{C}$ there is an increasing sequence $(E_k)_{k=1}^{\infty}$ of (ψ,n)-nowhere dense and (ψ,n)-closed sets such that $\lambda_n\left(\bigcup_{k=1}^{\infty} E_k\right) = 1$.*

P r o o f. Assume first that $n = 1$ and fix $\psi \in \mathcal{C}$. By The Second Taylor's Theorem, there is a perfect nowhere dense set $E_1 \subset [0,1]$ such that $\lambda(E_1) = \frac{1}{2}$ and $\limsup_{I \to 0} \frac{\lambda(E_1' \cap I)}{\lambda(I)\psi(\lambda(I))} = \infty$ for all $x \in E_1$. Obviously, E_1 is $\mathcal{T}_\psi$-nowhere dense and $\mathcal{T}_\psi$-closed. On every connected component (a_i, b_i) of the set $[0,1] \setminus E_1$ we construct an analogous set E_2^i and we put $E_2 = E_1 \cup \bigcup_{i=1}^{\infty} E_2^i$. The set E_2 is $\mathcal{T}_\psi$-nowhere dense, $\mathcal{T}_\psi$-closed and $\lambda(E_2) = \frac{1}{2} + \frac{1}{4}$. Inductively we construct the increasing sequence of $\mathcal{T}_\psi$-nowhere dense and $\mathcal{T}_\psi$-closed sets E_k. It is easy to observe that $\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = 1$.

If n is an arbitrary positive integer and $\psi \in \mathcal{C}$, then we use the first part of the lemma for the function $\psi^*(t) = \psi(t^n)$, and $\left(E_k \times [0,1]^{n-1}\right)_{k=0}^{\infty}$ is the requested sequence of sets. □

Clearly, null sets are (ψ,n)-closed and (ψ,n)-nowhere dense. Directly from the above lemma, we obtain:

Proposition 4. *For any $n \in \mathbb{N}$ and $\psi \in \mathcal{C}$ the set $\mathbb{R}^n$ is a $\mathcal{T}_{(\psi,n)}$-first category set. Consequently, $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is not a Baire space.*

Later, we will use Lemma 3 once more to show that $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is not regular. Making use of the results obtained for topologies $\mathcal{T}_\psi$, we have the following observations.

Observation 5. Let $\psi_1, \psi_2 \in \mathcal{C}$. If $\limsup_{x \to 0} \frac{\psi_1(x)}{\psi_2(x)} < \infty$, then $\mathcal{T}_{(\psi_1,n)} \subset \mathcal{T}_{(\psi_2,n)}$. If simultaneously $\liminf_{x \to 0} \frac{\psi_1(x)}{\psi_2(x)} > 0$, then $\mathcal{T}_{(\psi_1,n)} = \mathcal{T}_{(\psi_2,n)}$.

P r o o f. Assume that $\limsup_{x \to 0} \frac{\psi_1(x)}{\psi_2(x)} < \infty$. Take $A \in \mathcal{T}_{(\psi_1,n)}$ and $x \in A$. Then, $\lim_{r \to 0} \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi_1(r^n)} = 0$. From the equality

$$\frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi_2(r^n)} = \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi_1(r^n)} \cdot \frac{\psi_1(r^n)}{\psi_2(r^n)}$$

we immediately obtain that

$$\lim_{r \to 0} \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi_2(r^n)} = 0, \quad \text{hence} \quad A \in \mathcal{T}_{(\psi_2,n)}.$$

In the same way, we show that if $\liminf_{x \to 0} \frac{\psi_1(x)}{\psi_2(x)} > 0$, then $\mathcal{T}_{(\psi_2,n)} \subset \mathcal{T}_{(\psi_1,n)}$. □

Observation 6. Let $\psi_1, \psi_2 \in \mathcal{C}$. If $\lim_{x\to 0} \frac{\psi_2(x)}{\psi_1(x)} = \infty$, then $\mathcal{T}_{(\psi_1,n)} \subsetneq \mathcal{T}_{(\psi_2,n)}$ for any $n \in \mathbb{N}$.

P r o o f. From the condition $\lim_{x\to 0} \frac{\psi_2(x)}{\psi_1(x)} = \infty$, it follows that

$$\limsup_{x\to 0} \frac{\psi_1(x)}{\psi_2(x)} < \infty$$

and from Observation 5, we obtain the inclusion $\mathcal{T}_{(\psi_1,n)} \subset \mathcal{T}_{(\psi_2,n)}$. We will show that this inclusion is proper. Put $\psi_i^*(r) = \psi_i(r^n)$, $i = 1, 2$. Then, $\lim_{x\to 0} \frac{\psi_2^*(x)}{\psi_1^*(x)} = \infty$. From [11, Theorem 8], there exists a set $A \in \mathcal{T}_{\psi_2^*} \setminus \mathcal{T}_{\psi_1^*}$. Then,

$$A \times \mathbb{R}^{n-1} \in \mathcal{T}_{(\psi_2,n)} \setminus \mathcal{T}_{(\psi_1,n)}.$$ □

There are some properties of topologies $\mathcal{T}_{(\psi,n)}$ which depend on the so--called Δ_2 condition.

Definition 7. We will say that $\psi \in \mathcal{C}$ fulfills (Δ_2) condition ($\psi \in \Delta_2$) if

$$\limsup_{h\to 0^+} \frac{\psi(2h)}{\psi(h)} < \infty. \qquad (\Delta_2)$$

It is easy to observe that

$$\psi \in \Delta_2 \quad \text{if and only if} \quad \limsup_{h\to 0^+} \frac{\psi(\alpha h)}{\psi(h)} < \infty \quad \text{for any } \alpha > 0 \text{ (see [11])}.$$

The name of this condition is taken from the theory of Orlicz spaces. Functions of the form $\varphi(x) = x^p$, $p \geq 1$, play an important role in these spaces because for such functions the Orlicz space $L^\varphi(\mu)$ becomes $L^p(\mu)$ space. In the theory of (ψ, n)-density topologies, we also obtain interesting results for functions of similar form: $\psi(x) = x^\alpha$, $\alpha > 0$.

Obviously, for any $\psi \in \mathcal{C}$, (ψ, n)-density topology is invariant under translation.

Observation 8. The topology $\mathcal{T}_{(\psi,n)}$ is invariant under multiplication by nonzero numbers if and only if $\psi \in \Delta_2$.

P r o o f. Observe that for any $\alpha > 0$ and $A \in \mathcal{L}_n$

$$\frac{\lambda_n(\alpha A' \cap Q(x,r))}{r^n \cdot \psi(r^n)} = \frac{\alpha^n \lambda_n(A' \cap Q(x, \frac{r}{\alpha}))}{\alpha^n \cdot \left(\frac{r}{\alpha}\right)^n \cdot \psi\left(\left(\frac{r}{\alpha}\right)^n\right)} \cdot \frac{\psi\left(\left(\frac{r}{\alpha}\right)^n\right)}{\psi(r^n)}.$$

If $\psi \in \Delta_2$ and $A \in \mathcal{T}_{(\psi,n)}$, then, from the above equality, $\alpha A \in \mathcal{T}_{(\psi,n)}$ for any $\alpha > 0$. To show the opposite implication, assume that there is $a > 0$ such that $\limsup_{h\to 0^+} \frac{\psi(ah)}{\psi(h)} = \infty$. Put $\psi^*(h) = \psi(h^n)$. Then, $\limsup_{h\to 0^+} \frac{\psi^*(ah)}{\psi^*(h)} = \infty$. From [19], it follows that then, there exists an interval set $A = \bigcup_{i=1}^{\infty} [a_i, b_i]$ with $0 < a_i < b_i < a_{i+1}$, $i \in \mathbb{N}$, such that 0 is a ψ^*-density point of A and is not a ψ^*-density point of aA. Then, 0 is a (ψ, n)-density point of $E = A \times [0,1]^{n-1}$ and is not a (ψ, n)-density point of aE. □

In consequence, if $\psi_1 \in \Delta_2$ and $\psi_2 \notin \Delta_2$, then $\mathcal{T}_{(\psi_1,n)} \neq \mathcal{T}_{(\psi_2,n)}$.

Observation 9. For any function $\psi \in \Delta_2$ there exist functions $\psi_1, \psi_2 \in \Delta_2$ such that $\mathcal{T}_{(\psi_1,n)} \subsetneq \mathcal{T}_{(\psi,n)} \subsetneq \mathcal{T}_{(\psi_2,n)}$.

Proof. Let $\psi_1(x) = x\psi(x)$ and $\psi_2(x) = \sqrt{\psi(x)}$. Then, $\psi_1, \psi_2 \in \Delta_2$ and

$$\lim_{x\to 0^+} \frac{\psi(x)}{\psi_1(x)} = \lim_{x\to 0^+} \frac{\psi_2(x)}{\psi(x)} = \infty.$$

From Observation 6, we obtain $\mathcal{T}_{(\psi_1,n)} \subsetneq \mathcal{T}_{(\psi,n)} \subsetneq \mathcal{T}_{(\psi_2,n)}$. □

Some information on ψ-density topologies on $\mathbb{R}$ which are generated by $\psi \notin \Delta_2$ one may find in [11]. In particular, it was shown that for any $\alpha > 0$ there is a function $\psi_\alpha \notin \Delta_2$ such that $\mathcal{T}_{x^{\alpha+1}} \subsetneq \mathcal{T}_{\psi_\alpha} \subsetneq \mathcal{T}_{x^\alpha}$. Based on this observation, we can show an analogous result for (ψ, n)-density topologies: for any number $\alpha > 0$ there exists a function $\psi_\alpha \notin \Delta_2$ for which $\mathcal{T}_{(x^{\alpha+1},n)} \subsetneq \mathcal{T}_{(\psi_\alpha,n)} \subsetneq \mathcal{T}_{(x^\alpha,n)}$.

In Definition 2, we may use various differentiation basis. It is interesting that for $\psi \in \Delta_2$ the notion of (ψ, n)-density point with respect to cubes coincide with the following notion.

Definition 10. Let $\psi \in \mathcal{C}$, $x \in \mathbb{R}^n$, $A \in \mathcal{L}_n$. We will say that x is an (ψ, n)--density point of A with respect to balls if

$$\lim_{r\to 0} \frac{\lambda_n(A' \cap B(x,r))}{|B(x,r)| \cdot \psi(|B(x,r)|)} = 0.$$

As the volume of the ball is described by the formula $|B(x,r)| = \omega(n) r^n$, where $\omega(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ (Γ is the Euler gamma function), equivalently we may consider the condition

$$\lim_{r\to 0} \frac{\lambda_n(A' \cap B(x,r))}{r^n \cdot \psi(\omega(n) r^n)} = 0.$$

Proposition 11. *Assume that $\psi \in \Delta_2$. Then, x is a (ψ, n)-density point of A with respect to balls if and only if it is a (ψ, n)-density point of A with respect to cubes.*

Proof. Clearly, for any cube $Q(x,r)$ there exist numbers $a, b > 0$ such that $Q(x,r) \subset B(x,ar)$ and $B(x,r) \subset Q(x,br)$. Assume that x is a (ψ, n)-density point of A with respect to balls. Then

$$\frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi(r^n)} \cdot \frac{\psi(r^n)}{\psi(\omega(n) a^n r^n)} = \frac{\lambda_n(A' \cap Q(x,r))}{r^n \cdot \psi(\omega(n) a^n r^n)} \leq \frac{\lambda_n(A' \cap B(x,ar))}{r^n \cdot \psi(\omega(n) a^n r^n)}.$$

As $\psi \in \Delta_2$, $\limsup_{r\to 0^+} \frac{\psi(r^n)}{\psi(\omega(n)a^n r^n)} < \infty$ and x is a (ψ, n)-density point of A with respect to cubes.

Analogously, if x is a (ψ, n)-density point of A with respect to cubes, then

$$\frac{\lambda_n(A' \cap B(x,r))}{r^n \cdot \psi(b^n r^n)} = \frac{\lambda_n(A' \cap B(x,r))}{r^n \cdot \psi(\omega(n) r^n)} \cdot \frac{\psi(\omega(n) r^n)}{\psi(b^n r^n)} \le \frac{\lambda_n(A' \cap Q(x,br))}{r^n \cdot \psi(b^n r^n)}.$$

□

Theorem 12. *Let $\psi \notin \Delta_2$. Then*

(a) *there exists a set $A \subset \mathbb{R}^n$ such that 0 is a (ψ, n)-density point of A with respect to balls and it is not a (ψ, n)-density point of A with respect to cubes;*

(b) *there exists a set $B \subset \mathbb{R}^n$ such that 0 is a (ψ, n)-density point of B with respect to cubes and it is not a (ψ, n)-density point of B with respect to balls.*

P r o o f. We describe the construction for $n = 2$. The proof for bigger n is analogous. First, we will construct the measurable set $A \subset \mathbb{R}^2$ such that 0 is not its $(\psi, 2)$-dispersion point with respect to cubes and 0 is its $(\psi, 2)$-dispersion point with respect to balls.

Let

$$\Theta = (0,0), \quad r > 0, \quad F(A, r) = \frac{\lambda_2(A \cap Q(\Theta, r))}{r^2 \psi(r^2)}$$

and

$$G(A, r) = \frac{\lambda_2(A \cap B(\Theta, r))}{\pi r^2 \psi(\pi r^2)}.$$

The set $X_r = Q(\Theta, r) \setminus B\big(\Theta, \sqrt{\tfrac{2}{5}} r\big)$ has a positive Lebesgue measure and

$$\left| B\left(\Theta, \sqrt{\tfrac{2}{5}} r\right) \right| = \tfrac{2\pi}{5} |Q(\Theta, r)|.$$

Moreover, $\lambda_2(X_r) > 4P$, where P denotes the area of the isosceles right triangle included in X_r with hypotenuse tangent to the ball $B(\Theta, r)$. Hence

$$\lambda_2\big(X_r \cap Q(\Theta, r)\big) = \lambda_2(X_r) > \left(\sqrt{2} - 2\sqrt{\frac{2}{5}}\right)^2 r^2. \tag{3}$$

Let us denote $\alpha = \big(\sqrt{2} - 2\sqrt{\tfrac{2}{5}}\big)^2$. Since $\psi \notin \Delta_2$, then there exists $h_1 > 0$ such that $\psi(h_1) \le \alpha$ and

$$\frac{\psi(h_1)}{\psi(\frac{2\pi}{5} h_1)} < \frac{1}{5}. \tag{4}$$

Let $r_1 = \sqrt{h_1}$. By (3), we obtain

$$F(X_{r_1}, r_1) = \frac{\lambda_2(X_{r_1} \cap Q(\Theta, r_1))}{r_1^2 \psi(r_1^2)} > \frac{\alpha r_1^2}{r_1^2 \psi(r_1^2)} \ge 1.$$

Let A_1 be a measurable subset of X_{r_1} such that $\frac{\lambda_2(A_1)}{r_1^2\psi(r_1^2)} = 1$. For $0 < r < \sqrt{\frac{2}{5}}r_1$ the ball $B(\Theta, r)$ is disjoint with A_1, hence $G(A_1, r) = 0$. For $r \geq \sqrt{\frac{2}{5}}r_1$ from (4) we have

$$G(A_1, r) = \frac{\lambda_2(A_1 \cap B(\Theta, r))}{\pi r^2 \psi(\pi r^2)} \leq \frac{\lambda_2(A_1)}{\frac{2\pi}{5}r_1^2 \cdot \psi\left(\frac{2\pi}{5}r_1^2\right)}$$
$$= \frac{\lambda_2(A_1)}{r_1^2 \cdot \psi\left(r_1^2\right)} \cdot \frac{\psi\left(r_1^2\right)}{\psi\left(\frac{2\pi}{5}r_1^2\right)} \cdot \frac{5}{2\pi} < \frac{1}{5}.$$

Hence, $G(A_1, r) < \frac{1}{5}$ for any $r > 0$.

Assume that we defined positive numbers $r_1 > r_2 > \cdots > r_{k-1}$ and measurable disjoint sets $A_1, A_2, \ldots, A_{k-1}$ such that for any $i = 1, 2, \ldots, k-1$:

$$A_i \subset Q(\Theta, r_i) \setminus B\left(\Theta, \sqrt{\frac{2}{5}}r_i\right), \quad F(A_i, r_i) = 1 \quad \text{and} \quad G(A_i, r) < \frac{1}{5^i} \quad \text{for } r > 0.$$

There exists $r_k < \sqrt{\frac{2}{5}}r_{k-1}$ such that $|Q(\Theta, r_k)| \leq \lambda_2(A_{k-1})$ and $\frac{\psi(r_k^2)}{\psi(\frac{2\pi}{5}r_k^2)} < \frac{1}{5^k}$. Analogously, as we did it before, we choose the measurable set $A_k \subset Q(\Theta, r_k) \setminus B\left(\Theta, \sqrt{\frac{2}{5}}r_k\right)$ such that $F(A_k, r_k) = 1$. Then: for $0 < r < \sqrt{\frac{2}{5}}r_k$ the ball $B(\Theta, r_k)$ is disjoint with A_k, so $G(A_k, r) = 0$ and for $r \geq \sqrt{\frac{2}{5}}r_1$

$$G(A_k, r) = \frac{\lambda_2(A_k \cap B(\Theta, r))}{\pi r^2 \psi(\pi r^2)} \leq \frac{\lambda_2(A_k)}{r_k^2 \cdot \psi\left(r_k^2\right)} \cdot \frac{\psi\left(r_k^2\right)}{\psi\left(\frac{2\pi}{5}r_k^2\right)} \cdot \frac{5}{2\pi} < \frac{1}{5^k}.$$

Therefore, for any $r > 0$ we obtain $G(A_k, r) < \frac{1}{5^k}$.

Put $A = \bigcup_{k=1}^{\infty} A_k$. Then, 0 is not a $(\psi, 2)$-dispersion point of A with respect to cubes (because for any $k \in \mathbb{N}$, we have $F(A_k, r_k) = 1$). Simultaneously, 0 is a $(\psi, 2)$-dispersion point of A with respect to balls. Indeed, first observe that for $i \geq k+1$ all sets A_i are included in the cube

$$Q(\Theta, r_{k+1}) \quad \text{and} \quad |Q(\Theta, r_{k+1})| \leq \lambda_2(A_k).$$

Moreover, those sets are included in $B\left(\Theta, \sqrt{\frac{2}{5}}r_k\right)$, hence

$$\lambda_2\big(A \cap B(\Theta, r_{k-1})\big) \leq \lambda_2\left(A \cap B\left(\Theta, \sqrt{\frac{2}{5}}r_k\right)\right)$$
$$\leq \sum_{i=k}^{\infty} \lambda_2(A_i) \leq \sum_{i=k+1}^{\infty} \lambda_2(A_i) + \lambda_2(A_k)$$
$$\leq Q(\Theta, r_{k+1})| + \lambda_2(A_k) \leq 2\lambda_2(A_k).$$

Take $r > 0$. Then, there is a number $k \in \mathbb{N}$ such that $r \in (r_k, r_{k-1})$. From monotonicity of ψ, we have $\frac{\psi(r_k^2)}{\psi(\pi r_k^2)} \leq \frac{\psi(r_k^2)}{\psi(\frac{2\pi}{5} r_k^2)} \leq \frac{1}{5^k}$ and consequently,

$$G(A,r) \leq \frac{\lambda_2(A \cap B(\Theta, r_{k-1}))}{\pi r_k^2 \cdot \psi\left(\pi r_k^2\right)} \leq \frac{2\lambda_2(A_k)}{r_k^2 \cdot \psi\left(r_k^2\right)} \cdot \frac{\psi\left(r_k^2\right)}{\psi\left(\pi r_k^2\right)} \cdot \frac{1}{\pi} < \frac{1}{5^k},$$

which finishes the proof of the first part of the theorem.

The proof of (b) is analogous. For any $r > 0$, the set

$$X_r = B(\Theta, r) \setminus Q\left(\Theta, \sqrt{\frac{7}{2}} r\right)$$

has a positive Lebesgue measure and $\frac{\lambda_2(X_r)}{|B(\Theta,r)|}$ is constant. Denote $\alpha = \frac{\lambda_2(X_r)}{|B(\Theta,r)|}$. Since $\psi \notin \Delta_2$, then there exists a positive number h_1 such that

$$\frac{\psi(\pi h_1)}{\psi(\frac{7}{2} h_1)} < \frac{1}{5} \quad \text{and} \quad \psi(\pi h_1) \leq \alpha.$$

Put $r_1 = \sqrt{h_1}$. Then

$$G(X_{r_1}, r_1) = \frac{\lambda_2(X_{r_1} \cap B(\Theta, r_1))}{\pi r_1^2 \psi(\pi r_1^2)} = \frac{\lambda_2(X_{r_1})}{|B(\Theta, r_1)|} \cdot \frac{1}{\psi(\pi r_1^2)} \geq 1.$$

Let V_1 be a measurable subset of X_{r_1} for which

$$\frac{\lambda_2(V_1)}{|B(\Theta, r_1)|} \cdot \frac{1}{\psi(\pi r_1^2)} = 1.$$

Then, $F(V_1, r) = 0$ for any $0 < r < \sqrt{\frac{7}{2}} r_1$ (because $V_1 \cap Q(\Theta, r) = \emptyset$). For $r \geq \sqrt{\frac{7}{2}} r_1$ we have

$$F(V_1, r) = \frac{\lambda_2(V_1 \cap Q(\Theta, r))}{r^2 \psi(r^2)} \leq \frac{\lambda_2(V_1)}{\frac{7}{2} r_1^2 \cdot \psi\left(\frac{7}{2} r_1^2\right)} = \frac{\lambda_2(V_1)}{\pi r_1^2 \cdot \psi\left(\pi r_1^2\right)} \cdot \frac{\psi\left(\pi r_1^2\right)}{\psi\left(\frac{7}{2} r_1^2\right)} \cdot \frac{2\pi}{7} < \frac{1}{5}.$$

Hence, $F(V_1, r) < \frac{1}{5}$ for any $r > 0$. We fix $k \in \mathbb{N}$ and assume that we defined positive numbers $r_1, r_2, \ldots, r_{k-1}$ and the sets $V_1, V_2, \ldots, V_{k-1}$ such that for any $i = 1, 2, \ldots, k-1$:

$$V_i \subset B(\Theta, r_i) \setminus Q\left(\Theta, \sqrt{\frac{7}{2}} r_i\right), \; G(V_i, r_i) = 1 \quad \text{and} \quad F(V_i, r) < \frac{1}{5^i} \; \text{ for } \; r > 0.$$

Take $r_k < r_{k-1}$ for which

$$|B(\Theta, r_k)| \leq \lambda_2(V_{k-1}) \quad \text{and} \quad \frac{\psi(\pi r_k^2)}{\psi(\frac{7}{2} r_k^2)} < \frac{1}{5^k}.$$

Repeating the previous reasoning, we choose the measurable set

$$V_k \subset B(\Theta, r_k) \setminus Q\left(\Theta, \sqrt{\frac{7}{2}} r_k\right) \quad \text{such that} \quad G(V_k, r_k) = 1.$$

Then,

$$F(V_k, r) < \frac{1}{5^k} \quad \text{for any } r > 0.$$

By puting $V = \bigcup_{k=1}^{\infty} V_k$, we obtain, in the same way as in the previous part of the proof, that 0 is its $(\psi, 2)$-dispersion point of V with respect to cubes and 0 is not a $(\psi, 2)$-dispersion point of V with respect to balls. □

It is evident that functions of the form $\psi(x) = x^\alpha$ with $\alpha > 0$ fulfill (Δ_2) condition. Hence, for such functions, (ψ, n)-density with respect to balls means the same as with respect to cubes and it is equivalent to the notion of m--density examined in [5]. From Observation 6, we obtain continuum different m-density topologies $\mathcal{T}_m$ for different m and if $m_1 > m_2$, then $\mathcal{T}_{m_1} \subsetneq \mathcal{T}_{m_2}$. From Proposition 2 [13], it follows that for any $m \in [n, \infty)$ there is a function $\psi \in \mathcal{C}$ such that $\mathcal{T}_m$ is not similar to $\mathcal{T}_{(\psi,n)}$. Moreover, for any $m \in [n, \infty)$ there exists a number $k > m$ such that $\mathcal{T}_m$ is not similar to $\mathcal{T}_k$. Indeed, for $m = n$ the topology $\mathcal{T}_n$ coincides with the density topology on $\mathbb{R}^n$ and it is not similar to any topology $\mathcal{T}_m$ for $m > n$. Based on Theorem 22 and Proposition 24 from [13], for any $\alpha > 1$ there exists a perfect set $E \subset [0, 1]$ of positive measure which has nonempty interior in the topology $\mathcal{T}_{\psi_1}$ for $\psi_1(x) = x^{\alpha-1}$ and has the empty interior in $\mathcal{T}_{\psi_2}$ with $\psi_2(x) = x^{3\alpha-1}$. Making use of the set $E \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$, one can check that for any $m \in (n, \infty)$ the topology $\mathcal{T}_m$ is not similar to $\mathcal{T}_{3m}$. We do not know if topologies $\mathcal{T}_m$ and $\mathcal{T}_{2m}$ are similar.

From Proposition 4, we have that $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is not a Baire space. Since $\mathcal{T}_{(\psi,n)}$ is finer than the natural topology on $\mathbb{R}^n$, $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is a Hausdorff space. Repeating the reasoning from [16] (Sections 4. A and 6. D), using the notion of $\mathcal{G}_\delta$-insertion property and Theorem 4.6, one may conclude that $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is not regular. For the convenience of the reader, we will prove it straightforward using Lemma 3.

Theorem 13. *For any $n \in \mathbb{N}$ and $\psi \in \mathcal{C}$ the space $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is not regular.*

P r o o f. Observe that the function $\psi^*(t) = \psi(t^n)$ belongs to $\mathcal{C}$, too. Let $(E_k)_{k=1}^{\infty}$ be the increasing sequence of $\mathcal{T}_{\psi_*}$-nowhere dense and $\mathcal{T}_{\psi_*}$-closed subsets of $[0, 1]$ constructed in Lemma 3. Let us consider the set $A = \big((\bigcup_{k=1}^{\infty} E_k) \cap (0, 1)\big) \times \mathbb{R}^{n-1}$. It is open in (ψ, n)-density topology because $\lambda_n((0, 1)^n \setminus A) = 0$. We will prove that $(\mathbb{R}^n, \mathcal{T}_{(\psi,n)})$ is not regular by showing that for any nonempty set $U \subset A$ open in (ψ, n)-density topology, the $\mathcal{T}_{(\psi,n)}$-closure of U is not contained in A.

Let us fix a subset $U \in \mathcal{T}_{(\psi,n)}$ of A. We will find a sequence $\big(x^{(k)}\big)_{k\in\mathbb{N}}$ of elements of U such that $x = \lim_{k\to\infty} x^{(k)} \notin A$ and

$$\limsup_{r\to 0} \frac{\lambda_n(U \cap Q(x,r))}{r^n} \geq \frac{1}{2}.$$

Since $E_1 \times \mathbb{R}^{n-1}$ is $\mathcal{T}_\psi$-nowhere dense and $U \in \mathcal{T}_{(\psi,n)}$, there is a point

$$x^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)}) \in U \setminus \big(E_1 \times \mathbb{R}^{n-1}\big).$$

The sequence $(E_k)_{k\in\mathbb{N}}$ is increasing and $x_1^{(1)} \in \bigcup_{k=1}^{\infty} E_k$, so there is $p_1 \in \mathbb{N}$ such that $x_1^{(1)} \in E_{p_1} \setminus E_{p_1-1}$. We denote by (a_1, b_1) this component of $\mathbb{R} \setminus E_{p_1-1}$ which contains $x_1^{(1)}$ and we put $\varepsilon_1 = \min\{x_1^{(1)} - a_1, b_1 - x_1^{(1)}\}$. Since $x^{(1)}$ is a (ψ, n)-density point of U (so, it is a density point of U), we can choose a number $r_1 \in (0, \varepsilon_1)$ such that $\lambda_n\big(U \cap Q(x^{(1)}, r_1)\big) > r_1^n \cdot \big(1 - \frac{1}{2^{n+1}}\big)$. Then,

$$\lambda_n \Big(Q(x^{(1)}, r_1) \setminus U\Big) < r_1^n \cdot \frac{1}{2^{n+1}} = \Big(\frac{r_1}{2}\Big)^n \cdot \frac{1}{2}.$$

Note that if $y \in Q(x^{(1)}, \frac{r_1}{4})$, then $Q(y, \frac{r_1}{2}) \subset Q(x^{(1)}, r_1)$. Therefore, $\lambda_n\Big(U \cap Q\big(y, \frac{r_1}{2}\big)\Big) > (\frac{r_1}{2})^n \cdot \frac{1}{2}$. Moreover, for any $y \in Q(x^{(1)}, \frac{r_1}{4})$ the distance between y and $E_{p_1-1} \times \mathbb{R}^{n-1}$ is greater than $\frac{3}{4} r_1$.

The set $U_1 = U \cap Q(x^{(1)}, r_1)$ has a nonempty $\mathcal{T}_{(\psi,n)}$-interior, so there exists point $x^{(2)} \in U_1 \setminus \big(E_{p_1} \times \mathbb{R}^{n-1}\big)$. Denote by p_2 a positive integer such that $x_1^{(2)} \in E_{p_2} \setminus E_{p_2-1}$ and by (a_2, b_2) the component of $\mathbb{R} \setminus E_{p_2-1}$ such that $x_1^{(2)} \in (a_2, b_2)$.

We now proceed by induction and we find a sequence $(x^{(k)})_{k\in\mathbb{N}}$ of points of $\mathbb{R}^n$, a sequence $(p_k)_{k\in\mathbb{N}}$ of positive integers, and a decreasing sequence $(r_k)_{k\in\mathbb{N}}$ tending to zero such that for any $k \in \mathbb{N}$ and any $y \in Q(x^{(k)}, \frac{r_k}{4})$

$$x^{(k)} \in U \cap Q\left(x^{(k-1)}, \frac{r_k}{4}\right), \tag{5}$$

$$\lambda_n\bigg(U \cap Q\Big(y, \frac{r_k}{2}\Big)\bigg) > \Big(\frac{r_1}{2}\Big)^n \cdot \frac{1}{2}, \tag{6}$$

$$\operatorname{dist}\big(y, E_{p_k-1} \times \mathbb{R}^{n-1}\big) > \frac{3}{4} r_{p_k-1}. \tag{7}$$

From (5), we know that the sequence $(x^{(k)})_{k\in\mathbb{N}}$ is convergent. By (7),

$$x = \lim_{k\to\infty} x^{(k)} \notin A.$$

Finally, for any $k \in \mathbb{N}$ and $x \in Q(x^{(k)}, \frac{r_k}{4})$ from (6)

$$\limsup_{r\to 0} \frac{\lambda_n(U \cap Q(x,r))}{r^n} \geq \limsup_{k\to\infty} \frac{\lambda_n(U \cap Q(x, \frac{r_k}{2}))}{\big(\frac{r_k}{2}\big)^n} \geq \frac{1}{2}. \qquad \square$$

At the end of the article, we show that the generalized densities are used in such distant areas of mathematics as theory of sets of finite perimeter and Sobolev spaces.

3. Applications

According to the legend, ancient Carthage was founded on the land which can be encompassed by a single ox hide. This hide was cut into thin strips which were put end to end to encircle the hill. It was the source of the first isoperimetric problem: what is the curve which encloses the maximum area for a given perimeter? To formulate analogous problem in $\mathbb{R}^n$, we need a notion of volume of a set and a notion of perimeter of a set. Those two notions are basic for geometric measure theory and the sets of finite perimeter are the basic geometric tools in the theory of currents, of rectifiable sets, of functions with bounded variation.

First results connected with volume were obtained by H. Lebesgue in 1901--1902. In the 1920s, R. Cacciopoli gave the first modern definition of perimeter and later, in the 1950s, it was developed by Cacciopoli and E. De Giorgi. The definition of perimeter of a Borel set $E \subset \mathbb{R}^n$ given by Cacciopoli was geometric and intuitive. It was equal to

$$P(E) = \inf \left\{ \liminf_{n\to\infty} \quad \mathrm{Area}(\delta E_n),\ E_n \quad \text{polyhedra},\ E_n \to E \text{ in } L^1_{(\mathrm{loc})} \right\}.$$

It is obvious that E has finite perimeter if $P(E) < \infty$.

Following [8], we present the definition of De Giorgi.

Definition 14. Let U be an open set in $\mathbb{R}^n$, $f\colon \mathbb{R}^n \to \mathbb{R}^m$. A variation of a Lebesgue integrable function f on U is given by

$$Df(U) = \sup \left\{ \int_U (\operatorname{div} G) f \ \mathrm{d}x \colon G \in C^1_0(U, \mathbb{R}^n),\ |G(x)|_{\mathbb{R}^n} \le 1 \quad \text{for all } \ x \in U \right\},$$

where $\operatorname{div}(G) = \sum_{i=1}^n \frac{\delta G_i}{\delta x_i}$, $C^1_0(U,\mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued functions G on U such that G is continuously differentiable and G vanishes outside a certain compact subset of U, $|\cdot|_{\mathbb{R}^n}$ denotes the Euclidean norm on $\mathbb{R}^n$. If $Df(U) < \infty$, then we say that f is of finite variation. We say that $f \in L^1_{(\mathrm{loc})}(U)$ has locally bounded variation in U if for each open set $V \subset\subset U$ (this means that V is compactly contained in U)

$$Df(U) = \sup \left\{ \int_U (\operatorname{div} G) f \, \mathrm{d}x \colon G \in C^1_0(V, \mathbb{R}^n),\ |G(x)|_{\mathbb{R}^n} \le 1 \quad \text{for all } \ x \in V \right\}.$$

We will write then $f \in BV_{(\mathrm{loc})}(U)$.

We say that a Lebesgue measurable bounded set $E \subset U$ has a locally finite perimeter (denoted by $P(E)$) in U if $\chi_E \in BV_{(\mathrm{loc})}(U)$, where χ_E is the indicator function of a set E.

The next step of defining the perimeter was done by H. Federer in the 1960s. The notion of perimeter was a generalization of $\mathcal{H}^{n-1}$-measure of the boundary of a set E. The topological boundary of the set of finite perimeter can be very irregular, it can even have full Lebesgue measure. The primeter of a set is invariant under modifications by sets of Lebesgue measure zero (in $\mathbb{R}^n$), although such a set may essentially change (increase) the size of topological boundary.

H. Federer characterized the sets of finite perimeter via Hausdorf measure $\mathcal{H}^{n-1}$ of (so-called) the essential boundary of a set.

Definition 15. The essential (measure-theoretic) boundary of the set E is defined by

$$\delta^* E = \left\{ x \in \mathbb{R}^n\colon \limsup_{r\to 0} \min\left\{ \frac{\lambda_n(B(x,r)\cap E)}{|B(x,r)|}, \frac{\lambda_n\big(B(x,r)\setminus E\big)}{|B(x,r)|} \right\} > 0 \right\}.$$

Theorem 16 ([9])**.** *E has a finite perimeter if and only if $\mathcal{H}^{n-1}(\delta^* E) < \infty$.*

Moreover, Federer proved that $P(E) = \mathcal{H}^{n-1}(\delta^* E)$. He also showed that if a set $E \subset \mathbb{R}^n$ is of finite perimeter, then $\mathcal{H}^{n-1}(\delta^* E \setminus \Sigma_{\frac{1}{2}} E) = 0$, where the boundary $\Sigma_{\frac{1}{2}} E$ consists of such points where both E and its complement E' have density exactly $\frac{1}{2}$. Moreover, according to [1], he proved that there exists a set N such that $\mathcal{H}^{n-1}(N) = 0$ and $\mathbb{R}^n = E^1 \cup \delta^*(E) \cup E^0 \cup N$ (E^1 and E^0 are the interior and exterior of E in density topology, respectively).

Recently, in 2020 in [15], there was shown that Federer's characterization given in Theorem 16 remains true if the essential boundary of E is exchanged by the smaller boundary (called strong boundary) consisting of those points where the lower densities of sets E and E' are at least a given number.

In 2012 in [4], there was presented a definition of enhanced density points and enhanced density sets. Delladio transfered superdensity from the real line into $\mathbb{R}^n$. It is a case of (ψ, n)-density with respect to balls for $\psi = \mathrm{id}$. Observe that such ψ fulfills Δ_2 condition, so this notion is equivalent to (ψ, n)-density with respect to cubes.

Definition 17 ([4])**.** $x \in \mathbb{R}^n$ is an enhanced density point of $A \subset \mathbb{R}^n$ if

$$\lim_{r\to 0} \frac{\lambda_n^*(A' \cap B(x,r))}{r^{n+1}} = 0.$$

In the same paper, Delladio proved the following theorem.

Theorem 18. *Let E be a locally finite perimeter subset of $\mathbb{R}^n$, $n \geq 2$. Then*

$$\lim_{r\to 0} \frac{\lambda_n^*(A' \cap B(x,r))}{r^{n+\frac{n}{n-1}}} = 0 \quad \text{at a.e. } x \in E.$$

In particular, E is an enhanced density set. With denotations from Section 2, enhanced density set means that $\lambda_n(E \setminus \Phi_{(x,n)}(E)) = 0$. From the above theorem, it follows that the family of enhanced density sets includes locally finite perimeter sets which have some applications in differential forms, rectifiable sets, partially differencial equations.

The properties of enhanced density proved among others in [4] follow directly from the properties of (ψ, n)-density as a special case. It is interesting, then although the results are the same, the presented proofs show quite different tools and methods.

In 2015, Delladio generalized his definition of enhanced density (see [5]). He showed the connection between the theory of sets of finite perimeter and fine topology methods, which have some applications in potential theory.

Definition 19. Let $m \in [n, \infty)$; $x \in \mathbb{R}^n$ is an m-density point of $A \subset \mathbb{R}^n$ if

$$\lim_{r\to 0} \frac{\lambda_n^*(A' \cap B(x,r))}{r^m} = 0.$$

Directly from the observation

$$\lim_{r\to 0} \frac{\lambda_n^*(A' \cap B(x,r))}{r^m} = 0 \iff \lim_{r\to 0} \frac{\lambda_n^*(A' \cap B(x,r))}{|B(x,r)| \cdot r^{m-n}} = 0,$$

we have the next remark.

Remark 20. If $\psi(x) = x^\alpha$ for $\alpha \geq 0$, then (ψ, n)-density point with respect to balls is equivalent to the notion of m-density examined by Delladio (in particular, for $\alpha = 1$ it is the enhanced density).

Theorem 21 ([5])**.** *Let E be a set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$. Then*

$$\lim_{r\to 0} \frac{\lambda_n^*(A' \cap B(x,r))}{r^{m_0}} = 0 \quad \text{at a.e. } x \in E, \quad \text{with } m_0 = n + 1 + \frac{1}{n-1}.$$

That means that almost every point of a set of finite perimeter is its m_0--density point (a half of the Lebesgue Density Theorem). Moreover, m_0 is the maximum order of density common to all sets of finite perimeter.

Theorem 22 ([5])**.** *For all $m > m_0$ there is a compact set F_m of finite perimeter in $\mathbb{R}^n$ such that $\lambda_n^*(F_m) > 0$ and the set of m-density points of F_m is empty.*

Sets of finite perimeter have become a tool to study many problems involving surfaces and interfaces, in areas such as materials science, fluid mechanics, surface physics, image processing, oncology, and computer vision.

Sobolev spaces are inextricably associated with partial differential equations. For us, it was a real surprise when we found out that in this part of mathematics there are many elements of real functions theory linked to different kinds of densities and approximately continuities.

For the convenience of the reader, we recall some basic definitions and denotations. Sobolev functions on $\mathbb{R}^n$ are the functions $f\colon U \to \mathbb{R}^m$ (U is an open subset of $\mathbb{R}^n$) such that their weak first derivatives belong to some L^p space, $1 \le p \le \infty$.

Definition 23 ([8]). Assume that $f \in L^1_{(\mathrm{loc})}(U)$, $1 \le i \le n$. We say that $g_i \in L^1_{(\mathrm{loc})}(U)$ is the weak partial derivative of f with respect to x_i in U if

$$\int_U f \frac{\partial \varphi}{\partial x_i} = -\int_U g_i \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in C^1_0(U).$$

We will say that f belongs to the Sobolev space $W^{1,p}(U)$ if $f \in L^p(U)$ and the weak partial derivatives $\frac{\partial f}{\partial x_i}$ exist and belong to $L^p(U)$ for $i = 1, \dots, n$. The function f belongs to $W^{1,p}_{(\mathrm{loc})}(U)$ if $f \in W^{1,p}(V)$ for each open set $V \subset U$. We say that f is a Sobolev function if $f \in W^{1,p}_{(\mathrm{loc})}(U)$ for some $1 \le p \le \infty$. Each Sobolev function has locally bounded variation (but not contrary). Every Sobolev space is a Banach space.

The notion of approximately continuous functions was introduced at the beginning of the 20th century. In 1915, A. Denjoy [7] stated that a real valued function f is approximately continuous at a point x_0 if and only if there exists a measurable set A_{x_0} such that

$$\lim_{h\to 0+} \frac{m(A_{x_0} \cap [x_0 - h, x_0 + h])}{2h} = 1 \qquad \text{and} \qquad f(x_0) = \lim_{\substack{x\to x_0 \\ x\in A_{x_0}}} f(x).$$

This first definition did not involve the concept od density topology, which was defined later in the 1960s. Now, a point x_0 which fulfills the first from above conditions is called a density point of a set A_{x_0}. Almost simultaneously, Denjoy also defined the approximate upper limit of f at x_0 as the greatest lower bound of all members y for which $\lim_{h\to 0+} \frac{m(B\cap[x_0-h,x_0+h])}{2h} = 0$, where $B = \{x \in \mathbb{R}\colon f(x) > y\}$. Analogously, one can define the approximate lower limit of f at x_0. When both values are equal, then we say that there exists the approximate limit denoted by $(\mathrm{ap})\lim_{x\to x_0} f(x)$. It is proved ([20]) that these two definitions of approximately continuity are equivalent and we can say that f is approximately continuous at a point x_0 if and only if $f(x_0) = (\mathrm{ap})\lim_{x\to x_0} f(x)$. Making use of approximate limit, one may define approximate differentiation not only for real functions but also for functions from $\mathbb{R}^n$ into $\mathbb{R}^m$.

Definition 24 ([8])**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$. We say that f is approximately differentiable at $x \in \mathbb{R}^n$ if there is a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$(\mathrm{ap}) \lim_{y \to x} \frac{|f(y) - f(x) - L(y-x)|}{|y-x|} = 0.$$

Theorem 25 ([8])**.** *Assume that $f \in BV_{(\mathrm{loc})}(\mathbb{R}^n)$. Then, f is approximately differentiable almost everywhere with respect to λ_n (we will write λ_n-a.e.).*

Since $W^{1,p}_{(\mathrm{loc})}(\mathbb{R}^n) \subset BV_{(\mathrm{loc})}(\mathbb{R}^n)$ $(1 \le p \le \infty)$, any Sobolev function is approximately differentiable λ_n-a.e. Its approximate derivative is equal to its weak derivative λ_n-a.e. The above result was widen onto m-approximately continuity (differentiability) by Delladio in [6].

Definition 26 ([6])**.** A function $f\colon A \to \mathbb{R}$ $(A \subset \mathbb{R}^n)$ is m-approximately continuous (differentiable) at x if there is a set $E \subset A$ such that $x \in E$ is a point of m-density of E and $f|_E$ is continuous (differentiable).

Theorem 27 ([6])**.** *Let E be an open set in $\mathbb{R}^n$, $n \ge 2$, $f \in W^{1,p}_{(\mathrm{loc})}(E)$ with $1 \le p < n$. Then, f is $(n+p^*)$-approximately continuous a.e. in E ($p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p).*

A set of the form $L^-(f) = \{x\colon f(x) \le c,\ x \in \mathbb{R}^n,\ c \in \mathbb{R}\}$ is called a sublevel of a function f. By exchanging $\le$ into $\ge$ in this denotation we obtain the definition of superlevel of f.

Theorem 28 ([6])**.** *If E is a sublevel (superlevel) set of a function $f \in W^{1,p}_{(\mathrm{loc})}(E)$, then almost every point of E (with respect to λ_n) is its $(n+p^*)$-density point.*

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