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HOW TO OBTAIN MAXIMAL AND MINIMAL SUBRANGES OF TWO-DIMENSIONAL VECTOR MEASURES

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ABSTRACT. Let $(X, \mathcal{F})$ be a measurable space with a nonatomic vector measure $\mu = (\mu_1, \mu_2)$. Denote by $R(Y)$ the subrange $R(Y) = \{\mu(Z) : Z \in \mathcal{F}, Z \subseteq Y\}$. For a given $p \in \mu(\mathcal{F})$ consider a family of measurable subsets $\mathcal{F}_p = \{Z \in \mathcal{F} : \mu(Z) = p\}$. Dai and Feinberg proved the existence of a maximal subset $Z^* \in \mathcal{F}_p$ having the maximal subrange $R(Z^*)$ and also a minimal subset $M^* \in \mathcal{F}_p$ with the minimal subrange $R(M^*)$. We present a method of obtaining the maximal and the minimal subsets. Hence, we get simple proofs of the results of Dai and Feinberg.

1. Introduction

Let $\mu = (\mu_1, \mu_2)$ be a finite nonatomic vector measure defined on a measurable space $(X, \mathcal{F})$, where $\mu_1 \neq \mu_2$. For simplicity and without loss of generality, we assume that $\mu_1(X) = \mu_2(X) = 1$. For each $Y \in \mathcal{F}$ denote by $R(Y)$ the subrange $R(Y) = \{\mu(Z) : Z \in \mathcal{F}, Z \subseteq Y\} \subseteq \mu(\mathcal{F})$. It follows from the famous Lyapunov convexity theorem (see Lyapunov [6]) that the subranges $R(Y)$ are convex and compact in $\mathbb{R}^2$ for all $Y \in \mathcal{F}$.

For each $p = (p_1, p_2) \in \mu(\mathcal{F}) = R(X)$ denote by $\mathcal{F}_p \subseteq \mathcal{F}$ the following family of measurable subsets of X:

$$\mathcal{F}_p = \{Z \in \mathcal{F} : \mu(Z) = p\}.$$

Dai and Feinberg [3] defined maximal and minimal subsets of X with the measure p.

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Definition 1.1. A subrange $R(Z^*)$ of $Z^* \in \mathcal{F}_p$ is called the maximal if $Z \in \mathcal{F}_p$ implies that $R(Z) \subseteq R(Z^*)$. We call the set $Z^* \in \mathcal{F}_p$ the maximal subset of X with the measure p.

Definition 1.2. A subrange $R(M^*)$ of $M^* \in \mathcal{F}_p$ is called the minimal if $M \in \mathcal{F}_p$ implies that $R(M^*) \subseteq R(M)$. We call the set $M^* \in \mathcal{F}_p$ the minimal subset of X with the measure p.

Dai and Feinberg [3] proved the existence of the maximal and the minimal subsets of X and showed geometric construction of the maximal subranges. We show a method of obtaining such subsets and give another construction of the maximal and also the minimal subranges. Finally, we present a simple example to illustrate the method.

A similar problem concerning a Chebyshev measure was considered by Bianchini et. al. [1]. They proved that a strictly convex, centrally symmetric and compact subset of $\mathbb{R}^2$, whose boundary contains the origin, is the range of a Chebyshev measure.

2. Main results

Denote by $f_i = d\mu_i/dm, i = 1, 2$ the Radon-Nikodym derivatives with respect to the measure $m = \mu_1 + \mu_2$. Legut and Wilczyński [4] used some results of Candeloro and Martellotti [2] to prove the existence of an increasing family of sets $\{A(t)\}_{t\in[0,1]} \subseteq \mathcal{F}$ satisfying

$$\mu_1\big(A(t)\big) = t, \;\; \mu_2\big(A(t)\big) = \max\{\mu_2(A) : A \in \mathcal{F}, \mu_1\big(A(t)\big) = t\} \;\; t \in [0, 1], \quad (2.1)$$

$$\text{if} \quad t_1 < t_2,\ t_1, t_2 \in [0, 1], \quad \text{then} \quad A(t_1) \subset A(t_2). \quad (2.2)$$

Moreover, it follows from the Neyman-Pearson lemma (see Lehmann and Romano [5], cf. [4]) that there exists a number $y \geq 0$ depending on t, $t \in [0, 1]$, such that

$$I_{A(t)}(x) = \begin{cases} 1 & \text{if} \quad f_2(x) > yf_1(x)\,, \\ 0 & \text{if} \quad f_2(x) < yf_1(x)\,, \end{cases}$$

where $I_{A(t)}(x)$ stands for the indicator function of $A(t)$. Legut and Wilczyński [4] noticed that the range $R(X)$ can be described by the convex, continuous and nondecreasing function $G : [0, 1] \to [0, 1]$ defined by $G(t) = \mu_2\big(A(t)\big)$, $t \in [0, 1]$ as follows

$$R(X) = \big\{(t, s) \in [0, 1]^2 : 0 \leq t \leq 1, \quad 1 - G(1 - t) \leq s \leq G(t)\big\}\,. \quad (2.3)$$

For given $p = (p_1, p_2) \in R(X)$, we assume further that $p_2 \geq p_1$. The case of $p_2 < p_1$ can be solved in a similar way.

THEOREM 2.1. *Let $p = (p_1, p_2) \in int\big(R(X)\big)$. Then, there exists a number $t_* \in (0,1)$ such that $Z^* = A(t_*) \cup \big(X \setminus A(1-p_1+t_*)\big)$ is the maximal subset with the measure p. If $p = (p_1, p_2) \in \partial R(X)$, then $Z^* = A(p_1)$ is the maximal subset.*

P r o o f. Assume first that $p = (p_1, p_2) \in int\big(R(X)\big)$. It follows from the properties of the function G that there exists $t_* \in (0, p_1)$ such that

$$G(t_* + 1 - p_1) + p_2 - 1 = G(t_*). \tag{2.4}$$

Denote by $Z^* = A(t_*) \cup \big(X \setminus A(1-p_1+t_*)\big)$ and define the convex and continuous function $G_p : [0, p_1] \to [0, p_2]$ by

$$G_p(t) = \begin{cases} G(t) & \text{if} \quad 0 \le t \le t_* \,, \\ G(t + 1 - p_1) + p_2 - 1 & \text{if} \quad t_* < t \le p_1 \,. \end{cases} \tag{2.5}$$

It can be easily verified (cf. [4]) that $Z^* \in \mathcal{F}_p$ and

$$R(Z^*) = \big\{(t,s) \in [0,1]^2 : 0 \le t \le p_1, \quad p_2 - G_p(p_1 - t) \le s \le G_p(t)\big\} \,. \tag{2.6}$$

We show that Z^* is the maximal subset with the measure p. Suppose that there exists a set $Z \in \mathcal{F}_p$ such that $R(Z) \setminus R(Z^*) \neq \emptyset$. Let

$$q = (q_1, q_2) \in R(Z) \setminus R(Z^*) \subset [0, p_1] \times [0, p_2].$$

There are two possible cases:

$$q_2 > G_p(q_1) = G(q_1 + 1 - p_1) + p_2 - 1, \quad q_1 > t_* \,, \tag{2.7}$$

and

$$q_2 < p_2 - G_p(p_1 - q_1), \quad q_1 > p_1 - t_* \,. \tag{2.8}$$

Consider the first case. It follows from (2.7) that

$$p_2 - q_2 < 1 - G(1 - p_1 + q_1). \tag{2.9}$$

Since the set $R(Z)$ is centrally symmetric with the center $(p_1/2, p_2/2)$ (cf. [3]), then we must have $(p_1 - q_1, p_2 - q_2) \in R(Z) \subset R(X)$. But this contradicts the inequality (2.9) which means, by (2.3), that $(p_1 - q_1, p_2 - q_2) \notin R(X)$.

The second case can be considered in a similar way, so we omit it.

Assume now that $p = (p_1, p_2)$ belongs to the boundary of the range $R(X)$. It means that $p_2 = G(p_1)$. Using the similar argumentation as presented above for the function $G_p : [0, p_1] \to [0, p_2]$ defined by

$$G_p(t) = G(t), \quad 0 \le t \le p_1,$$

one can easily verify that $Z^* = A(p_1)$ is the maximal subset with the measure p. $\square$

Theorem 2.2. *Let $p = (p_1, p_2) \in int\big(R(X)\big)$. Then, there exists a number $t_* \in (0,1)$ such that $M^* = A(1-p_1+t_*) \setminus A(t_*)$ is the minimal subset with the measure $(1-p_1, 1-p_2)$.*

Proof. Let $Z^* = A(t_*) \cup \big(X \setminus A(1-p_1+t_*)\big)$ be the maximal subset with the measure p defined in Theorem 2.1. Denote $M^* = X \setminus Z^* = A(1-p_1+t_*) \setminus A(t_*)$. It is known (cf. [7]) that the range $R(X)$ can be decomposed as follows

$$R(X) = R(Z^*) \oplus R(M^*), \tag{2.10}$$

where "$\oplus$" denotes the Minkowski addition. Suppose that there exists a set $M \in \mathcal{F}_w$, where $w = (1-p_1, 1-p_2)$, such that

$$R(M) \subset R(M^*) \quad \text{and} \quad R(M^*) \setminus R(M) \neq \emptyset. \tag{2.11}$$

Consider another decomposition of the range $R(X)$

$$R(X) = R(X \setminus M) \oplus R(M). \tag{2.12}$$

It follows from the decompositions (2.10), (2.12) and the inclusion (2.11) that we must have

$$R(Z^*) \subset R(X \setminus M) \quad \text{and} \quad R(X \setminus M) \setminus R(Z^*) \neq \emptyset.$$

Since $(X \setminus M) \in \mathcal{F}_p$, we get a contradiction that Z^* is the maximal subset with the measure p, which completes the proof. $\square$

From Theorem 2.1 and 2.2 it follows

Proposition 2.3. *Let $p = (p_1, p_2) \in \partial R(X)$. Then, $Z^* = A(p_1)$ is at the same time the maximal subset with measure p and the minimal subset with measure $w = (1-p_1, 1-p_2)$.*

Dai and Feinberg [3] presented an interesting counterexample which shows that for three-dimensional nonatomic vector measures $\mu = (\mu_1, \mu_2, \mu_3)$ the maximal and minimal subsets with given measure $p \in \mu(\mathcal{F})$ do not have to exist.

3. Example

Let $(X, \mathcal{F}) = ([0,1], \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra defined on the unit interval $[0,1]$. Consider vector measure $\mu = (\mu_1, \mu_2)$ defined on $([0,1], \mathcal{B})$ where μ_1, μ_2 are two nonatomic probability measures with the following density functions:

$$f_1(x) \equiv 1, \quad f_2(x) = 2 - 2x, \quad x \in [0,1].$$

It is easy to see that the family $\{A(t)\}_{t \in [0,1]}$ defined by $A(t) = [0, t)$ satisfies (2.1) and (2.2). Hence $G(t) = 2t - t^2$, the range $R(X) = \mu(\mathcal{B})$ can be given by

$$R(X) = \left\{(t,s) \in [0,1]^2 : 0 \le t \le 1, \quad t^2 \le s \le 2t - t^2\right\}.$$

Let $p = \left(\frac{1}{2}, \frac{5}{8}\right)$. Solving the equation (cf. (2.4))

$$-t^2 + t + \frac{3}{8} = 2t - t^2$$

for the function $G(t) = 2t - t^2$ with respect to t, we get the solution $t_* = \frac{3}{8}$. Then, it follows from Theorem 2.1 that the maximal subset $Z^*$with the measure $\left(\frac{1}{2}, \frac{5}{8}\right)$ is given by

$$Z^* = \left[0, \frac{3}{8}\right) \cup \left[\frac{7}{8}, 1\right].$$

In turn, using Theorem 2.2, the minimal subset M^* with the measure $(1,1) - p = \left(\frac{1}{2}, \frac{3}{8}\right)$ is given by

$$M^* = \left[\frac{3}{8}, \frac{7}{8}\right).$$

Now, we construct the maximal subrange $\mu(Z^*)$ with the measure $\left(\frac{1}{2}, \frac{5}{8}\right)$ and also minimal subrange $\mu(M^*)$ with the measure $\left(\frac{1}{2}, \frac{3}{8}\right)$. Using (2.5), we obtain the function G_p

$$G_p(t) = \begin{cases} 2t - t^2 & \text{if } \ 0 \le t \le \frac{3}{8}, \\ -t^2 + t + \frac{1}{8} & \text{if } \ \frac{3}{8} < t \le \frac{1}{2}. \end{cases}$$

Hence by (2.6), we get the range $R(Z^*)$ presented in Fig. 1. It can be verified (cf. [4]) that

$$R(M^*) = \left\{ (t,s) \in [0,1]^2 : 0 \le t \le \frac{1}{2}, \quad \frac{1}{4}t + t^2 \le s \le \frac{5}{4}t - t^2 \right\}.$$

Instead of $R(M^*)$, for better perspicuity, the set $R(M^*) + p$ is shown in Fig. 1.

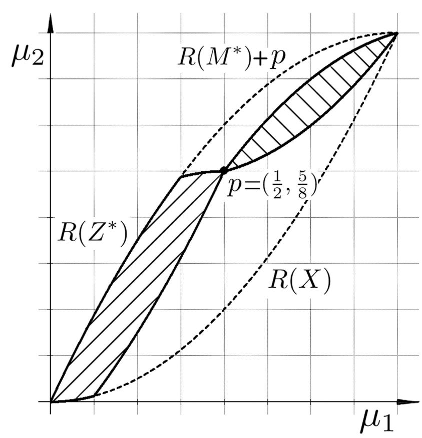

FIGURE 1. The maximal subrange $R(Z^*)$ with the measure $\left(\frac{1}{2}, \frac{5}{8}\right)$ and the set $R(M^*) + p$, where $R(M^*)$ is the minimal subrange with the measure $\left(\frac{1}{2}, \frac{3}{8}\right)$.

REFERENCES

[1] BIANCHINI, S.—CERF, R.—MARICONDA, C.: *Two-dimensional zonoids and Chebyshev measures*, J. Math. Anal. Appl. **211** (1997), 512–526.

[2] CANDELORO, D.—MARTELLOTTI, A.: *Geometric properties of the range of two--dimensional quasi-measures with respect to the Radon-Nikodym property*, Adv. Math. **93** (1992), 9–24.

[3] DAI, P .—FEINBERG, A.: *On maximal ranges of vector measures for subsets and purification of transition probabilities*, Proc. Amer. Math. Soc. **139** (2011), 4497–4511.

[4] LEGUT, J.—WILCZYŃSKI, M.: *How to obtain a range of a nonatomic vector measure in* $\mathbb{R}^2$, J. Math. Anal. Appl. **394** (2012), 102–111.

[5] LEHMANN, E. L.—ROMANO, J. P.: *Testing Statistical Hypotheses*. Springer Science + Business Media. Inc., New York, 1971.

[6] LYAPOUNOFF, A. A.: *Sur les fonctions-vecteurs complètement additives*, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4** (1940), 465–478. (In Russian)

[7] NEYMAN, A.: *Decomposition of ranges of vector measures*, Israel J. Math. **40** (1981), 54–64.

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