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STRICTLY INCREASING ADDITIVE GENERATORS OF THE SECOND KIND OF ASSOCIATIVE BINARY OPERATIONS

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ABSTRACT. The class of strictly increasing additive generators of the second kind is defined and analyzed. Necessary and sufficient conditions for a binary operation generated by a strictly increasing additive generator of the second kind to be associative are introduced. The relation between the class of strictly increasing additive generators of the second kind of associative binary operations and the class of discrete upper additive generators of associative binary operations is revealed.

1. Introduction

We define a new class of non-continuous additive generators, namely the class of additive generators of the second kind.

In the first part, we introduce conditions under which additive generators of the second kind generate associative operations. As we will see, these conditions lead to discrete upper additive generators of associative operations.

In the second part, we analyze a relation between the classes of additive generators of the second kind of associative operations and discrete upper additive generators of associative operations.

In literature there are many examples of strictly increasing non-continuous additive generators which are left-continuous and generate associative operations but that is not the case of strictly increasing non-continuous additive generators which are right-continuous and generate associative operations. As we will see in the paper, all the strictly increasing additive generators of the second kind are right-continuous.

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The definition of an additive generator of the first kind [11] and the definition of an additive generator of the second kind are analogous. Although there are some similarities, there are also many differences between the classes of additive generators of the second kind and additive generators of the first kind. It was showed [11] that conditions under which additive generators of the first kind generate associative operations lead to discrete additive generators of associative operations, and that the classes of additive generators of the first kind of associative operations and discrete additive generators of associative operations are closely related.

Although the resulting associative operations generated by additive generators of the second kind are not triangular conorms, a slight modification of the resulting associative operations leads to triangular conorms. Triangular conorms and their dual operations triangular norms are important classes of aggregation operations which play an important role in the theory of fuzzy sets and fuzzy logics. A detailed treatment of triangular norms and triangular conorms can be found in monographs [1] and [2]. Triangular norms on discrete settings are studied, for instance, in work [6].

2. Upper additively generated operations

Before we define an upper additively generated operation, we recall some basic definitions. The set of all non-negative integers is denoted by $\mathbb{N} \cup \{0\}$. The symbol $\leq$ denotes the standard linear order on $\mathbb{R} \cup \{-\infty, \infty\}$. The same symbol also denotes the restriction of $\leq$ to a non-empty set $X \subseteq \mathbb{R} \cup \{-\infty, \infty\}$.

Definition 2.1. Let $X, Y \subseteq \mathbb{R} \cup \{-\infty, \infty\}$ be non-empty linearly ordered sets with the usual linear order $\leq$.

(i) A binary operation $\odot : X^2 \to X$ is *non-decreasing* if $x \odot y \leq u \odot v$ for all $x, y, u, v \in X$, $x \leq u$, $y \leq v$.

(ii) The non-decreasing binary operations $\odot : X^2 \to X$ and $\oplus : Y^2 \to Y$ are *isomorphic* if there exists a strictly increasing bijection $f : X \to Y$ such that $x \odot y = f^{-1}\big(f(x) \oplus f(y)\big)$ for all $x, y \in X$ where f^{-1} is the standard inverse of f. The function f is an *isomorphism* of $\odot$ and $\oplus$.

(iii) Suppose that there exists a minimum $\min X$ of X. A binary operation $\odot : X^2 \to X$ is a *t-conorm* if it is non-decreasing, commutative, associative, and $\min X$ is its neutral element.

Let $\mathcal{F}$ denote the family of all strictly increasing functions $f : X \to [0, \infty]$ where $X \subseteq \mathbb{R} \cup \{-\infty, \infty\}$ is either the closed unit interval $[0, 1]$ or a non-empty finite set or an infinite countable set which can be expressed in the form of $\big\{x_l \mid l \in \mathbb{N} \cup \{0\}\big\} \cup \{x\}$, where $x_l < x_{l+1}$ for all $l \in \mathbb{N} \cup \{0\}$ and $\lim_{l \to \infty} x_l = x$.

Definition 2.2. A binary operation $F : X^2 \to X$ is *upper additively generated* if there exists $f \in \mathcal{F}$ with the domain X such that

$$F(x, y) = f^{(-1)}\big(f(x) + f(y)\big), \tag{1}$$

where $f^{(-1)} : [0, \infty] \to X$ is the pseudo-inverse of f given by

$$f^{(-1)}(t) = \inf\{z \in X \mid f(z) \geq t\}, \inf \emptyset = \max X. \tag{2}$$

We say that F is *upper additively generated* by f and that f is a *strictly increasing upper additive generator* (briefly, *upper additive generator*) of F. If X is finite, we say that f and F are *discrete.*

Remark 1. Let F be an upper additively generated operation. Then, F is always non-decreasing, commutative and $F(x, y) \geq \max\{x, y\}$ for all $x, y \in X$. In general, F need not be associative, and $\min X$ need not be a neutral element of F. Further, F has a neutral element $\min X$ if and only if $f(\min X) = 0$ or $X = \{x_0\}$.

Example. (i) The function $f : [0, 1] \to [0, \infty]$, $f(x) = x$ is an upper additive generator of Łukasiewicz t-conorm $F : [0, 1]^2 \to [0, 1]$, $F(x, y) = \min\{x + y, 1\}$.
(ii) Let $X = \big\{\frac{l}{n} \mid l \in \{0, 1, \dots, n\}\big\}$, $n \in \mathbb{N}$. The function $f : X \to [0, \infty]$, $f(\frac{l}{n}) = \frac{l}{n}$ is a discrete upper additive generator of a discrete Łukasiewicz t-conorm $F : X^2 \to X$, $F(\frac{i}{n}, \frac{j}{n}) = \min\{\frac{i}{n} + \frac{j}{n}, 1\}$.

3. Upper additively generated operations and additively generated operations

Before we look at the relation between an upper additively generated operation and an additively generated operation both generated by the same function, we recall the definition of an additively generated operation.

The next definition covers the definition of a discrete additive generator [5] and is a slight generalization of the definition of an additive generator used by Viceník [11, Definition 1] where only additive generators acting on finite sets or the unit closed interval $[0, 1]$ were considered.

Definition 3.1. A binary operation $F : X^2 \to X$ is *additively generated* if there exists $f \in \mathcal{F}$ with the domain X such that

$$F(x, y) = f^*\big(f(x) + f(y)\big), \tag{3}$$

where $f^* : [0, \infty] \to X$ is the pseudo-inverse of f given by

$$f^*(t) = \sup\{z \in X \mid f(z) \leq t\}, \sup \emptyset = \min X. \tag{4}$$

We say that F is *additively generated* by f and that f is a *strictly increasing additive generator* (briefly, *additive generator*) of F. If X is finite, we say that f and F are *discrete.*

Note that discrete additive generators of discrete associative operations are studied in works [3] – [5], and non-continuous additive generators acting on the closed unit interval $[0, 1]$ of associative operations are studied in works [7] – [10].

Let the symbol $R(f)$ denote the range of a function f in the paper.

Lemma 3.2. *Let X be the domain of $f \in \mathcal{F}$ and let the pseudo-inverses $f^{(-1)}$ and f^* of f be given by* (2) *and* (4)*, respectively. Then, the following hold:*

(i) $f^{(-1)}(t) \geq f^*(t)$ *for all* $t \in [0, \infty]$.

(ii) *Suppose* $X = [0, 1]$. *Then,* $f^{(-1)}(t) = f^*(t)$ *for all* $t \in [0, \infty]$.

(iii) *Suppose* $X \neq [0, 1]$. *For all* $t \in [0, \infty]$, $f^{(-1)}(t) > f^*(t)$ *if and only if* $t \in [\min R(f), \max R(f)]$ *and* $t \notin R(f)$.

Proof. Let $t \in [0, \infty]$. Write $X_-(t) = \{x \in X \mid f(x) \leq t\}$ and $X_+(t) = \{x \in X \mid f(x) \geq t\}$.

(i) We will consider the following four cases: Suppose that $t \in [0, \min R(f)[$. On the one hand, $X_-(t) = \emptyset$ and $f^*(t) = \min X$ by convention. On the other hand, $X_+(t) = X$, and so $f^{(-1)}(t) = \min X$.

Suppose that $t \in]\max R(f), \infty]$. On the one hand, $X_-(t) = X$, and so $f^*(t) = \max X$. On the other hand, $X_+(t) = \emptyset$ and $f^{(-1)}(t) = \max X$ by convention.

Suppose that $t \in [\min R(f), \max R(f)]$ and $t \in R(f)$. Then, there is one and only one point $p \in X$ such that $f(p) = t$. On the one hand, p is a maximum of the set $X_-(t)$, and so $f^*(t) = p$. On the other hand, p is a minimum of the set $X_+(t)$, and so $f^{(-1)}(t) = p$.

Suppose that $t \in [\min R(f), \max R(f)]$ and $t \notin R(f)$. Then, both sets $X_-(t)$ and $X_+(t)$ are non-empty, $X_-(t) \cap X_+(t) = \emptyset$ and $X_-(t) \cup X_+(t) = X$. For arbitrary $x_1 \in X_-(t)$ and $x_2 \in X_+(t)$ it is $f(x_1) < f(x_2)$ implying $x_1 < x_2$. Hence, $f^*(t) \leq f^{(-1)}(t)$.

(ii) Suppose that $X = [0, 1]$. With respect to the proof of (i), it is sufficient to show that $f^*(t) = f^{(-1)}(t)$ for all $t \in [\min R(f), \max R(f)]$, $t \notin R(f)$. The proof is by contradiction. Suppose that $f^*(t) < f^{(-1)}(t)$ for some $t \in [\min R(f), \max R(f)]$, $t \notin R(f)$. Fix $q \in]f^*(t), f^{(-1)}(t)[\subseteq [0, 1]$. Obviously, $q \notin X_-(t) \cup X_+(t)$ which is a contradiction to $X_-(t) \cup X_+(t) = [0, 1]$.

(iii) Suppose that $X \neq [0, 1]$, that is, X is finite or $X = \{x_l \mid l \in \mathbb{N} \cup \{0\}\} \cup \{x\}$ where $x_l < x_{l+1}$ for all $l \in \mathbb{N} \cup \{0\}$ and $\lim_{l \to \infty} x_l = x$. With respect to the proof of (i), it is sufficient to show that $f^*(t) < f^{(-1)}(t)$ for all $t \in [\min R(f), \max R(f)]$, $t \notin R(f)$. Fix $t \in [\min R(f), \max R(f)]$, $t \notin R(f)$. Obviously, there exists a maximum $\max X_-(t)$ of $X_-(t)$ and $\max X_-(t) = f^*(t)$, and similarly, there exists a minimum $\min X_+(t)$ of $X_+(t)$ and $\min X_+(t) = f^{(-1)}(t)$. Further, for arbitrary $x_1 \in X_-(t)$ and $x_2 \in X_+(t)$ it is $x_1 < x_2$. Hence, $f^*(t) < f^{(-1)}(t)$. $\square$

Proposition 3.3. *Let X be the domain of $f \in \mathcal{F}$ and let F be upper additively generated by f, and G be additively generated by f. Then, the following hold:*

(i) $F(x, y) \geq G(x, y)$ *for all* $x, y \in X$.

(ii) *Suppose* $X = [0, 1]$. *Then,* $F(x, y) = G(x, y)$ *for all* $x, y \in X$.

(iii) *Suppose* $X \neq [0, 1]$. *For all* $x, y \in X$, $F(x, y) > G(x, y)$ *if and only if* $f(x) + f(y) \leq \max R(f)$ *and* $f(x) + f(y) \notin R(f)$.

P r o o f. Our assertion follows from Lemma 3.2 and Definitions 2.2 and 3.1. $\square$

Remark 2. We have shown that $f \in \mathcal{F}$ whose domain is the interval $[0, 1]$ leads via the formulas (1) and (3) to an upper additively generated operation and an additively generated operation which are identical, and therefore, in this case, we can say an *additively generated operation* instead of an *upper additively generated operation* and an *additive generator* instead of an *upper additive generator*.

Example. The function $f \in \mathcal{F}$ with the domain $X = \{0, \frac{1}{2}, 1\}$ defined by $f(0) = 1$ and $f(\frac{1}{2}) = 3$ and $f(1) = 5$ leads to the upper additively generated operation $F(x, y) = 1$ for all $(x, y) \in X^2 \setminus \{(0, 0)\}$ and $F(0, 0) = \frac{1}{2}$, and the additively generated operation $G(x, y) = 1$ if $(x, y) \in X^2 \setminus \{(0, 0), (0, \frac{1}{2})(\frac{1}{2}, 0)\}$ and $G(0, \frac{1}{2}) = G(\frac{1}{2}, 0) = \frac{1}{2}$ and $G(0, 0) = 0$.

4. Upper additions

We begin by defining upper additions which are, as we will see in Section 6, closely related to upper additively generated operations.

Let us denote by $\mathcal{R}$ the family of all sets $X \subseteq [0, \infty]$ such that there exists a maximum $\max X$ of X and $\inf\{x \in X \mid x \geq t\} \in X$ for all $t \in [0, \max X]$. Obviously, if $X \in \mathcal{R}$ then there exists a minimum $\min X$ of X.

Definition 4.1. Let $X \in \mathcal{R}$. A binary operation $\oplus : X^2 \to X$ is an *upper addition* on X if $x \oplus y = S(x + y)$, where $S : [0, \infty] \to X$, $S(t) = \inf\{z \in X \mid z \geq t\}$, $\inf \emptyset = \max X$. The function S is the *upper function determined* by X.

The next remark describes the basic properties of upper functions and upper additions. Their proofs are straightforward.

Remark 3. Let $X \in \mathcal{R}$, S be the upper function determined by X, and $\oplus$ be the upper addition on X.

(i) Then, $S(t) = \min\{x \in X \mid x \geq t\} \geq t$ for all $t \in [0, \max X]$. Observe that $S(t) = \max X < t$ for all $t \in] \max X, \infty]$. Obviously, $S(t) = \min\{x \in X \mid x \geq t\} \geq t$ for all $t \in [0, \infty]$ if and only if $\infty \in X$. Further, for all $t \in [0, \infty]$, $S(t) = t$ if and only if $t \in X$. Furthermore, S is non-decreasing on $[0, \infty]$. Note that the upper function determined by X is uniquely determined by X.

(ii) Then, $x \oplus y = \min\{z \in X \mid z \geq x+y\} \geq x+y$ for all $x, y \in X$, $x+y \leq \max X$. Observe that $x \oplus y = \max X < x+y$ for all $x, y \in X$, $x+y > \max X$. Obviously, $x \oplus y = \min\{z \in X \mid z \geq x+y\} \geq x+y$ for all $x, y \in X$ if and only if either $\infty \in X$ or $X = \{0\}$. Further, for all $x, y \in X$, $x \oplus y = x+y$ if and only if $x+y \in X$. Furthermore, $\oplus$ is non-decreasing, commutative and $\max\{x, y\} \leq x \oplus y$ for all $x, y \in X$. In general, $\oplus$ need not be associative, and $\min X$ need not be a neutral element of $\oplus$. The operation $\oplus$ on X has a neutral element $\min X$ if and only if $0 \in X$ or $X = \{x_0\}$. Note that the upper addition on X is uniquely determined by X.

(iii) Then, for all $x \in X$ if $\min X < x < \max X$, then $x < x \oplus x$.

Example. The upper addition $\oplus$ on $[0, \infty]$ coincides with the usual addition $+$ on $[0, \infty]$, that is, $x \oplus y = x + y$ for all $x, y \in [0, \infty]$. The upper addition $\oplus$ on $[0, 1]$ is given by $x \oplus y = \min\{x + y, 1\}$ for all $x, y \in [0, 1]$.

Let $X, Y \subseteq R \cup \{-\infty, \infty\}$ be non-empty sets and $X \subseteq Y$. Suppose that $f : Y \to R \cup \{-\infty, \infty\}$ is a function. The restriction of f from Y to X is denoted by f/X, that is, $f/X : X \to R \cup \{-\infty, \infty\}$, $f/X(x) = f(x)$ for all $x \in X$. Suppose that $F : Y^2 \to Y$ is a binary operation on Y. If $F(x, y) \in X$ for all $x, y \in X$, then F can be restricted from Y to X and this restriction is denoted by F/X^2, that is, $F/X^2 : X^2 \to X$, $F/X^2(x, y) = F(x, y)$ for all $x, y \in X$.

Proposition 4.2. *Let $X, Y \in \mathcal{R}$ and let $X \subseteq Y$. If the upper addition on Y can be restricted to X then this restriction coincides with the upper addition on X.*

Proof. Let $\odot$ and $\oplus$ be the upper additions on X and Y, respectively. Suppose that the upper addition on Y can be restricted to X, that is, $x \oplus y \in X$ for all $x, y \in X$. Let us denote the restriction of $\oplus$ to X by $\oplus/X^2$. We can write $x \oplus y = x(\oplus/X^2)y$ for all $x, y \in X$.

First, we will prove that if $X \neq \{0\}$, then $\max X = \max Y$. Suppose that $X \neq \{0\}$. Since $X, Y \in \mathcal{R}$ then there exists a maximum $\max X$ of X and a maximum $\max Y$ of Y, and since $X \subseteq Y$, it is $\max X \leq \max Y$. Moreover, $0 < \max X$ since $X \neq \{0\}$. The proof is by contradiction. Suppose that $\max X < \max Y$. Then, $\max X < \max X \oplus \max X$ by definition of $\oplus$, and so, $\max X \oplus \max X \notin X$, which would be a contradiction. Thus, $\max X = \max Y$.

Now, we will prove that $\oplus/X^2$ coincides with the upper addition $\odot$ on X. If $X = \{0\}$, the assertion is obviously true. Suppose that $X \neq \{0\}$. We have already proved that $\max X = \max Y = m$. We will consider the following two cases.

Suppose that $x, y \in X$, $x + y > m$. Then, $\{z \in X \mid z \geq x + y\} = \emptyset$ and $\{z \in Y \mid z \geq x + y\} = \emptyset$ implying $x \odot y = m = x \oplus y$ by definition.

Suppose that $x, y \in X$, $x+y \leq m$. On the one hand, $x \odot y = \min\{z \in X \mid z \geq x + y\} \geq \min\{z \in Y \mid z \geq x + y\} = x \oplus y$. On the other hand, since $x \oplus y \in X$ and $x \oplus y \geq x + y$, it is $x \oplus y \in \{z \in X \mid z \geq x + y\}$ implying $x \oplus y \geq x \odot y$. Hence, $x \odot y = x \oplus y$.

Finally, for all $x, y \in X$, we have $x \odot y = x \oplus y$, and since $x \oplus y = x(\oplus/X^2)y$, we obtain $x \odot y = x(\oplus/X^2)y$. □

Remark 4. (i) Let us recall the definition of a lower addition [11, Definition 3]: Assuming that there exists a minimum $\min X$ of $X \subseteq [0, \infty]$ and $\sup\{z \in X \mid z \leq t\} \in X$ for all $t \in [\min X, \infty]$, the lower addition $\oplus : X^2 \to X$ is defined by $x \oplus y = \sup\{z \in X \mid z \leq x + y\}$ for all $x, y \in X$.
(ii) Additions [7, Definition 6] were defined on sets $X \in \mathcal{M}$ where $\mathcal{M}$ denotes the family of all sets $X \subseteq [0, \infty]$ such that there exists a strictly increasing function $f : [0, 1] \to [0, \infty]$ with the range $R(f) = X$. Note that the upper addition and the addition coincide on every set $X \in \mathcal{M} \cap \mathcal{R}$.

Example. Let $X = \big\{2l + 1 \mid l \in \mathbb{N} \cup \{0, \infty\}\big\}$.
(i) The upper addition $\oplus$ on X is defined by $(2i+1) \oplus (2j+1) = 2(i+j+1)+1$ for all $i, j \in \mathbb{N} \cup \{0, \infty\}$.
(ii) The lower addition $\oplus$ on X is defined by $(2i+1) \oplus (2j+1) = 2(i+j+1)-1$ for all $i, j \in \mathbb{N} \cup \{0, \infty\}$.

Note that the lower addition and the upper addition on X in the example above are not isomorphic.

5. Upper additions on intervals' sets of the second kind

Suppose that a set $M \subseteq [0, \infty]$ is in the form of

$$M = \left(\bigcup_{l \in L} [a_l, b_l[\right) \cup \{\infty\}, \quad L = \{0, 1, \ldots, n\}, \;\; n \in \mathbb{N} \cup \{0\} \tag{5}$$

or

$$M = \left(\bigcup_{l \in L} [a_l, b_l[\right) \cup \{\infty\}, \quad L = \mathbb{N} \cup \{0\}, \quad \lim_{l \to \infty} a_l = \infty, \tag{6}$$

where $a_l, b_l \in [0,\infty]$, $a_l < b_l$ for all $l \in L$, and $b_l < a_{l+1}$ for all $l, l+1 \in L$. Suppose that for all $i,j,k \in L$, if $([a_i,b_i[+[a_j,b_j[) \cap [a_k,b_k[\neq \emptyset$, then

$$\max\{a_i+b_j, b_i+a_j\} \leq a_k \text{ and } b_i+b_j \leq b_k. \tag{7}$$

Definition 5.1.

(i) A set $M \subseteq [0,\infty]$ is *given by a sequence of intervals* $\{[a_l,b_l[\}_{l\in L}$ if it is in the form of (5) or (6). Writing $A = \{a_l \mid l \in L\}$ the set $A \cup \{\infty\}$ is a *gap-point* set of M.

(ii) A set $M \subseteq [0,\infty]$ is an *intervals' set of the second kind* if it is in the form of (5) and satisfies (7).

(iii) A strictly increasing function $f : [0,1] \to [0,\infty]$ is a *strictly increasing additive generator of the second kind* (briefly, *additive generator of the second kind*) if its range is an intervals' set of the second kind.

Every intervals' set of the second kind is an element of $\mathcal{R}$ and every additive generator of the second kind is an element of $\mathcal{F}$.

Example. The set $M = (\bigcup_{l\in\mathbb{N}\cup\{0\}}[2l+1, 2l+2[) \cup \{\infty\}$ is given by a sequence of intervals $\{[2l+1,2l+2[\}_{l\in\mathbb{N}\cup\{0\}}$ and satisfies (7).

Remark 5. Let us recall the definition of an additive generator of the first kind [11, Definition 4]: A strictly increasing function $f : [0,1] \to [0,\infty]$ is a *strictly increasing additive generator of the first kind* (briefly, *additive generator of the first kind*) if its range $R(f) = M$ is in the form of

$$M = \left(\bigcup_{l\in L}]a_l, b_l]\right) \cup \{0\}, L = \{0,1,\dots,n\}, n \in \mathbb{N}\cup\{0\},$$

where $a_l, b_l \in [0,\infty]$, $a_l < b_l$ for all $l \in L$, and $b_l < a_{l+1}$ for all $l, l+1 \in L$, and the following is satisfied: for all $i,j,k \in L$, if $(]a_i,b_i]+]a_j,b_j]) \cap]a_k,b_k] \neq \emptyset$, then

$$a_k \leq a_i + a_j \quad \text{and} \quad b_k \leq \min\{a_i+b_j, b_i+a_j\}.$$

Lemma 5.2. *Let $M \subseteq [0,\infty]$ be given by a sequence of intervals $\{[a_l,b_l[\}_{l\in L}$ and satisfy* (7)*. Then, $a_l, b_l \in]0,\infty[$, $b_l - a_l \leq a_0$ for all $l \in L$, and in particular, $b_0 \leq 2a_0$.*

Proof. First, we will prove that $b_l - a_l \leq a_0$ for all $l \in L$. The proof is by contradiction. Suppose that $b_l - a_l > a_0$ for some $l \in L$. Then, $a_l \leq a_0 + a_l < b_l$ implying that $([a_0,b_0[+[a_l,b_l[) \cap [a_l,b_l[\neq \emptyset$. By (7), it is $a_0 + b_l \leq \max\{a_0 + b_l, b_0 + a_l\} \leq a_l$, and consequently, $b_l \leq a_l$ which contradicts to $a_l < b_l$.

Now, we will prove that $b_0 \leq 2a_0$. From $b_l - a_l \leq a_0$ for all $l \in L$, we have $b_0 - a_0 \leq a_0$, or equivalently, $b_0 \leq 2a_0$.

Finally, we will prove that $a_l, b_l \in]0,\infty[$. Since $0 < b_0$, from $b_0 \leq 2a_0$ it is $0 < a_0$ implying $a_l, b_l \in]0,\infty]$. It remains to prove that $b_l < \infty$ for all $l \in L$.

The proof is by contradiction. Suppose that $b_l = \infty$ for some $l \in L$. Then, $([a_l, b_l[+[a_l, b_l[) \cap [a_l, b_l[\neq \emptyset$, and by (7), $\infty = \max\{a_l + b_l, b_l + a_l\} \leq a_l$, that is, $b_l \leq a_l$ which contradicts to $a_l < b_l$. □

The next lemma describes some specific properties of upper additions acting on the sets which are given by a sequence of intervals and satisfy (7).

Let $M \subseteq [0, \infty]$ be given by a sequence of intervals $\{[a_l, b_l[\}_{l \in L}$, satisfy (7), and let $\oplus$ be the upper addition on M. Put

$$c_0 = 0, c_l = b_{l-1} \text{ for all } l \in L \setminus \{0\}, \text{ and } c_\infty = b_n \text{ if } n = \max L. \tag{8}$$

Write $a_\infty = \infty$. Observe that for arbitrary $x, y \in M$ with $x + y \notin M$, there exists an index $l \in L \cup \{\infty\}$ such that $x + y \in [c_l, a_l]$. Since $[c_l, a_l] \cap M = \{a_l\}$, we obtain $x \oplus y = \min\{z \in M \mid z \geq x + y\} = a_l$.

Lemma 5.3. *Let M be given by a sequence of intervals $\{[a_l, b_l[\}_{l \in L}$ and let satisfy* (7). *Let $\oplus$ be the upper addition on M and let numbers c_l be given by* (8). *Suppose that $i, j \in L$, $x \in [a_i, b_i[$ and $y \in [a_j, b_j[$. Write $a_\infty = b_\infty = \infty$ and $a_k = \min\big\{a_l \mid a_l \geq a_i + a_j,\ l \in L \cup \{\infty\}\big\}$. Then*

(i) *$c_k \leq a_i + a_j < \max\{a_i + b_j, b_i + a_j\} \leq a_k$ and $b_i + b_j \leq b_k$,*

(ii) *if $k \in L$, then $x \oplus y \in [a_k, b_k[$, and if $k = \infty$ then $x \oplus y = \infty$,*

(iii) *if $x + y \notin \cup_{l \in L}[a_l, b_l[$, then $x \oplus y = a_i \oplus a_j = a_k$,*

(iv) *if $x = a_i$ or $y = a_j$, then $x \oplus y = a_i \oplus a_j = a_k$.*

Proof. Let $i, j \in L$, $x \in [a_i, b_i[$, $y \in [a_j, b_j[$ and $a_k = \min\{a_l \mid a_l \geq a_i + a_j, l \in L \cup \{\infty\}\}$ Recall $a_l, b_l \in]0, \infty[$ for all $l \in L$ by Lemma 5.2.

(i) Clearly, $a_i + a_j < \max\{a_i + b_j, b_i + a_j\}$. Recall that $a_i + a_j \leq a_k$ by definition of a_k. We will prove that $\max\{a_i + b_j, b_i + a_j\} \leq a_k$ and $b_i + b_j \leq b_k$. We will consider the following two cases:

Suppose that $k \in L$. Then, $\max\{a_i + b_j, b_i + a_j\} \leq a_k$ and $b_i + b_j \leq b_k$, since if $\max\{a_i + b_j, b_i + a_j\} > a_k$ or $b_k < b_i + b_j$, then we would have $k \in L$ and $([a_i, b_i[+[a_j, b_j[) \cap [a_k, b_k[\neq \emptyset$, and by (7), $\max\{a_i + b_j, b_i + a_j\} \leq a_k$ and $b_i + b_j \leq b_k$, which would be a contradiction.

Suppose that $k = \infty$. Then, $a_\infty = b_\infty = \infty$, and obviously,

$$\max\{a_i + b_j, b_i + a_j\} \leq a_\infty \quad \text{and} \quad b_i + b_j \leq b_\infty.$$

It remains to prove that $c_k \leq a_i + a_j$. We will consider the following two cases:

Suppose that $k \in L$. Then, $k \neq 0$, since if $k = 0$ then from $a_i + a_j \leq a_0$ we would have $i = j = 0$ and $a_0 = 0$, which would contradict to $a_0 > 0$. Thus, $k \in L \setminus \{0\}$, and $a_{k-1} < a_i + a_j$ by definition of a_k. Further, $b_{k-1} \leq a_i + a_j$ since if $a_i + a_j < b_{k-1}$, then $([a_i, b_i[+[a_j, b_j[) \cap [a_{k-1}, b_{k-1}[\neq \emptyset$, and by (7), it would be $a_i + a_j \leq \max\{a_i + b_j, b_i + a_j\} \leq a_{k-1}$, which would be a contradiction. Thus, $c_k = b_{k-1} \leq a_i + a_j$.

Suppose that $k = \infty$. Then, $a_\infty = \infty$ and $a_l < a_i + a_j < a_\infty$ for all $l \in L$ by definition of $a_k = a_\infty$. Further, the non-empty set L is finite since if $L = \mathbb{N} \cup \{0\}$, then $\lim_{l\to\infty} a_l = \infty > a_i + a_j$ which would contradict to $a_l < a_i + a_j < \infty$ for all $l \in L$. So, there exists $\max L = n$ and $a_n < a_i + a_j$. Further, $b_n \le a_i + a_j$ since if $a_i + a_j < b_n$, then $([a_i, b_i[+[a_j, b_j[) \cap [a_n, b_n[\neq \emptyset$, and by (7), it would be $a_i + a_j \le \max\{a_i + b_j, b_i + a_j\} \le a_n$, which would be a contradiction. Hence, $c_\infty = b_n \le a_i + a_j$.
(ii) Clearly, $a_i + a_j \le x + y < b_i + b_j$. By (i), $x + y \in [c_k, b_k[$. We will consider the following two cases:

Suppose that $k \in L$. Then, $x+y \in [c_k, a_k[$ or $x+y \in [a_k, b_k[$. If $x+y \in [c_k, a_k[$, then $x \oplus y = a_k$ because of $[c_k, a_k] \cap M = \{a_k\}$, and if $x + y \in [a_k, b_k[$, then $x \oplus y = x + y$. Hence, $x \oplus y \in [a_k, b_k[$.

Suppose that $k = \infty$. Then, $a_\infty = \infty$ and $x+y \in [c_\infty, \infty[$, and so, $x \oplus y = \infty$ because of $[c_\infty, \infty] \cap M = \{\infty\}$.
(iii) Let $x + y \notin \cup_{l\in L}[a_l, b_l[$. By (i), $x + y \in [c_k, b_k[$. Hence, $x + y \in [c_k, a_k[$. By (i), $a_i + a_j \in [c_k, a_k[$. Hence, $x \oplus y = a_i \oplus a_j = a_k$ because of $[c_k, a_k] \cap M = \{a_k\}$.
(vi) Suppose that $x = a_i$ or $y = a_j$. By (i), $c_k \le a_i+a_j \le x+y < \max\{a_i+b_j, b_i+a_j\} \le a_k$, and so, $x+y \in [c_k, a_k[$ and $a_i+a_j \in [c_k, a_k[$. Hence, $x\oplus y = a_i \oplus a_j = a_k$ because of $[c_k, a_k] \cap M = \{a_k\}$. □

6. The relation between upper additions and upper additively generated operations

First, we look at the relation between $\mathcal{R}$ and $\{R(f) \mid f \in \mathcal{F}\}$. Observe that $[0,1] \cup [2,3] \in \mathcal{R}$ but $[0,1] \cup [2,3] \notin \{R(f) \mid f \in \mathcal{F}\}$, and that $[0,1] \cup]2,3] \notin \mathcal{R}$ but $[0,1] \cup]2,3] \in \{R(f) \mid f \in \mathcal{F}\}$. It is clear that if X is the domain of $f \in \mathcal{F}$ and $X \neq [0,1]$, then $R(f) \in \mathcal{R}$.

Lemma 6.1. *Let $[0,1]$ be the domain of $f \in \mathcal{F}$. Then, $R(f) \in \mathcal{R}$ if and only if f is right-continuous.*

P r o o f. ($\Rightarrow$) The proof is by a contradiction. Suppose that $R(f) \in \mathcal{R}$ and that there exists $p \in [0,1[$ such that $\lim_{x\to p^+} f(x) = l > f(p)$. Obviously, $l \notin \mathcal{R}(f)$. Choose $t \in]f(p), l]$. Then, $\inf\{x \in R(f) \mid x \ge t\} = l \notin R(f)$, which contradicts to $R(f) \in \mathcal{R}$.
($\Leftarrow$) The proof is by contradiction. Suppose that f is right-continuous and that there exists $t \in [0,\infty]$ such that $\inf\{x \in R(f) \mid x \ge t\} = i \notin R(f)$. The set $\{x \in R(f) \mid x \ge t\}$ is non-empty, since if it is empty, then $i = \max R(f) \in R(f)$, which would be a contradiction. Obviously, i is an accumulation point of $R(f)$

from the right. Write $p = \inf\{x \in [0,1] \mid f(x) \geq i\}$. Obviously, $p \in [0,1[$ and $\lim_{x\to p^+} f(x) = i > f(p)$, which contradicts to the right-continuity of f. $\square$

Theorem 6.2. *Let X be the domain of $f \in \mathcal{F}$ and $R(f) \in \mathcal{R}$. Let $F : X^2 \to X$ be upper additively generated by f and let $\oplus : R(f)^2 \to R(f)$ be the upper addition on $R(f)$. Then, the strictly increasing bijection $f : X \to R(f)$ is an isomorphism of F and $\oplus$, that is, $F(x,y) = f^{-1}(f(x) \oplus f(y))$ for all $x, y \in X$.*

Proof. First, we will prove that $f^{-1}\big(S(t)\big) = f^{(-1)}(t)$ for all $t \in [0,\infty]$, where S is the upper function determined by $R(f)$. If $t \in]\max R(f), \infty]$, then $f^{-1}\big(S(t)\big) = \max X = f^{(-1)}(t)$ by definitions. Fix $t \in [0, \max R(f)]$. Since $R(f) \in \mathcal{R}$, it is $S(t) = \inf\{x \in R(f) \mid x \geq t\} \in R(f)$, that is, there exists one and only one $p \in X$ such that $S(t) = f(p)$, and $f^{-1}\big(S(t)\big) = p$. It remains to proof that $f^{(-1)}(t) = p$. On the one hand, $S(t) \geq t$, that is, $f(p) \geq t$ implying $p \in \{x \in X \mid f(x) \geq t\}$, and so, $f^{(-1)}(t) = \inf\{x \in X \mid f(x) \geq t\} \leq p$. On the other hand, for all $x \in X$, $x < p$, it is $f(x) < t$ since if $f(x) \geq t$, for some $x \in X$, $x < p$, we would have $f(p) > f(x) \geq S(t)$ which would contradict to $f(p) = S(t)$. Thus, $\{x \in X \mid f(x) \geq t\} \subseteq [p, \infty]$, and so, $f^{(-1)}(t) = \inf\{x \in X \mid f(x) \geq t\} \geq p$. We have already proved $f^{(-1)}(t) = p$.

For all $x, y \in X$, it is $F(x,y) = f^{(-1)}\big(f(x) + f(y)\big) = f^{-1}\big(S(f(x) + f(y))\big) = f^{-1}\big(f(x) \oplus f(y)\big)$ by the definitions of F and $\oplus$. $\square$

Let $\mathcal{D}$ denote the set of all non-empty sets $X \subseteq [0,\infty]$, where X is a finite set or it is an infinite countable set which can be expressed in the form of $\{x_l \mid l \in \mathbb{N} \cup \{0\}\} \cup \{x\}$ where $x_l < x_{l+1}$ for all $l \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} x_l = x$. Clearly, $\mathcal{D} \subseteq \mathcal{R}$.

Proposition 6.3. *Let X be the domain of $f \in \mathcal{F}$ and let $R(f) \in \mathcal{R}$. If the upper addition on $R(f)$ can be restricted to a set $Z \subseteq R(f)$ and $Z \in \mathcal{D}$, then the operation F upper additively generated by f can be restricted to $Y = f^{-1}(Z)$, and this restriction $F/Y^2 : Y^2 \to Y$ is upper additively generated by f/Y.*

Proof. By Theorem 6.2, the strictly increasing bijection $f : X \to R(f)$ is an isomorphism of the operation $F : X^2 \to X$ upper additively generated by f and the upper addition $\oplus : R(f)^2 \to R(f)$ on $R(f)$ and $F(x,y) = f^{-1}\big(f(x) \oplus f(y)\big)$ for all $x, y \in X$. If the upper addition on $R(f)$ can be restricted to a non-empty set $Z \subseteq R(f)$, then F can be restricted to $Y = f^{-1}(Z)$, and the strictly increasing bijection $f/Y : Y \to Z$ is an isomorphism of $F/Y^2 : Y^2 \to Y$ and $\oplus/Z^2 : Z^2 \to Z$ such that $F/Y^2(x,y) = (f/Y)^{-1}\big(f/Y(x)(\oplus/Z^2)f/Y(y)\big)$ for all $x, y \in Y$. Clearly, $Z \in \mathcal{R}$ because $Z \in \mathcal{D} \subseteq \mathcal{R}$. By Proposition 4.2, $\oplus/Z^2$ coincides with the upper addition $\oplus_Z : Z^2 \to Z$ on Z, that is, $f/Y(x)(\oplus/Z^2)f/Y(y) = f/Y(x) \oplus_Z f/Y(y)$ for all $x, y \in Y$. Hence, $F/Y^2(x,y) = (f/Y)^{-1}(f/Y(x) \oplus_Z f/Y(y))$ for all $x, y \in Y$. Obviously, Y is the domain of a strictly increasing function f/Y and $R(f/Y) = Z \in \mathcal{R}$. If $f/Y \in \mathcal{F}$, then, by Theorem 6.2, F/Y^2 is upper additively generated by f/Y. It remains to prove that $f/Y \in \mathcal{F}$. We will consider the following two cases.

Suppose that a non-empty set Z is finite. Then, the set Y is non-empty and finite, and obviously, $f/Y \in \mathcal{F}$.

Suppose that $Z = \{z_l \mid l \in \mathbb{N} \cup \{0\}\} \cup \{z\}$, $z_l < z_{l+1}$ for all $l \in \mathbb{N} \cup \{0\}$, $\lim_{l\to\infty} z_l = z$. Write $f^{-1}(z_l) = y_l$ for all $l \in \mathbb{N} \cup \{0\}$ and $f^{-1}(z) = y$. Then, $Y = \{y_l \mid l \in \mathbb{N} \cup \{0\}\} \cup \{y\}$, $y_l < y_{l+1}$ for all $l \in \mathbb{N} \cup \{0\}$ and $\lim_{l\to\infty} y_l \le y$. In order to complete the proof, we should show that $\lim_{l\to\infty} y_l = y$. Because $Y \subseteq X$, the domain X of f can be either $\{x_l \mid l \in \mathbb{N} \cup \{0\}\} \cup \{x\}$, $x_l < x_{l+1}$ for all $l \in \mathbb{N} \cup \{0\}$, $\lim_{l\to\infty} x_l = x$ or $[0,1]$. In the former case, the sequence $\{y_l\}$ is a subsequence of $\{x_l\}$, and so, $\lim_{l\to\infty} y_l = \lim_{l\to\infty} x_l = x$. Since $\lim_{l\to\infty} y_l \le y \le x$, it is $\lim_{l\to\infty} y_l = y$. In the latter case, it is $\lim_{l\to\infty} y_l = y$ since if $\lim_{l\to\infty} y_l < y$, then for an arbitrary $w \in]\lim_{l\to\infty} y_l, y[$, we would have $z = \lim_{l\to\infty} z_l = \lim_{l\to\infty} f(y_l) \le f(w) < f(y)$ which would contradict to $z = f(y)$. $\square$

7. Restrictions of additive generators of the second kind to discrete upper additive generators

We will show that every additive generator of the second kind of an associative operation can always be restricted to a discrete upper additive generator of a discrete associative operation.

Lemma 7.1. *Let M be given by a sequence of intervals $\{[a_l, b_l[\}_{l\in L}$, satisfy (7), and let $A \cup \{\infty\}$ be its gap-point set where $A = \{a_l \mid l \in L\}$. Then, the upper addition on M can be restricted to $A \cup \{\infty\}$ and this restriction coincides with the upper addition on $A \cup \{\infty\}$. Moreover, the upper addition on M is associative if and only if the upper addition on $A \cup \{\infty\}$ is associative.*

Proof. Put $a_\infty = \infty$. Then, $A \cup \{\infty\} = \{a_l \mid l \in L \cup \{\infty\}\}$. It is obvious that $M, A \cup \{\infty\} \in \mathcal{R}$. Let us denote the upper addition on M by $\oplus$. From Lemma 5.3 (iv) and from the fact that $a_\infty \oplus x = x \oplus a_\infty = a_\infty$ for all $x \in M$, it follows that $a_i \oplus a_j \in A \cup \{\infty\}$ for all $i, j \in L \cup \{\infty\}$, that is, the upper addition on M can be restricted to $A \cup \{\infty\}$, and moreover, this restriction coincides with the upper addition on $A \cup \{\infty\}$ by Proposition 4.2. Obviously, if the upper addition on M is associative then so is the upper addition on $A \cup \{\infty\}$.

Suppose that the upper addition on $A \cup \{\infty\}$ is associative. Since the upper addition on $A \cup \{\infty\}$ coincides with the restriction of the upper addition $\oplus$ from M to $A \cup \{\infty\}$, we can write $(a_i \oplus a_j) \oplus a_k = a_i \oplus (a_j \oplus a_k)$ for all $i, j, k \in L \cup \{\infty\}$. We will prove that the upper addition $\oplus$ on M is associative, that is, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in M$. The last equality is obviously true if x or y or z is ∞. Suppose that $x \in [a_i, b_i[$, $y \in [a_j, b_j[$, $z \in [a_k, b_k[$ for some $i, j, k \in L$. Let S be the upper function determined by M. We will consider the following three cases:

Suppose that $x + y, y + z \in \bigcup_{l \in L}[a_l, b_l[$. Obviously, $x \oplus y = x + y$, and so $(x \oplus y) \oplus z = S((x \oplus y) + z) = S(x + y + z)$. Similarly, $x \oplus (y \oplus z) = S(x + y + z)$.

Suppose that $x+y, y+z \notin \bigcup_{l \in L}[a_l, b_l[$. By Lemma 5.3 (iii), $x \oplus y = a_i \oplus a_j = a_m$ where $a_m = \min\{a_l \in A \cup \{\infty\} \mid a_l \geq a_i + a_j\}$, and by Lemma 5.3 (iv), $a_m \oplus z = a_m \oplus a_k$. Hence, $(x \oplus y) \oplus z = (a_i \oplus a_j) \oplus a_k$. Similarly, $x \oplus (y \oplus z) = a_i \oplus (a_j \oplus a_k)$.

Suppose that $x + y \in \bigcup_{l \in L}[a_l, b_l[$ and $y + z \notin \bigcup_{l \in L}[a_l, b_l[$. As in the first case, $(x \oplus y) \oplus z = S(x + y + z)$, and as in the second case, $x \oplus (y \oplus z) = a_i \oplus (a_j \oplus a_k)$. Recall that $(a_i \oplus a_j) \oplus a_k = a_i \oplus (a_j \oplus a_k)$. On the one hand, $x \geq a_i$, $y \geq a_j$ and $z \geq a_k$, and because of the monotonicity of $\oplus$,

$$(x \oplus y) \oplus z \geq (a_i \oplus a_j) \oplus a_k = x \oplus (y \oplus z).$$

On the other hand, since $\infty \in M$, it is $y \oplus z = \min\{w \in M \mid w \geq y + z\} \geq y + z$, and because of the monotonicity of S,

$$x \oplus (y \oplus z) = S\big(x + (y \oplus z)\big) \geq S(x + y + z) = (x \oplus y) \oplus z. \qquad \square$$

Example. (i) The set $M = \big(\bigcup_{l \in \mathbb{N} \cup \{0\}}[2l+1, 2l+2[\big) \cup \{\infty\}$ is given by a sequence of intervals $\{[2l+1, 2l+2[\}_{l \in \mathbb{N} \cup \{0\}}$ and satisfies (7). By Lemma 7.1, the upper addition on M is associative since the upper addition $\oplus$ on the gap-point set $\{2l+1 \mid l \in \mathbb{N} \cup \{0, \infty\}\}$ given by $(2i+1) \oplus (2j+1) = 2(i+j+1)+1$ is associative. In fact, for all $i, j, k \in \mathbb{N} \cup \{0, \infty\}$, on the one hand, $\big((2i+1) \oplus (2j+1)\big) \oplus (2k+1) = 2(i+j+k+2)+1$ and on the other hand, $(2i+1) \oplus \big((2j+1) \oplus (2k+1)\big) = 2(i+j+k+2)+1$.

(ii) The set $M = \big(\bigcup_{l \in \{0,1,\ldots,n\}}[2l+1, 2l+2[\big) \cup \{\infty\}$, $n \in \mathbb{N} \cup \{0\}$ is an intervals' set of the second kind. By Lemma 7.1, the upper addition on M is associative since the upper addition $\oplus$ on the gap-point set $\{2l+1 \mid l \in \{0, 1, \ldots, n\} \cup \{\infty\}\}$ given by

$$(2i+1) \oplus (2j+1) = \begin{cases} 2(i+j+1)+1 & \text{if } 2(i+j+1)+1 \leq 2n+1. \\ \infty & \text{if } 2(i+j+1)+1 > 2n+1. \end{cases}$$

is associative.

Theorem 7.2. *Let $f : [0,1] \to [0,\infty]$ be an additive generator of the second kind of an associative operation F with the gap-point set $A \cup \{\infty\}$ of the range $R(f)$. Then, F can be restricted to $X = f^{-1}(A \cup \{\infty\})$, and this restriction $F/X^2 : X^2 \to X$ is a discrete associative operation upper additively generated by f/X.*

P r o o f. By Theorem 6.2, the strictly increasing bijection $f : [0,1] \to R(f)$ is an isomorphism of F and $\oplus$ where $\oplus$ is the upper addition on $R(f)$. Since F is associative, so is $\oplus$. By Lemma 7.1, the upper addition $\oplus$ can be restricted to $A \cup \{\infty\}$ and this restriction $\oplus/(A \cup \{\infty\})^2$, which is obviously associative, coincides with the upper addition on $A \cup \{\infty\}$. By Proposition 6.3, the operation F can be restricted to $X = f^{-1}(A \cup \{\infty\})$ and this restriction F/X^2 is upper additively generated by the restriction f/X of f to X. Since f/X is an isomorphism of F/X^2 and $\oplus/(A \cup \{\infty\})^2$ by Theorem 6.2, the operation F/X^2 is associative. $\square$

EXAMPLE. Let $L = \{0,1,\dots,n\}$, $n \in \mathbb{N}$ and let $f : [0,1] \to [0,\infty]$, $f(x) = n(x - \frac{l}{n}) + (2l+1)$ for all $x \in [\frac{l}{n}, \frac{l+1}{n}[$ and $l \in L \setminus \{n\}$, and $f(1) = \infty$. Observe that $R(f) = \big(\bigcup_{l \in L \setminus \{n\}} [2l+1, 2l+2[\big) \cup \{\infty\}$. Clearly, the function f is a strictly increasing additive generator of the second kind of F. The upper addition on $R(f)$ is associative, and by Theorem 6.2, so is the operation F additively generated by f. Write $X = f^{-1}\big(\{2l+1 \mid l \in L \setminus \{n\}\} \cup \{\infty\}\big) = \{\frac{l}{n} \mid l \in L\}$. The restriction f/X of f to X is a discrete upper additive generator of an associative operation $G = F/X^2$ by Theorem 7.2.

8. Extensions of discrete upper additive generators to additive generators of the second kind

We introduce conditions under which a discrete upper additive generator of a discrete associative operation can be extended to an additive generator of the second kind of an associative operation. The construction is based on constructing of an intervals' set of the second kind from a non-empty finite set $X \subseteq]0,\infty[$ of the certain properties.

Suppose that a set $X \subseteq [0,\infty[$ is in the form of

$$X = \{x_l \in [0,\infty[\mid l \in L\},\ x_l < x_{l+1} \text{ for all } l, l+1 \in L,$$

where $L = \{0,1,\dots,n\}$, $n \in \mathbb{N} \cup \{0\}$ or $L = \mathbb{N} \cup \{0\}$. Write

$$\delta(X) = \begin{cases} \inf P & \text{if } P \neq \emptyset, \\ \infty & \text{if } P = \emptyset, \end{cases} \tag{9}$$

where $P = \{x_l - x_{l-1} \mid l \in L \setminus \{0\}\}$, and

$$\varepsilon(X) = \begin{cases} \inf Q & \text{if } Q \neq \emptyset, \\ \infty & \text{if } Q = \emptyset, \end{cases} \tag{10}$$

where $Q = \{|x_i + x_j - x_k| \mid x_i + x_j - x_k \neq 0,\ i,j,k \in L\}$. Clearly, $\delta(X) \geq 0$ and $\varepsilon(X) \geq 0$. If a non-empty set X is finite, then $\delta(X) > 0$ and $\varepsilon(X) > 0$. It is obvious that $\delta(X) = \infty$ if and only if $X = \{x_0\}$, and $\varepsilon(X) = \infty$ if and only if

$X = \{0\}$. If a non-empty set X is finite and $X \neq \{x_0\}$, then $\delta(X) = \min P > 0$ and $\varepsilon(X) = \min Q > 0$.

Definition 8.1. A non-empty set $X \subseteq [0, \infty]$ is *anti-additive* if $x + y \notin X$ for all $x, y \in X$.

Lemma 8.2. *Let M be given by a sequence of intervals $\{[a_l, b_l[\}_{l\in L}$ and let satisfy (7). Then, the set $A = \{a_l \mid l \in L\}$ is anti-additive.*

Proof. Recall that $A \subseteq]0, \infty[$ by Lemma 5.2. The proof is by contradiction. Suppose that $a_i+a_j = a_k$ for some $i, j, k \in L$. Then, $([a_i, b_i[+[a_j, b_j[)\cap[a_k, b_k[\neq \emptyset$, and by (7), $\max\{a_i + b_j, b_i + a_j\} \leq a_k$ implying $a_i + a_j < a_k$ which is a contradiction. □

The next lemma is crucial for our construction.

Lemma 8.3. *Let $A = \{a_l \in]0, \infty[\mid l \in L\}$, $a_l < a_{l+1}$ for all $l, l + 1 \in L$ where $L = \{0, 1, \ldots, n\}$, $n \in \mathbb{N} \cup \{0\}$ or $L = \mathbb{N} \cup \{0\}$. Let $\delta = \delta(A)$ and $\varepsilon = \varepsilon(A)$ be given by (9) and (10), respectively. Suppose that A is anti-additive and $\delta > 0$ and $\varepsilon > 0$. Then, the set*

$$M = \left(\bigcup_{l\in L}[a_l, a_l + d_l[\right) \cup \{\infty\},$$

where $\{d_l\}_{l\in L}$ is a non-decreasing sequence of positive real numbers such that $d_l < \delta$ and $d_l \leq \varepsilon$ for all $l \in L$, is given by a sequence of intervals $\{[a_l, b_l[\}_{l\in L}$, satisfies (7), and $A \cup \{\infty\}$ is its gap-point set.

Proof. In order to show that M is given by a sequence of intervals $\{[a_l, b_l[\}_{l\in L}$, satisfies (7), and $A \cup \{\infty\}$ is its gap-point set, we should prove the following: $a_{i-1} + d_{i-1} < a_i$ for all $i \in L \setminus \{0\}$, and that for all $i, j, k \in L$, if $([a_i, a_i + d_i[+[a_j, a_j + d_j[) \cap [a_k, a_k + d_k[\neq \emptyset$, then $\max\{a_i + d_i + a_j, a_i + a_j + d_j\} \leq a_k$ and $a_i + d_i + a_j + d_j \leq a_k + d_k$.

For all $i \in L \setminus \{0\}$, from $0 < d_{i-1} < \delta \leq a_i - a_{i-1}$ we obtain $a_{i-1} + d_{i-1} < a_i$. Suppose that $([a_i, a_i + d_i[+[a_j, a_j + d_j[) \cap [a_k, a_k + d_k[\neq \emptyset$ for some $i, j, k \in L$ for the rest of the proof.

First, we will prove that $a_i + a_j < a_k$. Clearly, $a_i + a_j \neq a_k$ because A is anti-additive. If $a_k < a_i+a_j$, from $d_k \leq \varepsilon \leq |a_i+a_j-a_k| = a_i+a_j-a_k$, we would have $a_k+d_k \leq a_i+a_j$, and consequently, $([a_i, a_i+d_i[+[a_j, a_j+d_j[)\cap[a_k, a_k+d_k[= \emptyset$, which would be a contradiction. Thus, $a_i + a_j < a_k$, and consequently, $i < k$ and $j < k$.

We will prove that $\max\{a_i + d_i + a_j, a_i + a_j + d_j\} \leq a_k$. From $\max\{d_i, d_j\} \leq \varepsilon \leq |a_i + a_j - a_k| = a_k - (a_i + a_j)$, we obtain $\max\{a_i + d_i + a_j, a_i + a_j + d_j\} = a_i + a_j + \max\{d_i, d_j\} \leq a_k$.

Finally, we will prove that $a_i+d_i+a_j+d_j \leq a_k+d_k$. Since $i < k$ and $j < k$, it is $d_i \leq d_k$ and $d_j \leq d_k$. Thus, from $a_i+d_i+a_j \leq \max\{a_i+d_i+a_j, a_i+a_j+d_j\} \leq a_k$, it is $a_i + d_i + a_j + d_j \leq a_k + d_k$. $\square$

Example. Let $A = \big\{2l + 1 \mid l \in \mathbb{N} \cup \{0\}\big\}$. The set A is anti-additive and $\delta(A) = 2$ and $\varepsilon(A) = 1$. The upper addition on $A \cup \{\infty\}$ is associative.
(i) By Lemma 8.3, the set $M = (\bigcup_{l\in\mathbb{N}\cup\{0\}}[2l + 1, 2l + 2[) \cup \{\infty\}$ is given by a sequence of intervals, satisfies (7), and $A\cup\{\infty\}$ is its gap-point set, and moreover, by Lemma 7.1, the upper addition on M is associative.
(ii) By Lemma 8.3, the set $M = (\bigcup_{l\in\mathbb{N}\cup\{0\}}[2l+1, 2l+2-\frac{1}{2^{l+1}}[)\cup\{\infty\}$ is given by a sequence of intervals $\{[2l+1, 2l+2-\frac{1}{2^{l+1}}[\}_{l\in\mathbb{N}\cup\{0\}}$, satisfies (7), and $A\cup\{\infty\}$ is its gap-point set, and moreover, by Lemma 7.1, the upper addition on M is associative.

Corollary 8.4. *Let $A \subseteq]0, \infty[$ be a non-empty finite set. Then, the following two assertions are equivalent:*

(i) *There exists an intervals' set M of the second kind with the gap-point set $A \cup \{\infty\}$ such that the upper addition on M is associative.*
(ii) *The set A is anti-additive and the upper addition on $A\cup\{\infty\}$ is associative.*

Proof. ($\Rightarrow$) Suppose that (i) holds. Then, A is anti-additive by Lemma 8.2, and the upper addition on $A \cup \{\infty\}$ is associative by Lemma 7.1.
($\Leftarrow$) Suppose that (ii) holds. An existence of an intervals' set M of the second kind with the gap-point set $A \cup \{\infty\}$ follows from Lemma 8.3, and the associativity of the upper addition on M follows from Lemma 7.1. $\square$

Theorem 8.5. *Let $X \subseteq [0, 1]$ be a finite set and $0, 1 \in X$. If $g : X \to [0, \infty]$, $g(1) = \infty$ is a discrete upper additive generator of a discrete associative operation G with an anti-additive set $R(g) \setminus \{\infty\}$, then there exists an additive generator $f : [0, 1] \to [0, \infty]$ of the second kind of an associative operation F such that $R(f)$ has the gap-point set $R(g)$, $f/X = g$ and $F/X^2 = G$.*

Proof. Suppose that $g : X \to [0, \infty]$, $g(1) = \infty$ is a discrete upper additive generator of a discrete associative operation G with an anti-additive set $R(g)\setminus\{\infty\}$. Write $X = \{x_l \mid l \in L\}$, $L = \{0, 1, \dots, n\}$, $n \in N$, where $x_l < x_{l+1}$ for all $l, l + 1 \in L$.

First, we will describe the construction of f. As in Lemma 8.3, we construct an intervals' set $M = \big(\bigcup_{l\in L\setminus\{n\}}[a_l, a_l + d_l[\big) \cup \{\infty\}$ of the second kind with the gap-point set $A \cup \{\infty\} = R(g) = \{a_l \mid a_l = g(x_l), l \in L\}$ where $0 < d_0 \leq \cdots \leq d_{n-1} < \delta(A)$ and $d_{n-1} \leq \varepsilon(A)$ and $A = \big\{a_l \mid a_l = g(x_l), l \in L \setminus \{n\}\big\}$. Then, we choose an arbitrary strictly increasing bijection $f : [0, 1] \to M$ with $f(x_l) = a_l$ for all $l \in L$, for instance, $f(x) = \frac{d_l}{(x_{l+1}-x_l)}(x - x_l) + a_l$ for all $x \in [x_l, x_{l+1}[$ and $l \in L \setminus \{n\}$ and $f(1) = \infty$.

Now, we will prove that F is an associative operation. Since the strictly increasing bijection $g : X \to A \cup \{\infty\}$ upper additively generates an associative operation G, by Theorem 6.2, the upper addition on $A \cup \{\infty\}$ is associative, and by Lemma 7.1, so is the upper addition on M. Thus, by Theorem 6.2, the strictly increasing bijection $f : [0, 1] \to M$ additively generates an associative operation F.

Finally, we will show that $f/X = g$ and $F/X^2 = G$. By construction of f, it is obvious that $f/X = g$. Since the upper addition on M can be restricted to $A \cup \{\infty\}$ by Lemma 7.1, the operation F can be restricted to $X = f^{-1}(A \cup \{\infty\})$, and this restriction F/X^2 is upper additively generated by f/X by Proposition 6.3. Clearly, $F/X^2 = G$ since $f/X = g$. □

Example. Let $A = \{5, 7, 15, 19, 25, 27, 29\}$. First, we will show that the upper addition $\oplus$ on $A \cup \{\infty\}$ is associative. Since $(5 \oplus 5) \oplus 15 = \infty$ and $5 \oplus (5 \oplus 15) = \infty$, the monotonicity and the commutativity of $\oplus$ imply that for all $x, y, z \in A \cup \{\infty\}$, if $\max\{x, y, z\} \geq 15$, then $(x \oplus y) \oplus z = (x \oplus (y \oplus z) = \infty$. For instance, $(19 \oplus 7) \oplus 5 \geq (15 \oplus 5) \oplus 5 = 5 \oplus (15 \oplus 5) = 5 \oplus (5 \oplus 15) = \infty$, and so $(19 \oplus 7) \oplus 5 = \infty$. Since $(5 \oplus 5) \oplus 5 = 25$ and $(7 \oplus 7) \oplus 7 = 25$, the monotonicity and the commutativity of $\oplus$ imply that for all $x, y, z \in A \cup \{\infty\}$, if $\max\{x, y, z\} < 15$ then $(x \oplus y) \oplus z = x \oplus (y \oplus z) = 25$. Thus, $\oplus$ on $A \cup \{\infty\}$ is associative.

The set A is anti-additive, $\delta(A) = 2$ and $\varepsilon(A) = 1$. By Lemma 8.3, the set $M = [5, 6[\cup [7, 8[\cup [15, 16[\cup [19, 20[\cup [25, 26[\cup [27, 28[\cup [29, 30[\cup \{\infty\}$ is an intervals' set of the second kind and $A \cup \{\infty\}$ is its gap-point set. By Lemma 7.1, the upper addition on M is associative.

Let $X = \left\{ \frac{l}{7} \mid l \in \{0, 1, \ldots, 7\} \right\}$. By Theorem 6.2, Lemma 7.1, and Proposition 6.3, the strictly increasing bijection $g : X \to A \cup \{\infty\}$ is a discrete upper additive generator of an associative operation G, and an arbitrary strictly increasing bijection $f : [0, 1] \to M$ with $f/X = g$ is an additive generator of the second kind of an associative operation F. Moreover, $F/X^2 = G$.

Remark 6. It is a matter of straightforward verification to show that if $f : X \to [0, \infty]$ is an upper additive generator of an associative operation $F : X^2 \to X$, then the function $g : X \to [0, \infty]$ given by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X \setminus \{\min X\}, \\ 0 & \text{if } x = \min X, \end{cases}$$

is an upper additive generator of t-conorm $G : X^2 \to X$ such that

$$G(x, y) = \begin{cases} F(x, y) & \text{if } (x, y) \in (X \setminus \{\min X\})^2, \\ \max\{x, y\} & \text{if } \min\{x, y\} = \min X. \end{cases}$$

With respect to Remark 6 from every upper additive generator of an associative operation, we can easily construct an upper additive generator of a t-conorm.

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