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PRODUCT OF MEASURABLE SPACES AND APPLICATIONS

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ABSTRACT. We deal with products of measurable spaces and relationships between measures on products and asymmetrical stochastic dependence/independence of one extended probability space on another one.

1. Introduction

At previous conferences on Real Functions held in (Stará Lesná 2016, 2018, Ustka 2017) we have presented our results related to the transition from classical probability (cf [16]) to its fuzzification (cf. [2], [3]). The classical probability space $(\Omega, \mathbf{A}, p)$ is extended to $(\Omega, \mathcal{M}(\mathbf{A}), \int(.)\,\mathrm{d}p)$, where the family of classical events $\mathbf{A}$ (a σ-field of subsets of Ω, we assume that $\{\omega\} \in \mathbf{A}$ for all $\omega \in \Omega$ and if $\omega', \omega \in \Omega$, $\omega \neq \omega'$, then there exists $A \in \mathbf{A}$ such that $\omega \in A$ and $\omega' \in (\Omega \setminus A)$) is extended to the family of measurable fuzzy events $\mathcal{M}(\mathbf{A})$ ($\mathbf{A}$-measurable $[0,1]$-valued functions, equipped with the pointwise order, convergence of sequences, algebraic operations, and Łukasiewicz logic), and the probability measure p on $\mathbf{A}$ is extended to the probability integral $\int(.)\,\mathrm{d}p$ on $\mathcal{M}(\mathbf{A})$ (*it will be condensed to* $\overline{p}$). Fundamental role is played by observables and their duals, called statistical maps. Let $(\Omega, \mathbf{A})$ and $(\Xi, \mathbf{B})$ be measurable

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spaces and let $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$ be the corresponding fuzzy events. Recall that $g\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is said to be an *observable* from $\mathcal{M}(\mathbf{B})$ to $\mathcal{M}(\mathbf{A})$ whenever:

(i) If $u \in \mathcal{M}(\mathbf{B})$ is the top, respectively, bottom, element, then $g(u)$ is the top, respectively, bottom, element in $\mathcal{M}(\mathbf{A})$;

(ii) if $u, v \in \mathcal{M}(\mathbf{B})$ and $u \leq v$, then $g(u) \leq g(v)$ and $g(v-u) = g(v) - g(u)$;

(iii) g is sequentially continuous (with respect to the point-wise convergence of sequences of functions).

Moreover, if $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ and $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ are two extended spaces and

(iv) for each $u \in \mathcal{M}(\mathbf{B})$, we have $\int u \, \mathrm{d}q = \int g(u) \, \mathrm{d}p$;

then the observable g yields a *stochastic channel* from $(\Omega, \mathcal{M}(\mathbf{A}), \overline{p})$ to $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$: for each $u \in \mathcal{M}(\mathbf{B})$, to get the stochastic information $\overline{q}(u)$, we send u to $g(u) \in \mathcal{M}\big(\mathbf{A}\big)$ and then, $\overline{q}(u) = \overline{p}\big(g(u)\big)$, i.e., $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ serves as a "stochastic source" for $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$. Further, if $\mathbf{A}$ is the trivial algebra $\mathbf{T} = \big\{\emptyset, \{\omega\}\big\}$, then we can identify the interval [0,1] and $\mathcal{M}(\mathbf{T})$ and, consequently, $\overline{q}$ *becomes an observable into* $\mathcal{M}(\mathbf{T})$.

Finally, each observable $g\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ defines a *statistical map* T_g on the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B})$ of all probability measures on $\mathbf{B}$: put $T_g(t) = s$ whenever $\overline{t} \circ g = \overline{s}$. Note that statistical maps generalize measurable maps (we identify points $\omega \in \Omega$ and Dirac measures δ_ω) and observables generalize preimage maps of measurable maps in the sense that if $f\colon\ \Omega \longrightarrow \Xi$ is a measurable map, then the preimage map $f^{\leftarrow}\colon\ \mathbf{B} \longrightarrow \mathbf{A}$, $f^{\leftarrow}(B) = \{\omega \in \Omega;\ f(\omega) \in B\}$, is a sequentially continuous Boolean homomorphism (we identify a set A and its indicator function χ_A and the convergence of sequences of sets amounts to the pointwise convergence of the corresponding sequences of indicator functions) (cf. [7], [13], [5]).

More information on generalized (fuzzified) probability theory can be found in: [26], [18], [15], [4], [6], [9], [8], [11], [10], [12], [14], [13], [19], [20], [21], [22], [23], [25], [24].

2. Probability measures on products

In this section, we collect some folkloristic facts about probability measures on the product of two measurable spaces.

Let $\big(\Omega \times \Xi, \mathbf{A} \times \mathbf{B}\big)$ be the usual product of measurable spaces $(\Omega, \mathbf{A})$ and $(\Xi, \mathbf{B})$, let $pr_1 : \Omega \times \Xi \longrightarrow \Omega$, $pr_2 : \Omega \times \Xi \longrightarrow \Xi$ be the usual projections $\big(pr_1(\omega, \xi) = \omega,\ pr_2(\omega, \xi) = \xi,\ \omega \in \Omega,\ \xi \in \Xi\big)$, let $pr_1^{\leftarrow}\colon\ \mathbf{A} \longrightarrow \mathbf{A} \times \mathbf{B}$ and $pr_2^{\leftarrow}\colon\ \mathbf{B} \longrightarrow \mathbf{A} \times \mathbf{B}$ be the corresponding measurable preimage maps. Let $A \in \mathbf{A}$, $B \in \mathbf{B}$. Then, $pr_1^{\leftarrow}(A) = A \times \Xi$, $pr_2^{\leftarrow}(B) = \Omega \times B$ and it is known that pr_1 and pr_2 are sequentially continuous Boolean homomorphisms.

Further, let $(\Lambda, \mathbf{C})$ be a measurable space and let $f\colon\ \Lambda \longrightarrow \Omega$, $g\colon\ \Lambda \longrightarrow \Xi$ be measurable maps. Then, *there is a unique measurable map* $h\colon\ \Lambda \longrightarrow \Omega \times \Xi$ such that $pr_1 \circ h = f$ and $pr_2 \circ h = g$, namely $h(\lambda) = \big(f(\lambda), g(\lambda)\big), \lambda \in \Lambda$. Indeed, $(\Omega \times \Xi, \mathbf{A} \times \mathbf{B})$, *along with the projections* pr_1 and pr_2, is the *categorical product* of measurable spaces $(\Omega, \mathbf{A})$ and $(\Xi, \mathbf{B})$. (Recall, see e.g. [1], that an object $\mathcal{O}_1 \times \mathcal{O}_2$, along with projections $\pi_i\colon\ \mathcal{O}_1 \times \mathcal{O}_2 \longrightarrow \mathcal{O}_i, i = 1, 2$, is the product of objects $\mathcal{O}_1$ and $\mathcal{O}_2$ whenever for each object $\mathcal{O}$ and each pair of morphisms $f_i\colon\ \mathcal{O} \longrightarrow \mathcal{O}_i, i = 1, 2$, there exists a unique morphism $f\colon\ \mathcal{O} \longrightarrow \mathcal{O}_1 \times \mathcal{O}_2$ such that $f_i = \pi_i \circ f, i = 1, 2$; all objects and morphisms belong to a given category.) Denote $h = f \otimes g$. Clearly, $(f \otimes g)^{\leftarrow} \circ pr_1^{\leftarrow} = f^{\leftarrow}$ and $(f \otimes g)^{\leftarrow} \circ pr_2^{\leftarrow} = g^{\leftarrow}$.

It is known that the preimage map of a measurable map composed with a probability measure is a probability measure. Consider the following commutative diagram (Figure 1), where s is a probability measure on $\mathbf{C}$.

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "pr_1^{\leftarrow}"] \arrow[dr, "f^{\leftarrow}"'] & \mathbf{A}\times\mathbf{B} \arrow[d, dashed, "\exists!"', "h^{\leftarrow}"] & \mathbf{B} \arrow[l, "pr_2^{\leftarrow}"'] \arrow[dl, "g^{\leftarrow}"] \\
& \mathbf{C} \arrow[d, "s"] & \\
& {[0,1]} &
\end{tikzcd}$$

FIGURE 1.

Then, s uniquely defines three probability measures: $p = s \circ f^{\leftarrow}$ on $\mathbf{A}$, $r = s \circ (f \otimes g)^{\leftarrow}$ on $\mathbf{A} \times \mathbf{B}$, and $q = s \circ g^{\leftarrow}$ on $\mathbf{B}$. Moreover, $p = r \circ pr_1^{\leftarrow}$ and $q = r \circ pr_2^{\leftarrow}$. Accordingly, we get another commutative diagram (Figure 2), where for $m \in \mathcal{P}(\mathbf{C})$ we define $T_{f^{\leftarrow}}(m) = m \circ f^{\leftarrow}$, $T_{(f\otimes g)^{\leftarrow}}(m) = m \circ (f \otimes g)^{\leftarrow}$ $T_{g^{\leftarrow}}(m) = m \circ g^{\leftarrow}$, and for $r \in \mathcal{P}(\mathbf{A} \times \mathbf{B})$ we define $L_1(r) = r \circ pr_1^{\leftarrow}$, i.e., $\big(L_1(r)\big)(A) = r(A \times \Xi)$, $A \in \mathbf{A}$, resp. $L_2(r) = r \circ pr_2^{\leftarrow}$, i.e., $\big(L_2(r)\big)(B) = r(\Omega \times B)$, $B \in \mathbf{B}$. The resulting maps $L_1\colon\ \mathcal{P}(\mathbf{A} \times \mathbf{B}) \longrightarrow \mathcal{P}(\mathbf{A})$ and $L_2\colon\ \mathcal{P}(\mathbf{A} \times \mathbf{B}) \longrightarrow \mathcal{P}(\mathbf{B})$ are called *lateral projections.*

Observe that the probability space $(\Lambda, \mathbf{C}, s)$ can serve, via the observables $f^{\leftarrow}$ and $g^{\leftarrow}$, as a source of stochastic information for both $(\Omega, \mathbf{A}, p)$ and $(\Xi, \mathbf{B}, q)$ and, in a broader context, this leads to the notion of a joint probability space.

DEFINITION 2.1. Let $(\Omega, \mathbf{A}, p)$ and $(\Xi, \mathbf{B}, q)$ be probability spaces. If $L_1(r) = p$ and $L_2(r) = q$, then $(\Omega \times \Xi, \mathbf{A} \times \mathbf{B}, r)$ is said to be a *joint probability space of* $(\Omega, \mathbf{A}, p)$ *and* $(\Xi, \mathbf{B}, q)$.

Indeed, if $(\Lambda, \mathbf{C}, m)$ serves as a source of stochastic information for both $(\Omega, \mathbf{A}, p)$ and $(\Xi, \mathbf{B}, q)$ in the sense that $m \circ f^{\leftarrow} = p$ and $m \circ g^{\leftarrow} = q$, then $\big(\Omega \times \Xi, \mathbf{A} \times \mathbf{B}, m \circ (f \otimes g)^{\leftarrow}\big)$, along with the projections pr_1 and pr_2, does the same. Denote $\mathcal{J}(p,q) = \{r \in \mathcal{P}(\mathbf{A} \times \mathbf{B});\ L_1(r) = p, L_2(r) = q\}$. Consequently, whenever we need a common source of stochastic information for $(\Omega, \mathbf{A}, p)$ and $(\Xi, \mathbf{B}, q)$, it suffices to consider joint probability spaces, i.e., probability spaces of the form $(\Omega \times \Xi, \mathbf{A} \times \mathbf{B}, r)$, $r \in \mathcal{J}(p,q)$. As we shall see, joint probability spaces play a key role in asymmetrical stochastic dependence/independence for extended probability spaces. Observe that $p \times q \in \mathcal{J}(p,q)$ and, in general, $\mathcal{J}(p,q) \neq \{p \times q\}$.

Lemma 2.1. *Let $r \in \mathcal{P}(\mathbf{A} \times \mathbf{B})$.*

(i) *Let $L_1(r) = \delta_\omega$, $\omega \in \Omega$, and $L_2(r) = q$. Then $r = \delta_\omega \times q$.*

(ii) *Let $L_2(r) = \delta_\xi$, $\xi \in \Xi$, and $L_1(r) = p$. Then $r = p \times \delta_\xi$.*

Proof. (i) Let $A \in \mathbf{A}$ and $B \in \mathbf{B}$. We have to prove that $r(A \times B) = \delta_\omega(A).q(B)$. From $\{\omega\} \in \mathbf{A}$ and $r\big((\Omega \setminus \{\omega\}) \times \Xi\big) = \delta_\omega(\Omega \setminus \{\omega\}) = 0$ we get $q(B) = r(\Omega \times B) = r(\{\omega\} \times B)$. Consequently, $r(A \times B) = 0 = \delta_\omega(A).q(B)$ for $\omega \in (\Omega \setminus A)$ and $r(A \times B) = r(\Omega \times B) = q(B) = \delta_\omega(A).q(B)$ for $\omega \in A$. This proves (i). The proof of (ii) is analogous. $\square$

Since $\mathcal{J}(p_1,q_1) \cap \mathcal{J}(p_2,q_2) = \emptyset$ whenever $(p_1,q_1) \neq (p_2,q_2)$, we get a partition of $\mathcal{P}(\mathbf{A} \times \mathbf{B})$ indexed by $\mathcal{P}(\mathbf{A}) \times \mathcal{P}(\mathbf{B})$ and hence an equivalence relation: $r \sim r'$ if and only if $L_1(r) = L_1(r')$ and $L_2(r) = L_2(r')$. Clearly, $\mathcal{J}(p,q) = \{r \in \mathcal{P}\big(\mathbf{A} \times \mathbf{B}\big);\ r \sim p \times q\}$. This leads to the following visualization of $\mathcal{P}(\mathbf{A} \times \mathbf{B})$. We identify Ω and $\delta_\Omega = \{\delta_\omega;\ \omega \in \Omega\}$, Ξ and $\delta_\Xi = \{\delta_\xi;\ \xi \in \Xi\}$, form a "base plane" $\mathcal{P}(\mathbf{A}) \times \mathcal{P}(\mathbf{B})$, and identify each pair $(p,q) \in \mathcal{P}(\mathbf{A} \times \mathbf{B})$ and the product measure $p \times q$. Then, $\mathcal{P}(\mathbf{A} \times \mathbf{B})$ can be visualized as the union of all $\mathcal{J}(p,q)$ indexed by the "base plane" consisting of all product measures.

In the next sections, we show that each observable g from $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ to $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ "picks" a probability measure $r_p \in \mathcal{J}(p,q)$ and this leads to an asymmetrical dependence/independence of $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ on $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$. Of course, stochastic independence implies $r_p = p \times q$.

3. Application to stochastic dependence/independence

This section is devoted to asymmetrical dependence/independence of an extended probability space $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ on an extended probability space $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$, introduced in [3].

Let $g\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be an observable such that $\overline{q} = \overline{p} \circ g$ and consider the resulting stochastic channel from $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ to $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$. Then, for each $u \in \mathcal{M}(\mathbf{B})$, the stochastic information $\overline{q}(u) = \overline{p}\big(g(u)\big)$ about u is determined by $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$. Consequently, in some sense, g establishes a stochastic dependence of $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ on $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$. Our goal is to discuss how this asymmetrical stochastic dependence is related to probability measures on $\mathbf{A}\times\mathbf{B}$.

Let $(\Omega, \mathbf{A}, p)$ be a classical probability space. Then, extended probability space $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ is said to be the fuzzification of $(\Omega, \mathbf{A}, p)$. Let $(\Omega\times\Xi, \mathbf{A}\times\mathbf{B}, r)$ be a classical joint probability space of $(\Omega, \mathbf{A}, p)$ and $(\Xi, \mathbf{B}, q)$. Then, $\big(\Omega\times\Xi, \mathcal{M}(\mathbf{A}\times\mathbf{B}), \overline{r}\big)$ is said to be a joint extended probability space of $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ and $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$.

Important assertions, on which our discussion is based, can be reformulated as Theorem 1 (cf. in [3, Proposition 2.8, Proposition 2.10, and Corollary 2.11]), see Fig. 2 where the canonical embeddings (constant prolongations) $e_1\colon\ \mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{A}\times\mathbf{B})$ and $e_2\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A}\times\mathbf{B})$ defined by $\tilde{v}(\omega,\xi) = (e_1(v))(\omega,\xi) = v(\omega)$, $\omega \in \Omega, \xi \in \Xi$, $v \in \mathcal{M}(\mathbf{A})$, resp. $\tilde{u}(\omega,\xi) = \big(e_2(u)\big)(\omega,\xi) = u(\xi)$, $\omega \in \Omega, \xi \in \Xi$, $u \in \mathcal{M}(\mathbf{B})$, are observables and the unique extensions of coprojections $pr_1^{\leftarrow}$ and $pr_2^{\leftarrow}$, respectively. If g is a map of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ such that, for each $u \in \mathcal{M}(\mathbf{B})$, $g(u)$ is a constant function on Ω the value of which is $\overline{q}(u)$, then g is an observable from $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ to $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ and it is said to be *degenerated*. Clearly, for all $s \in \mathcal{P}(\mathbf{A})$ we have $T_g(s) = q$ (cf. [3]). Note that a classical degenerated random variable becomes a special case (it maps all classical outcomes $\omega \in \Omega$ to the same real number and the preimage map maps every real event either to $\emptyset$ or to Ω, and maps each probability measure on the sample events to the same Dirac measure on the real Borel measurable sets).

Theorem 1. *Let* $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ *and* $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ *be extended probability spaces. Let* $\mathrm{id}\colon\ \mathcal{M}(\mathbf{A}) \longrightarrow \mathcal{M}(\mathbf{A})$ *be the identity observable, let* $g\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ *be an observable, and let* $T_{\mathrm{id}}\colon\ \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{A})$ *and* $T_g\colon\ \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{B})$ *be the corresponding statistical maps.*

(i) *Then, there exists a unique observable* $h\colon\ \mathcal{M}\big(\mathbf{A}\times\mathbf{B}\big) \longrightarrow \mathcal{M}(\mathbf{A})$ *such that* $h \circ e_1 = \mathrm{id}$ *and* $h \circ e_2 = g$, *where* h *is equal to the product* $\mathrm{id}\otimes g$ *of observables* id *and* g *defined as follows*

$$\big((\mathrm{id}\otimes g)(u)\big)(\omega) = \int u \,\mathrm{d}\big(T_{\mathrm{id}}(\delta_\omega)\times T_g(\delta_\omega)\big), \quad \omega\in\Omega, \quad u\in\mathcal{M}(\mathbf{A}\times\mathbf{B}); \qquad (\otimes)$$

(ii) *Let* $T_h\colon\ \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{A}\times\mathbf{B})$ *be the statistical map defined by* h. *Then,*

$$T_h(\delta_\omega) = T_{\mathrm{id}\otimes g}(\delta_\omega) = \delta_\omega \times T_g(\delta_\omega) \quad \textit{and} \quad L_1 \circ T_h = T_{\mathrm{id}}, \quad L_2 \circ T_h = T_g;$$

(iii) *If* (g, T_g) *is a degenerated stochastic channel from* $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ *to* $(\Xi, \mathcal{M}(\mathbf{B}), \overline{q})$, *i.e.,* $T_g(t) = q$ *for all* $t \in \mathcal{P}(\mathbf{A})$, *then* $T_h(t) = t\times q$ *for all* $t \in \mathcal{P}(\mathbf{A})$ *and, in particular,* $T_h(p) = p\times q$.

$$
\begin{tikzcd}
\mathcal{M}(\mathbf{A}) \arrow[r, "e_1"] \arrow[dr, "\mathrm{id}"'] & \mathcal{M}(\mathbf{A}\times\mathbf{B}) \arrow[d, dashed, "\exists!\, h"] & \mathcal{M}(\mathbf{B}) \arrow[l, "e_2"'] \arrow[dl, "g"] \\
& \mathcal{M}(\mathbf{A}) \arrow[d, "\overline{p}"] & \\
& \mathcal{M}(\mathbf{T}) &
\end{tikzcd}
$$

FIGURE 2.

A joint extended probability space $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r}\big)$, $r \in \mathcal{J}(p,q) \subseteq \mathcal{P}(\mathbf{A} \times \mathbf{B})$, is characterized by the requirement that it contains all stochastic information about its constituents transmitted via the lateral stochastic channels (e_1, L_1) and (e_2, L_2), respectively. Let $g\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be an observable such that $\overline{q} = \overline{p} \circ g$. From Theorem 1 it follows that there exists a unique joint extended probability space $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p}\big)$, $\overline{r_p} = \overline{p} \circ (\mathrm{id} \otimes g)$, which is "the best" of all joint extended probability spaces taking into account g. It is determined by the observable $h\colon\ \mathcal{M}(\mathbf{A} \times \mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$, $h = \mathrm{id} \otimes g$, and h is the unique observable satisfying two conditions: $h \circ e_1 = \mathrm{id}$ and $h \circ e_2 = g$. The first condition guarantees that, for each $v \in \mathcal{M}(\mathbf{A})$, h does not distort the stochastic information about $e_1(v) \in \mathcal{M}(\mathbf{A} \times \mathbf{B})$ (the same as the stochastic information about $v = \mathrm{id}(v)$). The second condition, factorization of g (cf. [5]), guarantees that, for each $u \in \mathcal{M}(\mathbf{B})$, h transmits the same stochastic information about $e_2(u)$ (the same stochastic information as about u) as g transmits about u: for each $u \in \mathcal{M}(\mathbf{B})$, $g(u) \in \mathcal{M}(\mathbf{A})$ provides stochastic information about u, $e_2(u) \in \mathcal{M}(\mathbf{A} \times \mathbf{B})$ provides stochastic information about u, and $h\big(e_2(u)\big) \in \mathcal{M}(\mathbf{A})$ provides stochastic information about u. For $h = \mathrm{id} \otimes g$ we have $g(u) = h\big(e_2(u)\big)$.

DEFINITION 3.1. Let (g, T_g) be a stochastic channel from $(\Omega, \mathcal{M}(\mathbf{A}), \overline{p})$ to $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$. Then, $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p}\big)$, $\overline{r_p} = \overline{p} \circ (\mathrm{id} \otimes g)$, is said to be the g-joint extended probability space of $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ and $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$.

Intuitively, the g-joint extended probability space is "the best" joint experiment which reflects the stochastic information transmitted via (g, T_g) from $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ to $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$. Namely, if (g, T_g) is degenerated, then $r_p = p \times q$. Further, $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ "looks from one side (via $\mathrm{id} \circ g$) like the g-joint extended probability space" and it "looks like $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ from the other side (via g)". Observe that (e_1, L_1), (e_2, L_2), (g, T_g), $(\mathrm{id}, T_{\mathrm{id}})$, and $(\mathrm{id} \otimes g, T_{\mathrm{id} \otimes g})$ are stochastic channels transmitting stochastic information between two of the three involved extended probability spaces. Factorizations $g = (\mathrm{id} \circ g) \circ e_2$ and $\mathrm{id} = (\mathrm{id} \circ g) \circ e_1$ guarantee that the experiment $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ provides all stochastic information about $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ and the g-joint extended probability space $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p}br)$, which uniquely models the relationship $g\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$.

If g of the g-joint model is degenerated, i.e., $T_g(s) = q$ for all $s \in \mathcal{P}(\mathbf{A})$, then the stochastic information about the events in $\mathcal{M}(\mathbf{B})$, transmitted via the degenerated stochastic channel (g, T_g), is the same independently of the choice of $s \in \mathcal{P}(\mathbf{A})$. Further, from (iii) in Theorem 1 it follows that $r_p = T_h(p) = p \times q$, hence the following definition is natural.

Definition 3.2. Let (g, T_g) be a stochastic channel from an extended probability space $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ to an extended probability space $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ and let $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p}\big)$ be their g-joint extended probability space. If (g, T_g) is degenerated, then $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ is said to be *stochastically independently joined to* $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$.

Indeed, the definition is consistent with the classical symmetrical stochastic independence of $(\Omega, \mathbf{A}, p)$ and $(\Xi, \mathbf{B}, q)$ in the following sense. We embed the two probability spaces into their joint probability space $\big(\Omega \times \Xi, \mathbf{A} \times \mathbf{B}, \overline{r_p}\big)$ and, since $r_p = p \times q$, the two corresponding fields of events $\mathbf{A} \times \Xi = \{A \times \Xi;\ A \in \mathbf{A}\}$ and $\Omega \times \mathbf{B} = \{\Omega \times B;\ B \in \mathbf{B}\}$ are stochastically independent in the joint probability space.

Moreover, let $\xi \in \Xi$ be a classical outcome and let $B \in \mathbf{B}$ be an event. Then, $\xi \in B$, resp. $\xi \in (\Xi \setminus B)$, can be interpreted as "ξ supports B", resp. "ξ supports $\Xi \setminus B$". Since g is degenerated, $g(\chi_B) \in \mathcal{M}(\mathbf{A})$ is a constant function. This can be interpreted as "each classical outcome $\omega \in \Omega$ supports $g(\chi_B)$ with the same logical strength $(g(\chi_B))(\omega) = \overline{q}(\chi_B) = q(B)$" and, consequently, as "the occurrence of ξ is not influenced by the occurrence of a particular outcome $\omega \in \Omega$". In other words, in the broader context of $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p}\big)$, "*the classical outcome $\xi \in \Xi$ is independent on the classical outcomes $\omega \in \Omega$*". Remember, this is exactly how the stochastic independence of one classical random experiment on another classical random experiment is intuitively understood.

In quantum physics, it is natural to assume that to a classical outcome $\omega \in \Omega$ there corresponds not a classical outcome $\xi \in \Xi$, but a probability measure on the outcomes in Ξ. Hence, it is natural to generalize a random variable to a statistical map mapping the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B})$ of all probability measures on $\mathbf{B}$. Accordingly, it is natural to consider probability measures as *extended outcomes* of an extended probability space. Once more, for each extended random event $u \in \mathcal{M}(\mathbf{B})$, "an extended outcome $s \in \mathcal{P}(\mathbf{B})$ supports u with logical strength $\overline{s}(u)$" and, if g is degenerated, then to $u \in \mathcal{M}(\mathbf{B})$ there corresponds a constant function $g(u) \in \mathcal{M}(\mathbf{A})$

and "each extended outcome $t \in \mathcal{P}(\mathbf{A})$ supports $g(u)$ with the same logical strength $\overline{t}\big(g(u)\big) = \overline{q}(u)$". Hence, "each extended outcome $s \in \mathcal{P}(\mathbf{B})$ is independent on the extended outcomes $t \in \mathcal{P}(\mathbf{A})$".

To sum up, an extended probability space $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ is stochastically independently joined to an extended probability space $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ exactly when "*the extended outcomes of the former space are independent on the extended outcomes of the latter space*".

4. Application to conditional probability

Let (g, T_g) be a stochastic channel from an extended probability space $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ to an extended probability space $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ and let $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p}\big)$ be their g-joint extended probability space. In this section, we show how g and, consequently $r_p \in \mathcal{P}(\mathbf{A} \times \mathbf{B})$, capture the asymmetrical dependence of $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$ on $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$.

Let $(\Lambda, \mathbf{C}, P)$ be a classical probability space, let $\mathbf{D}$ be a σ-field contained in $\mathbf{C}$, and let $P_{\mathbf{D}}$ be the restriction of P to $\mathbf{D}$. Let $\mathcal{E}$ be the family of all integrable $\mathbf{C}$-measurable functions. Clearly, $\mathcal{M}(\mathbf{C}) \subset \mathcal{E}$. Then (cf. [17]) for each $w \in \mathcal{E}$ there exists a $\mathbf{D}$-measurable function $E^{\mathbf{D}} w$, defined up to $P_{\mathbf{D}}$-equivalence by

$$\int_D \big(E^{\mathbf{D}} w\big) \, \mathrm{d}P_{\mathbf{D}} = \int_D w \, \mathrm{d}P, \quad D \in \mathbf{D}. \tag{$CE\mathbf{D}$}$$

The $\mathbf{D}$-measurable function $E^{\mathbf{D}} w$ is called the *conditional expectation of* w *given* $\mathbf{D}$. The restriction of $E^{\mathbf{D}}$ to indicator functions $\chi_C, C \in \mathbf{C}$, is called *conditional probability given* $\mathbf{D}$ and denoted $P^{\mathbf{D}} C = E^{\mathbf{D}}_{\chi_C}$.

We shall deal with a special case, where $\Lambda = \Omega \times \Xi$, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, $\mathbf{D} = \mathbf{A} \times \{\emptyset, \Xi\}$, $P = r_p$, and $w = \tilde{u} = e_2(u)$, $u \in \mathcal{M}(\mathbf{B})$. Our goal is to describe the relationship between an observable $g\colon\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ and the conditional expectation $E^{\mathbf{D}}$.

Proposition 4.1. *Let $\big(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p}\big)$ be the g-joint extended probability space of $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ and $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$. For $\Lambda = \Omega \times \Xi$, $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, $\mathbf{D} = \mathbf{A} \times \{\emptyset, \Xi\}$, $P = r_p$, let $E^{\mathbf{D}}$ be the corresponding conditional expectation given $\mathbf{D}$. Then (up to $P_{\mathbf{D}}$-equivalence)*

$$E^{\mathbf{D}} \tilde{u} = \widetilde{g(u)} = e_1\big(g(u)\big), \quad \tilde{u} = e_2(u), \quad u \in \mathcal{M}(\mathbf{B}).$$

P r o o f. Let $u \in \mathcal{M}(\mathbf{B})$. Since

$$\int\limits_{A\times\Xi} \left(E^{\mathbf{D}}\tilde{u}\right)\,\mathrm{d}P_{\mathbf{D}} = \int\limits_{A\times\Xi} \tilde{u}\,\mathrm{d}P,\ A \in \mathbf{A},$$

it suffices to prove that for each $A \in \mathbf{A}$ we have

$$\int\limits_{A\times\Xi} \tilde{u}\,\mathrm{d}r_p = \int\limits_{A\times\Xi} \widetilde{g(u)}\,\mathrm{d}P_{\mathbf{D}}.$$

Since $\overline{r_p} = \overline{p} \circ (\mathrm{id} \otimes g)$, we get

$$\int\limits_{A\times\Xi} \tilde{u}\,\mathrm{d}r_p = \int \chi_{A\times\Xi}.\tilde{u}\,\mathrm{d}r_p = \int (\mathrm{id} \circ g)(\chi_{A\times\Xi}.\tilde{u})\,\mathrm{d}p = \int (\mathrm{id} \circ g)(\chi_A.u)\,\mathrm{d}p.$$

Using ($\otimes$), we get

$$\big((\mathrm{id} \otimes g)(\chi_A.u)\big)(\omega) = \int \chi_A.u\,\mathrm{d}\big(\delta_\omega \times T_g(\delta_\omega)\big) = \int \chi_A\,\mathrm{d}(\delta_\omega) \int u\,\mathrm{d}\big(T_g(\delta_\omega)\big) =$$

$$\chi_A(\omega) . \int g(u)\,\mathrm{d}(\delta_\omega) = \chi_A(\omega)\Big(\big(g(u)\big)(\omega)\Big) = \big(\chi_A.g(u)\big)(\omega), \quad \omega \in \Omega,$$

and hence

$$\int\limits_{A\times\Xi} \tilde{u}\,\mathrm{d}r_p = \int \chi_A.g(u)\,\mathrm{d}p = \int\limits_{A\times\Xi} \widetilde{g(u)}\,\mathrm{d}P_{\mathbf{D}}.$$

Thus, $\widetilde{g(u)} = E^{\mathbf{D}}\tilde{u}$ (up to $P_{\mathbf{D}}$-equivalence). □

Since $\tilde{u}$ and $\widetilde{g(u)}$ are canonical embeddings of the events $u \in \mathcal{M}(\mathbf{B})$ and $g(u) \in \mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{A} \times \mathbf{B})$, the next definition is quite natural.

Definition 4.1. Let $(\Omega \times \Xi, \mathcal{M}(\mathbf{A} \times \mathbf{B}), \overline{r_p})$ be the g-joint experiment of $\big(\Omega, \mathcal{M}(\mathbf{A}), \overline{p}\big)$ and $\big(\Xi, \mathcal{M}(\mathbf{B}), \overline{q}\big)$. Then, the observable $g:\ \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is said to be the *conditional probability on* $\mathcal{M}(\mathbf{B})$ *given* $\mathcal{M}(\mathbf{A})$.

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