sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2020-0024
Tatra Mt. Math. Publ. **76** (2020), 127–142

SOLUTION OF THE FRACTIONAL BRATU-TYPE EQUATION VIA FRACTIONAL RESIDUAL POWER SERIES METHOD

Ali Khalouta — Abdelouahab Kadem

Ferhat Abbas Sétif University 1, 19000 Sétif, ALGERIA

ABSTRACT. In this paper, we present numerical solution for the fractional Bratu-type equation via fractional residual power series method (FRPSM). The fractional derivatives are described in Caputo sense. The main advantage of the FRPSM in comparison with the existing methods is that the method solves the nonlinear problems without using linearization, discretization, perturbation or any other restriction. Three numerical examples are given and the results are numerically and graphically compared with the exact solutions. The solutions obtained by the proposed method are in complete agreement with the solutions available in the literature. The results reveal that the FRPSM is a very effective, simple and efficient technique to handle a wide range of fractional differential equations.

1. Introduction

The fractional calculus which deals with derivatives and integrals of arbitrary orders plays a vital role in many fields of pure and applied mathematics [6, 14, 15, 17, 19]. In recent years, fractional differential equations are widely used in interpretation and modelling of many real matters and appear in applied mathematics and physics including fluid mechanics, diffusion processes, aerodynamics, electrodynamics, electrostatics, electrochemistry, control theory, mathematical biology, and so on. Since the exact solutions of fractional differential equations do not exist, many researchers focus on approximate solutions of this type of equations.

2010 Mathematics Subject Classification: Primary 34A08, 26A33. Secondary 34K28, 35C10.

Keywords: fractional Bratu-type equation, Caputo fractional derivative, fractional residual power series method.

There are several numerical methods given for solving fractional differential equations. The most used ones are: Adomian decomposition method (ADM) [8], variational iteration method (VIM) [16], new iterative method (NIM) [13], differential transform method (DTM) [5], homotopy perturbation method (HPM) [1] and homotopy analysis method (HAM) [11].

The main objective of this paper is to apply the FRPSM to study and construct an approximate solution to the fractional Bratu equation in the form

$$D^{2\alpha}u(x) + \lambda \exp\big(u(x)\big) = 0, \quad 0 < x < 1, \quad \lambda \in \mathbb{R}, \tag{1}$$

with the initial conditions

$$u(0) = u_0, D^{\alpha}u(0) = u_1, \tag{2}$$

where $D^{2\alpha} = D^{\alpha}D^{\alpha}$ is the Caputo fractional derivative operator of order 2α, $1/2 < \alpha \leq 1$. When $\alpha = 1$, the equation (1) reduces to the classical Bratu equation.

The Bratu equation was used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [7, 12, 20, 21].

The exact solution for the classical Bratu equation is given by [2, 3] as

$$u(x) = -2\ln\left[\frac{\cosh\left(\left(x - \frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh(\frac{\theta}{4})}\right], \tag{3}$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{4}$$

The problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$, and $\lambda < \lambda_c$, respectively, where the critical value λ_c satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right), \tag{5}$$

where

$$\lambda_c = 3.513830719. \tag{6}$$

The rest of the paper is organized as follows: In Section 2, we give some necessary definitions and properties of the fractional calculus theory and fractional power series (FPS). In Section 3, we introduce our results to solve the fractional Bratu-type equation (1) with initial conditions (2) using the FRPSM. In Section 4, we present three numerical examples to show the efficiency and effectiveness of this method. In Section 5, we discuss our obtained results represented by figures and tables, and verified with Matlab (version R2016a). Section 6 is devoted to the conclusions of the paper.

2. Basic definitions

In this section, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details see [6,14, 18,19].

Definition 2.1. A real function $u(x), x > 0$ is considered to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $u(x) = x^p f(x)$, where $f(x) \in C\big([0, +\infty)\big)$, and it is said to be in the space C_μ^n if $u^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of the function $u \in C_\mu, \mu \geq -1$ is defined as

$$I^\alpha u(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_0^x (x-\xi)^{\alpha-1} u(\xi) \mathrm{d}\xi, & \alpha > 0, \\ u(x), & \alpha = 0, \end{cases} \tag{7}$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 2.3. The Caputo fractional derivative operator of the function $u \in C_\mu^n, \mu \geq -1, n \in \mathbb{N}$ is defined as

$$D^\alpha u(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int\limits_0^x (x-\xi)^{n-\alpha-1} u^{(n)}(\xi) \mathrm{d}\xi, & n-1 < \alpha < n, \\ u^{(n)}(x), & \alpha = n. \end{cases} \tag{8}$$

For this definition we have the following properties:

1) $$D^\alpha (C) = 0, \text{ where } C \text{ is a constant.}$$

2) $$D^\alpha x^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & \text{if } \beta > n-1, \\ 0, & \text{if } \beta \leq n-1. \end{cases}$$

3) $$D^\alpha f\big(u(x)\big) = f'\big(u(x)\big) D^\alpha u(x).$$

Definition 2.4. [4] A power series representation of the form

$$\sum_{m=0}^{\infty} c_m (x-x_0)^{m\alpha} = c_0 + c_1 (t-t_0)^\alpha + c_2 (t-t_0)^{2\alpha} + \cdots, \tag{9}$$

where $n-1 < \alpha \leq n$ and $x \geq x_0$ is called a fractional power series (FPS) about x_0, where x is a variable and c_m are constants called the coefficients of the series.

Theorem 2.5 ([4]). *Suppose that u has a FPS representation at $x = x_0$ of the form*

$$u(x) = \sum_{m=0}^{\infty} c_m (x - x_0)^{m\alpha}, \quad x_0 \le x \le x_0 + R, \tag{10}$$

and R is the radius of convergence of the FPS.

If

$$u(x) \in C\,[x_0, x_0 + R) \quad \text{and} \quad D^{m\alpha} \in C\,(x_0, x_0 + R) \quad \text{for} \quad m = 0, 1, 2, \dots,$$

then the coefficients c_m will take the form of

$$c_m = \frac{D^{m\alpha} u(x_0)}{\Gamma(m\alpha + 1)}, \tag{11}$$

where $D^{m\alpha} = D^{\alpha} \cdot D^{\alpha} \cdots D^{\alpha}$ (m-times).

3. Analysis of the fractional residual power series method (FRPSM)

Theorem 3.1. *Let us consider the fractional Bratu-type equation* (1) *with the initial conditions* (2)*. Then by FRPSM the solution of equations* (1) *and* (2) *is given in the form of infinite series as follows*

$$u(x) = \sum_{n=0}^{\infty} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad 0 < \alpha \le 1, \quad 0 < x < R < 1, \tag{12}$$

where u_n are the coefficients of the series and R is the radius of convergence.

Proof. To achieve our goal, we consider the following fractional Bratu-type equation (1) with the initial conditions (2).

First, write the solution of equation (1) as fractional power series of the form

$$u(x) = \sum_{n=0}^{\infty} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)}. \tag{13}$$

Using the initial conditions in equation (2), the approximate solution to (13) can be written in the form of

$$u_k(x) = \sum_{n=0}^{k} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)} = u_0 + u_1 \frac{x^{\alpha}}{\Gamma(\alpha + 1)} + \sum_{n=2}^{k} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)}, \tag{14}$$

where

$$u_1(x) = u_0 + u_1 \frac{x^{\alpha}}{\Gamma(\alpha + 1)} \tag{15}$$

is considered as the first FRPS approximate solution of $u(x)$.

To determine the values of the coefficients $u_n, n = 2,3,4,\ldots,$ we solve the equation

$$D^{(n-2)\alpha}\mathrm{Res}_n(0) = 0, \quad n = 2,3,4,\ldots$$

where $\mathrm{Res}_k(x)$ is the k^{th} residual function [4] and is defined by

$$\mathrm{Res}_k(x) = D^{2\alpha}u_k(x) + \lambda\exp\big(u_k(x)\big).$$

To determine u_2 in equation (14), we substitute the second FRPS approximate solution

$$u_2(x) = u_0 + u_1\frac{x^\alpha}{\Gamma(\alpha+1)} + u_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)},$$

into

$$\begin{aligned}\mathrm{Res}_2(x) &= D^{2\alpha}u_2(x) + \lambda\exp\big(u_2(x)\big)\\ &= u_2 + \lambda\exp\left(u_0 + u_1\frac{x^\alpha}{\Gamma(\alpha+1)} + u_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\right).\end{aligned}$$

Then we solve $\mathrm{Res}_2(0) = 0$ to get

$$u_2 = -\lambda\exp\big(u_0\big).$$

To determine u_3 in equation (14), we substitute the third FRPS approximate solution

$$u_3(x) = u_0 + u_1\frac{x^\alpha}{\Gamma(\alpha+1)} + u_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + u_3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)},$$

into

$$\begin{aligned}\mathrm{Res}_3(x) &= D^{2\alpha}u_3(x) + \lambda\exp\big(u_3(x)\big)\\ &= u_2 + u_3\frac{x^\alpha}{\Gamma(\alpha+1)}\\ &\quad +\lambda\exp\left(u_0 + u_1\frac{x^\alpha}{\Gamma(\alpha+1)} + u_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + u_3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}\right).\end{aligned}$$

Then we solve $D^\alpha\mathrm{Res}_3(0) = 0$ to get

$$u_3 = -\lambda u_1\exp\big(u_0\big).$$

To determine u_4 in equation (14), we substitute the fourth FRPS approximate solution

$$u_4(x) = u_0 + u_1\frac{x^\alpha}{\Gamma(\alpha+1)} + u_2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + u_3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + u_4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)},$$

into

$$\begin{aligned}
\mathrm{Res}_4(x) &= D^{2\alpha}u_4(x) + \lambda \exp\big(u_4(x)\big) \\
&= u_2 + u_3 \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_4 \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \\
&\quad + \lambda \exp\left(u_0 + u_1 \frac{x^{\alpha}}{\Gamma(\alpha+1)} + u_2 \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \right. \\
&\qquad\qquad \left. + u_3 \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + u_4 \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} \right).
\end{aligned}$$

Then we solve $D^{2\alpha}\mathrm{Res}_4(0) = 0$ to get

$$u_4 = -\lambda u_2 \exp\big(u_0\big).$$

In general, to determine u_k, we substitute the kth FRPS approximate solution $u_k(x)$ into

$$\begin{aligned}
\mathrm{Res}_k(x) &= D^{2\alpha}u_k(x) + \lambda \exp\big(u_k(x)\big) \\
&= \sum_{n=0}^{k-2} u_{n+2} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} \\
&\quad + \lambda \exp\left(u_0 + u_1 \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \sum_{n=2}^{k} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} \right).
\end{aligned}$$

Afterwards we solve $D^{(k-2)\alpha}\mathrm{Res}_k(0) = 0$ to get

$$u_k = -\lambda u_{k-2} \exp\big(u_0\big), \quad k \geq 3.$$

Finally, the solution of equations (1) and (2) can be expressed by

$$u(x) = \lim_{k \longrightarrow \infty} u_k(x) = \sum_{n=0}^{\infty} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}.$$

The proof is complete. □

Theorem 3.2. *If there exists a constant* $0 < \gamma < 1$ *such that*

$$\|u_{n+1}(x)\| \leq \gamma \|u_n(x)\|, \quad n \in \mathbb{N}, \quad 0 < x < R < 1,$$

then the sequence of approximate solutions (12) *converges to the corresponding exact solution.*

P r o o f. For all $0 < x < R < 1$, we have

$$\begin{aligned}
\|u(x) - u_n(x)\| &= \left\| \sum_{k=n+1}^{\infty} u_k(x) \right\| \le \sum_{k=n+1}^{\infty} \|u_k(x)\| \\
&\le \sum_{k=n+1}^{\infty} \gamma \|u_{k-1}(x)\| \le \sum_{k=n+1}^{\infty} \gamma^2 \|u_{k-2}(x)\| \\
&\le \cdots \\
&\le \|u_0\| \sum_{k=n+1}^{\infty} \gamma^k = \frac{\gamma^{n+1}}{1-\gamma} \|u_0\| .
\end{aligned}$$

Since $0 < \gamma < 1$ and u_0 is bounded, then

$$\lim_{n \longrightarrow \infty} \|u(x) - u_n(x)\| = 0.$$

This completes the proof. □

4. Numerical examples

In this section, we present three numerical examples of fractional Bratu-type equations to validate the capability, reliability and efficiency of the FRPSM. Then we compare approximate solutions with corresponding exact ones for different values of the order fractional derivative $\alpha, 1/2 < \alpha \le 1$.

We define $E_k(\alpha)$ to be the absolute error between the exact solution u and the kth FRPS approximate solution u_k, where $k = 0, 1, 2, 3, \dots$, as follows

$$E_k(x, \alpha) = |u(x) - u_k(x)| .$$

Example 4.1. Consider the following fractional Bratu-type equation

$$D^{2\alpha} u(x) - 2 \exp\big(u(x)\big) = 0, \quad \frac{1}{2} < \alpha \le 1, \quad 0 < x < 1, \tag{16}$$

with the initial conditions

$$u(0) = D^{\alpha} u(0) = 0. \tag{17}$$

For $\alpha = 1$, the exact solution of equations (16) and (17) is (see. [12])

$$u(x) = -2 \ln(\cos x).$$

By applying the steps involved in FRPSM as presented in Section 3, we have the solution of equations (16) and (17) in the form

$$u(x) = \sum_{n=0}^{\infty} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)},$$

and

$$u_n = \begin{cases} u_{2p} = 2^p, & p \in \mathbb{N}^*, \\ u_{2p+1} = 0\,, & p \in \mathbb{N}^*. \end{cases}$$

Therefore, the approximate solution of equations (16) and (17) can be expressed by

$$u(x) = \frac{2}{\Gamma(2\alpha+1)} x^{2\alpha} + \frac{4}{\Gamma(4\alpha+1)} x^{4\alpha} + \frac{8}{\Gamma(6\alpha+1)} x^{6\alpha} + \cdots$$

which is exactly the same as the result obtained by ADM [9] and HPM [10].

TABLE 1. The numerical values of the 12th FRPS approximate solution, (BPs) [12] and the exact solution for Example 4.1 when $\alpha = 1$.

x	FRPSM	BPs	Exact
0.03448	0.001891	0.00118912	0.00118911
0.10345	0.010721	0.0107219	0.010721
0.17241	0.029873	0.0298804	0.0298737
0.24138	0.058832	0.0588668	0.058839
0.31034	0.097867	0.0979798	0.097897
0.37931	0.14736	0.147662	0.147465
0.44828	0.20778	0.208484	0.20807
0.51724	0.27968	0.281178	0.280393
0.58621	0.36378	0.366712	0.365339
0.65517	0.46085	0.466255	0.464004
0.72414	0.57184	0.581339	0.577847
0.79313	0.69784	0.713882	0.708731
0.86207	0.83990	0.866119	0.858899
0.93103	0.99951	1.04121	1.03165
1	1.17820	1.24298	1.23125

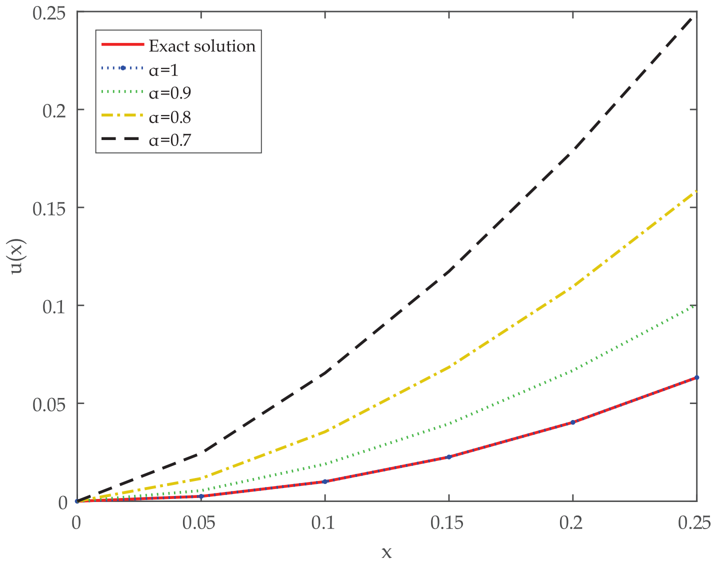

FIGURE 1. 2D plots for the approximate solution $u_6(x)$ by FRPSM and exact solution for Example 4.1.

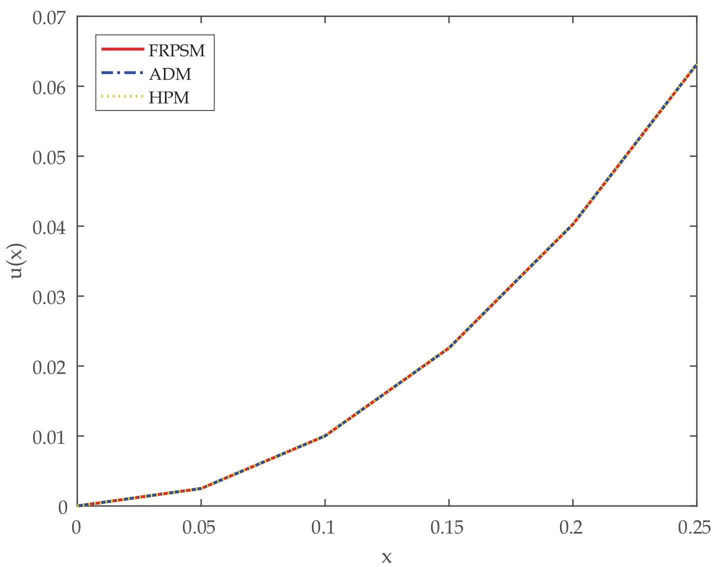

FIGURE 2. Comparison of approximate solutions FRPSM, ADM and HPM at $\alpha = 1$ for Example 4.1.

TABLE 2. Comparison of the absolute errors for the exact solution and approximate solution $u_6(x)$ for Example 4.1.

x	$E_6(\alpha = 0.8)$	$E_6(\alpha = 0.9)$	$E_6(\alpha = 1)$
0.05	9.1265×10^{-3}	2.9348×10^{-3}	5.2136×10^{-10}
0.1	2.5451×10^{-2}	8.9658×10^{-3}	3.3469×10^{-8}
0.15	4.5845×10^{-2}	1.6967×10^{-2}	3.8317×10^{-7}
0.2	6.9288×10^{-2}	2.6485×10^{-2}	2.1683×10^{-6}
0.25	9.5299×10^{-2}	3.7273×10^{-2}	8.3482×10^{-6}

EXAMPLE 4.2. Consider the following fractional Bratu-type equation

$$D^{2\alpha}u(x) - \pi^2 \exp\big(u(x)\big) = 0, \quad \frac{1}{2} < \alpha \le 1, \quad 0 < x < 1, \tag{18}$$

with the initial conditions

$$u(0) = 0, \quad D^{\alpha}u(0) = \pi. \tag{19}$$

For $\alpha = 1$, the exact solution of equations (18) and (19) is (see. [21])

$$u(x) = -\ln(1 - \sin \pi x).$$

By applying the steps involved in FRPSM as presented in Section 3, we have the solution of equations (18) and (19) in the form

$$u(x) = \sum_{n=0}^{\infty} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)},$$

and

$$u_n = \pi^n, \quad n = 2, 3, 4 \ldots$$

Therefore, the approximate solution of equations (18) and (19) can be expressed by

$$u(x) = \frac{\pi}{\Gamma(\alpha + 1)} x^{\alpha} + \frac{\pi^2}{\Gamma(2\alpha + 1)} x^{2\alpha} + \frac{\pi^3}{\Gamma(3\alpha + 1)} x^{3\alpha} + \cdots$$

which is exactly the same result as obtained by HPM [10].

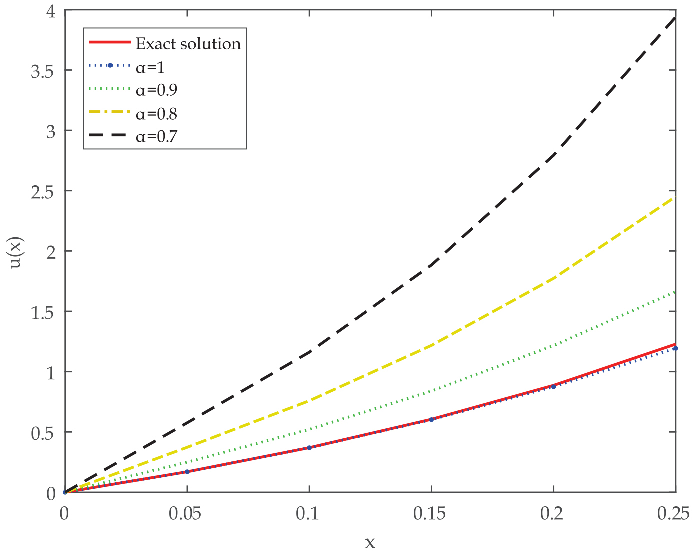

FIGURE 3. 2D plots for the approximate solution $u_6(x)$ by FRPSM and exact solution for Example 4.2.

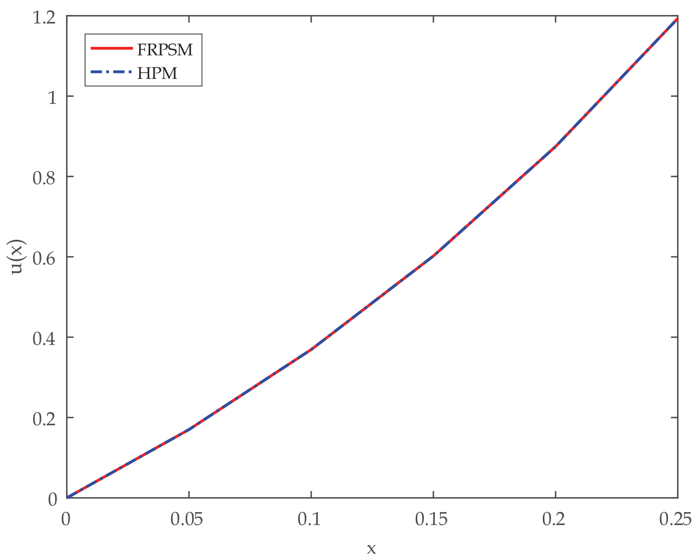

FIGURE 4. Comparison of approximate solutions FRPSM and HPM at $\alpha = 1$ for Example 4.2.

TABLE 3. Comparison of the absolute errors for the exact solution and approximate solution $u_6(x)$ for Example 4.2.

x	$E_6(\alpha = 0.8)$	$E_6(\alpha = 0.9)$	$E_6(\alpha = 1)$
0.05	2.0279×10^{-1}	7.9440×10^{-2}	2.8838×10^{-5}
0.1	3.8883×10^{-1}	1.5164×10^{-1}	5.3118×10^{-4}
0.15	6.1177×10^{-1}	2.3409×10^{-1}	3.1352×10^{-3}
0.2	8.8604×10^{-1}	3.2918×10^{-1}	1.1735×10^{-2}
0.25	12.209×10^{-1}	4.3443×10^{-1}	3.4620×10^{-2}

EXAMPLE 4.3. Consider the following fractional Bratu-type equation

$$D^{2\alpha}u(x) + \pi^2 \exp\big(-u(x)\big) = 0, \quad \frac{1}{2} < \alpha \leq 1, \quad 0 < x < 1, \tag{20}$$

subject to the initial conditions

$$u(0) = 0, D^{\alpha}u(0) = \pi. \tag{21}$$

For $\alpha = 1$, the exact solution of equations (20) and (21) is (see. [21])

$$u(x) = \ln(1 + \sin \pi x).$$

By applying the steps involved in FRPSM as presented in Section 3, we have the solution of equations (20) and (21) in the form

$$u(x) = \sum_{n=0}^{\infty} u_n \frac{x^{n\alpha}}{\Gamma(n\alpha + 1)},$$

and

$$u_n = (-1)^{n+1}\pi^n, \quad n = 2, 3, 4 \ldots$$

Therefore, the approximate solution of equations (20) and (21) can be expressed by

$$u(x) = \frac{\pi}{\Gamma(\alpha+1)}x^{\alpha} - \frac{\pi^2}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{\pi^3}{\Gamma(3\alpha+1)}x^{3\alpha} - \cdots$$

which is exactly the same result as obtained by HPM [10].

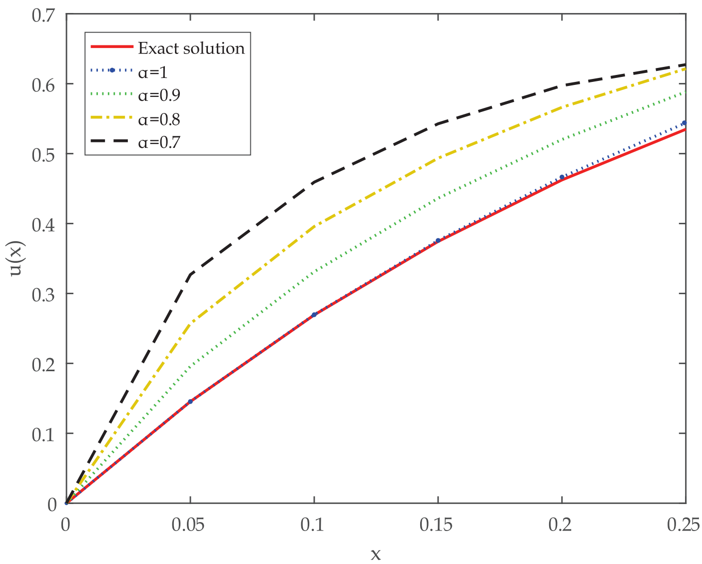

FIGURE 5. 2D plots for the approximate solution $u_6(x)$ by FRPSM and exact solution for Example 4.3.

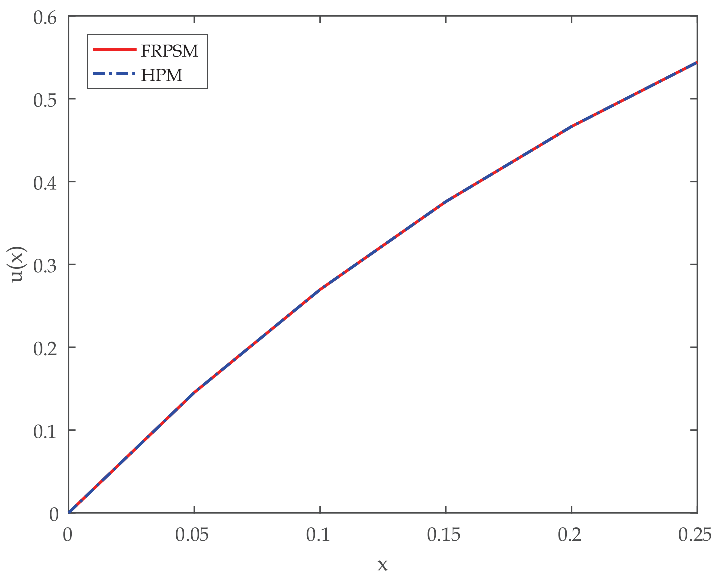

FIGURE 6. Comparison of approximate solutions FRPSM and HPM at $\alpha = 1$ for Example 4.3.

TABLE 4. Comparison of the absolute errors for the exact solution and approximate solution $u_6(x)$ for Example 4.3.

x	$E_6(\alpha = 0.8)$	$E_6(\alpha = 0.9)$	$E_6(\alpha = 1)$
0.05	1.1151×10^{-1}	5.0355×10^{-2}	2.2422×10^{-5}
0.1	1.2645×10^{-1}	6.1802×10^{-2}	3.2017×10^{-4}
0.15	1.1912×10^{-1}	6.1887×10^{-2}	1.4560×10^{-3}
0.2	1.0392×10^{-1}	5.7854×10^{-2}	4.1571×10^{-3}
0.25	8.6636×10^{-2}	5.3482×10^{-2}	9.2121×10^{-3}

5. Numerical results and discussion

In this section, we discuss and evaluate the numerical results of the approximate solutions for Examples 4.1, 4.2 and 4.3, respectively. Figures 1, 3 and 5 represent the behaviour of the exact solution and the approximate solution $u_6(x)$ for different values of α ($\alpha = 0.7, 0.8, 0.9, 1$). These figures affirm that when the order of the fractional derivative α approaches 1, the approximate solutions obtained by FRPSM approach the exact solutions.

Figures 2, 4 and 6 represent the comparison between the approximate solutions obtained by the FRPSM, ADM and HPM, for $\alpha = 1$. As it can be seen in figures, the solutions obtained by the proposed method are nearly identical with those provided by the ADM and HPM.

Table 1 represents the numerical values of the 12th FRPS approximate solution, (BPs) [12] and the exact solution for Example 4.1 when $\alpha = 1$.

Tables 2–4 represent the absolute errors between the exact solution and the approximate solution $u_6(x)$ for different values of x and α. These tables clarify the convergence of the approximate solutions to the exact ones.

6. Conclusion

In this paper, the fractional residual power series method (FRPSM) has been used to solve the fractional Bratu-type equation. In order to illustrate the accuracy and effectiveness of this method, we have applied it to solve three numerical examples and compared our results with those in the literature. The solutions obtained by the proposed method were in an agreement with the solutions available in the literature. The comparisons in tables and figures show that the approximate solutions converge very rapidly to the corresponding exact ones and the presented method is a powerful tool for solving this type of equation. Therefore, we can conclude that the FRPSM is practically well-suited for solving a wide range of fractional differential equations.

Acknowledgements. The authors would like to thank Professor Miroslava Růžičková (TMMP editor) as well as the anonymous referees who has made valuable and careful comments, which improved the paper considerably.

REFERENCES

[1] ABDULAZIZ, O.—HASHIM, I.—MOMANI, S.: *Solving systems of fractional differential equations by homotopy perturbation method*, Phys. Lett. A. **372** (2008), no. 4, 451–459.

[2] BOYD, J. P.: *An analytical two-dimensional Bratu equation*, J. Sci. Comput. **1** (1986), no. 2, 183–206.

[3] BOYD, J. P.: *Chebyshev polynomial expansions for simultaneous approximation of two branches of a function with application to the one-dimensional Bratu equation*, Appl. Math. Comput. **142** (2003), 189–200.

[4] EL-AJOU, A.—ABU ARQUB, O.—AL ZHOUR, Z.—MOMANI, S.: *New results on fractional power series: theories and applications*, Entropy **15** (2013), no. 12, 5305–5323.

[5] ARIKOGLU, A.—OZKOL, I.: *Solution of fractional differential equations by using differential transform method*, Chaos Solitons Fractals **34** (2007), no. 5, 1473–1481.

[6] BABAEI, A.—JAFARI, H.—AHMADI, M.: *A fractional order HIV/AIDS model based on the effect of screening of unaware infectives*, Math. Methods Appl. Sci. **42** (2019), no. 7, 2334–2343.

[7] BUCKMIRE, R.: *Application of a Mickens finite-difference scheme to the cylindrical Bratu–Gelfand problem*, Numer. Methods Partial Differential Equations **20** (2004), no. 3, 327–337.

[8] DAFTARDAR-GEJJI, V.—JAFARI, H.: *Solving a multi-order fractional differential equation using adomian decomposition*, Appl. Math. Comput. **189** (2007), 541–548.

[9] GHAZANFARI, B.—SEPAHVANDZADEH, A.: *Adomian Decomposition Method for Solving Fractional Bratu-type Equations*, J. Math. Comput. Sci. **8** (2014), 236–244.

[10] GHAZANFARI, B.—SEPAHVANDZADEH, A.: *Homotopy perturbation method for solving fractional Bratu-type equation*, J. Math. Model. **2** (2015), no. 2, 143–155.

[11] HASHIM, I.—ABDULAZIZ, O.—MOMANI, S.: *Homotopy analysis method for fractional IVPs*, Commun. Nonlinear Sci. Numer. Simul. **14** (2009), 674–684.

[12] JAFARI, H—TAJADODI, H.: *Electro–Spunorganic Nanofibers Elaboration Process Investigations using BPs Operational Matrices*, Iran. J. Math. Chem. **7** (2016), no. 1, 19–27.

[13] KHALOUTA, A.—KADEM, A.: *Comparison of new iterative method and atural homotopy perturbation method for solving nonlinear time-fractional wave-like quations with variable coefficients*, Nonlinear Dyn. Syst. Theory **19** (1-SI) (2019), 160–169.

[14] KILBAS, A.—SRIVASTAVA, H. M.—TRUJILLO, J. J.: *Theory and Application of Fractional Differential equations*. Elsevier, Amsterdam, 2006.

[15] MILLER, K. S.—ROSS, B.: *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley, New York, 1993.

[16] ODIBAT, Z. S.—MOMANI, S.: *Analytical comparison between the homotopy perturbation method and variational iteration method for differential equations of fractional order*. Internat. J. Modern Phys. B. **22** (2008), 4041–4058.

[17] PODLUBNY, I.: *Fractional Differential Equations*. Academic Press, New York, 1999.

[18] SAMKO, S. G.—KILBAS, A. A.—MARICHEV, O. I.: *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science, Yverdon, Switzerland, 1993.

[19] SINGH, A.—DAS, S.—ONG, S.-H.—JAFARI, H.: *Numerical Solution of Nonlinear Reaction-Advection-Diffusion Equation*, J. Comput. Nonlinear Dyn. **14** (2019), no. 4, 041003.

[20] SYAM, M. I.—HAMDAN, A.: *An efficient method for solving Bratu equations*, Appl. Math. Comput. **176** (2006), no. 2, 704–713.

[21] WAZWAZ, A. M.: *Adomian decomposition method for a reliable treatment of the Bratu-type equations*, Appl. Math. Comput. **166** (2005), no. 3, 652–663.

Received February 11, 2019

Laboratory of Fundamental and Numerical Mathematics
Department of Mathematics
Faculty of Sciences
Ferhat Abbas Sétif University 1
19000 Sétif
ALGERIA

E-mail: nadjibkh@yahoo.fr
abdelouahabk@yahoo.fr