

THE ABSOLUTELY STRONGLY STAR-HUREWICZ PROPERTY WITH RESPECT TO AN IDEAL

SUMIT SINGH¹ — BRIJ K. TYAGI² — MANOJ BHARDWAJ¹

¹Department of Mathematics, University of Delhi, New Delhi, INDIA

²Atma Ram Sanatan Dharma College, University of Delhi, New Delhi, INDIA

ABSTRACT. A space X is said to have the absolutely strongly star- \mathcal{I} -Hurewicz (ASSZH) property if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X and each dense subset Y of X , there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of Y such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \text{St}(F_n, \mathcal{U}_n)\} \in \mathcal{I}$, where \mathcal{I} is the proper admissible ideal of \mathbb{N} . In this paper, we investigate the relationship between the ASSZH property and other related properties and study the topological properties of the ASSZH property. This paper generalizes several results of Song [25] to the larger class of spaces having the ASSZH properties.

1. Introduction

In 1996, Scheepers [20] started the systematic study of selection principles in topology and their relation to game theory and Ramsey theory (see also [13]). Kočinac and Scheepers [16] studied the Hurewicz property in detail and found its relation with function spaces, game theory and Ramsey theory. Subsequently this topic became one of the most active areas of set theoretic topology. Maio and Kočinac [17] introduced the statistical analogues of certain types of open covers and selection principles using actually the ideal of asymptotic density zero sets of natural numbers. Das, Kočinac and Chandra [6, 7] extended this study to the arbitrary ideal of natural numbers. Using the notions of ideals, they started a more general approach to study certain results of open covers and selection principles. Further, Das et al. [8] studied the ideal version of the

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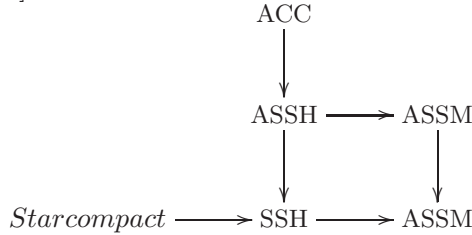
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SSH property called the strongly star- \mathcal{I} -Hurewicz (SSZH) property, where \mathcal{I} is the proper admissible ideal of \mathbb{N} . Singh, Tyagi and Bhardwaj [22, 27] studied the ideal version of star-K-Hurewicz and star-C-Hurewicz properties.

On the other hand, Fleischman [12] defined a space X to be starcompact if, for every open cover \mathcal{U} of X , there exists a finite subset F of X such that $\text{St}(F, \mathcal{U}) = X$, where $\text{St}(F, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. He proved that every countable compact space was starcompact. Douwen et al. in [9] showed that in a T_2 -space, every starcompact space is countably compact but not in a T_1 -space (see [23, Example 2.5]). Matveev [18] introduced absolute countably compact (ACC) space X if, for each open cover \mathcal{U} of X and each dense subset D of X , there exists a finite subset F of D such that $\text{St}(F, \mathcal{U}) = X$. It is clear that T_2 ACC space is countably compact. In [9], a starcompact space is called strongly starcompact. Kočinac [14, 15] defined the strongly star-Menger (SSM) property. Bonanzinga et al. [3] defined the strongly star-Hurewicz (SSH) property. Caserta, Di Maio and Kočinac [5] gave the selective version of the ACC spaces and introduced the absolutely strongly star-Menger (ASSM) property and the absolutely strongly star-Hurewicz (ASSH) property (see also [24, 25]). Song investigated the relationships among the above properties and gave the following implications [25].



The purpose of this paper is to study the ideal analogue of the ASSH property called the absolute strongly star- \mathcal{I} -Hurewicz (ASSZH) property. Further, we investigate the relationship of the ASSZH with the above related properties and study the topological properties of the ASSZH property.

Throughout the paper, the extent $e(X)$ of a space X is the smallest cardinal number κ such that the cardinality of every discrete closed subset of X is not greater than κ . Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, \mathfrak{c} the cardinality of the set of all real numbers. For a cardinal κ , let κ^+ be the smallest cardinal greater than κ . For each pair of ordinals

$$\alpha, \beta \quad \text{with} \quad \alpha < \beta,$$

we write

$$\begin{aligned}
 [\alpha, \beta] &= \{\gamma : \alpha \leq \gamma < \beta\}, & (\alpha, \beta] &= \{\gamma : \alpha < \gamma \leq \beta\}, \\
 (\alpha, \beta) &= \{\gamma : \alpha < \gamma < \beta\}, & [\alpha, \beta] &= \{\gamma : \alpha \leq \gamma \leq \beta\}.
 \end{aligned} \tag{1}$$

As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. For other terms and symbols we follow [10].

2. Preliminaries

Let (X, τ) or X and (Y, σ) or Y be topological spaces or spaces. We will denote by $\text{Cl}(A)$ and $\text{Int}(A)$ the closure of A and the interior of A , respectively, for a subset A of a space X . Throughout the paper, \mathbb{N} denotes the set of all positive integers. In this section, we consider basic definitions used in this paper.

Let A be a subset of X and \mathcal{P} be a collection of subsets of X , then

$$\text{St}(A, \mathcal{P}) = \bigcup \{U \in \mathcal{P} : U \cap A \neq \emptyset\}.$$

We usually write

$$\text{St}(x, \mathcal{P}) = \text{St}(\{x\}, \mathcal{P}).$$

A space X is said to have the strongly star-Hurewicz [3] (in short SSH) space if, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X , there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $x \in \text{St}(F_n, \mathcal{U}_n)$ for all but finitely many n .

A space X is said to have the absolutely strongly star-Hurewicz [25] (in short ASSH) space if, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X and each dense subset Y of X , there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of Y such that for each $x \in X$, $x \in \text{St}(F_n, \mathcal{U}_n)$ for all but finitely many n .

A space X is said to have the strongly star-Menger [14, 15] (in short SSM) space if, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X , there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\text{St}(F_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X .

A space X is said to have the absolutely strongly star-Menger [24] (in short ASSM) space if, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X and each dense subset Y of X , there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of Y such that $\{\text{St}(F_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X .

A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$, while an ideal is said to be admissible ideal or free ideal \mathcal{I} of Y if $\{y\} \in \mathcal{I}$ for each $y \in Y$.

\mathcal{I} is a proper ideal in Y if $Y \notin \mathcal{I}$. \mathcal{I} is a proper ideal in Y , then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y: \text{there exists } A \in \mathcal{I} : M = Y \setminus A\}$ is a filter on Y whereas the coideal of \mathcal{I} is $\mathcal{I}^+ = \{A \subset Y: A \notin \mathcal{I}\}$. Throughout the paper, \mathcal{I} will stand for proper admissible ideal of \mathbb{N} .

A space X is said to have the strongly star- \mathcal{I} -Hurewicz [8] (in short SSZH) space if, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X , there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \text{St}(F_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to have the absolutely star-Lindelöf property [2,4] if, for each open cover \mathcal{U} of X and each dense subset D of X , there exists a countable subset F of D such that $\text{St}(F, \mathcal{U}) = X$.

3. The absolutely strongly star- \mathcal{I} -Hurewicz and related properties

In this section, we introduce the notion of the absolutely strongly star- \mathcal{I} -Hurewicz property and give some examples to show the relationship between the absolutely strongly star- \mathcal{I} -Hurewicz and other related properties.

DEFINITION 3.1. A space X is said to have the absolutely strongly star- \mathcal{I} -Hurewicz (in short ASSZH) property if, for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X and each dense subset Y of X , there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of Y such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \text{St}(F_n, \mathcal{U}_n)\} \in \mathcal{I}$.

LEMMA 3.2. For a topological space X and an admissible ideal \mathcal{I} ,

$$\text{ASSZH} \Rightarrow \text{SSZH} \Rightarrow \text{SSM} \quad \text{and} \quad \text{ASSZH} \Rightarrow \text{ASSM}.$$

Proof. The proof is easy, thus omitted. □

We have the following diagram from the above definitions and Lemma 3.2.

$$\begin{array}{ccccccc} \text{ACC} & \longrightarrow & \text{ASSH} & \longrightarrow & \text{ASSZH} & \longrightarrow & \text{ASSM} \\ & & \downarrow & & \downarrow & & \downarrow \\ \text{Starcompact} & \longrightarrow & \text{SSH} & \longrightarrow & \text{SSZH} & \longrightarrow & \text{ASSM} \end{array}$$

The following example shows that the converse of implications in the above diagram need not be true.

EXAMPLE. There exists a Tychonoff space having the SSZH property but does not have the ASSZH property.

PROOF. Let $X = [0, \omega_1) \times [0, \omega_1]$ be the product of $[0, \omega_1)$ and $[0, \omega_1]$. Clearly, X is countably compact. Hence, X has the SSZH property. Now, we show that X does not have the ASSZH property. For each $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha) \times (\alpha, \omega_1] \quad \text{and} \quad Y = [0, \omega_1) \times [0, \omega_1).$$

For each $n \in \mathbb{N}$,

$$\mathcal{U}_n = \{U_\alpha : \alpha < \omega_1\} \cup \{Y\}$$

is a cover of X . Consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open cover of X and the dense subset Y of X . Let $(F_n : n \in \mathbb{N})$ be any sequence of finite subsets of Y . We only show that there exists a point $x \in X$ such that

$$\{n \in \mathbb{N} : x \notin \text{St}(F_n, \mathcal{U}_n)\} = \mathbb{N} \notin \mathcal{I}.$$

Let

$$\alpha_n = \sup\{\alpha : \alpha \in \pi(F_n)\},$$

where $\pi : [0, \omega_1) \times [0, \omega_1] \rightarrow [0, \omega_1]$ is the projection. Then, for each $n \in \mathbb{N}$, $\alpha_n < \omega_1$, since F_n is finite. Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then, $\beta < \omega_1$. Pick $\alpha' > \beta$, then

$$\{n \in \mathbb{N} : \langle \alpha', \omega_1 \rangle \notin \text{St}(F_n, \mathcal{U}_n)\} = \mathbb{N} \notin \mathcal{I},$$

since for every $U_\beta \in \mathcal{U}_n$, if $\langle \alpha', \omega_1 \rangle \in U_\beta$, then $\beta > \alpha'$ and for each $\beta > \alpha'$, $U_\beta \cap F_n = \emptyset$, which shows that X does not have the ASSZH property. \square

Recall that a family $\mathcal{A} \subset P(\mathbb{N})$ is said to be almost disjoint (in short *AD*) if every element of \mathcal{A} is infinite and the sets $A \cap B$ are finite for all distinct elements $A, B \in \mathcal{A}$. For an *AD* family \mathcal{A} , put $\psi(\mathcal{A}) = \mathcal{A} \cup \mathbb{N}$ and topologize $\psi(\mathcal{A})$ as follows: the natural numbers are isolated and for each element $A \in \mathcal{A}$ and each finite set $F \subset \mathbb{N}$, $\{A\} \cup (A \setminus F)$ is a basic open neighbourhood of A . The spaces constructed in this manner are called Isbell-Mrówka ψ -spaces [19]. It is well-known that \mathcal{A} is maximal almost disjoint (in short *MAD*) family if and only if $\psi(\mathcal{A})$ is pseudocompact.

Define $\mathbb{N}^{\mathbb{N}}$ as the set of all functions from \mathbb{N} to itself. For all $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ means $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means $f(n) \leq g(n)$ for all n). A subset B of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ is bounded if there is $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for each $f \in B$. The unbounding number, denoted by \mathfrak{b} , is the smallest cardinality of an unbounded subset of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. The dominating number, denoted by \mathfrak{d} , is the smallest cardinality of a cofinal subset of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. The cardinality of a set is denoted by $|A|$. For an ideal \mathcal{I} of \mathbb{N} , an ideal version of the unbounding number, denoted by $\mathfrak{b}(\mathcal{I})$, was introduced in [11], where

$$\mathfrak{b}(\mathcal{I}) = \min\{|B| : B \subset (\mathbb{N}^{\mathbb{N}}, \leq^*) \text{ and for all } g \in \mathbb{N}^{\mathbb{N}} \text{ there is an } f \in B \text{ such that } \{n \in \mathbb{N} : g(n) \leq f(n)\} \in \mathcal{I}^+\}.$$

Now, we give some examples of Tychonoff spaces such that: (1) The space has the ASSM property but does not have the ASSZH property. (2) The space has the SSM property but does not have the SSZH property. (3) The space has the ASSZH property but does not have the ASSH property. (4). The space has the absolutely star-Lindelöf property but does not have the ASSZH property. For these examples, we use the following results.

THEOREM 3.3 ([4, Proposition 2,], [24, Remark 2.5]). *Let X be a $\psi(\mathcal{A})$ space generated by an AD family \mathcal{A} of \mathbb{N} . Then, the following statements are equivalent:*

- (1) $\psi(\mathcal{A})$ has the SSM property.
- (2) $|\mathcal{A}| < \mathfrak{d}$.
- (3) $\psi(\mathcal{A})$ has the ASSM property.

THEOREM 3.4 ([4, Proposition 3], [25, Remark 2.6]). *Let X be a $\psi(\mathcal{A})$ space generated by an AD family \mathcal{A} of \mathbb{N} . Then, the following statements are equivalent:*

- (1) $\psi(\mathcal{A})$ has the SSH property.
- (2) $|\mathcal{A}| < \mathfrak{b}$.
- (3) $\psi(\mathcal{A})$ has the ASSH property.

THEOREM 3.5 ([8, Theorem 4.4]). *Let X be a $\psi(\mathcal{A})$ space generated by an AD family \mathcal{A} of \mathbb{N} . Then, the following statements are equivalent:*

- (1) $\psi(\mathcal{A})$ has the SSZH property.
- (2) $|\mathcal{A}| < \mathfrak{b}(\mathcal{I})$.

We prove the following theorem using Theorem 3.5.

THEOREM 3.6. *Let X be a $\psi(\mathcal{A})$ space generated by an AD family \mathcal{A} of \mathbb{N} . Then, the following statements are equivalent:*

- (1) $\psi(\mathcal{A})$ has the SSZH property.
- (2) $|\mathcal{A}| < \mathfrak{b}(\mathcal{I})$.
- (3) $\psi(\mathcal{A})$ has the ASSZH property.

Proof. (1) \Leftrightarrow (2) follows from Theorem 3.5 and (3) \Rightarrow (1) follows from definitions. We only have to show that (2) \Rightarrow (3). Let $|\mathcal{A}| < \mathfrak{b}(\mathcal{I})$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and Y be any dense set of X . For each $n \in \mathbb{N}$ and $A \in \mathcal{A}$, choose an element $U_{n,A}$ that contains A . Define a function $g_A \in \mathbb{N}^{\mathbb{N}}$ by $g_A(n) = \min\{k \in \mathbb{N} : k \in U_{n,A}\}$. Since $|\mathcal{A}| < \mathfrak{b}(\mathcal{I})$, there is a $f^* \in \mathbb{N}^{\mathbb{N}}$ such that for all $A \in \mathcal{A}$, $\{n \in \mathbb{N} : g_A(n) \geq f^*(n)\} \in \mathcal{I}$.

Let $A_n = \{1, 2, 3, \dots, \max\{f^*(n), n\}\}$. Since every dense set of X contains natural number. Thus, for each $n \in \mathbb{N}$, A_n is a finite subset of Y . Now, for each $A \in \mathcal{A}$,

$$\{n \in \mathbb{N} : g_A(n) < f^*(n)\} \subset \{n \in \mathbb{N} : A \in \text{St}(A_n, \mathcal{U}_n)\}.$$

As $\{n \in \mathbb{N} : g_A(n) < f^*(n)\} \in \mathcal{F}(\mathcal{I})$. Therefore, for each $A \in \mathcal{A}$,

$$\{n \in \mathbb{N} : A \in \text{St}(A_n, \mathcal{U}_n)\} \in \mathcal{F}(\mathcal{I}).$$

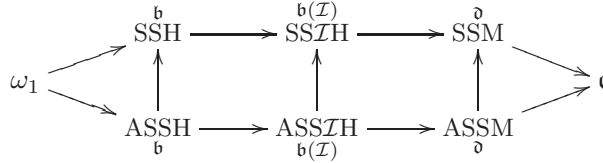
Next, let $m \in \mathbb{N}$. We add m in A_n for each $n \geq m + 1$ if m is not in A_n . Subsequently, for each $m \in \mathbb{N}$, the set $\{i \in \mathbb{N} : m \in \text{St}(A_i, \mathcal{U}_i)\}$ is cofinite. This completes the proof. \square

LEMMA 3.7. *For an admissible ideal, the inequality $\omega_1 \leq \mathfrak{b} \leq \mathfrak{b}(\mathcal{I}) \leq \mathfrak{d} \leq \mathfrak{c}$ holds.*

Proof. Let \mathcal{A} be an almost disjoint family of infinite subsets of \mathbb{N} . For the proof of inequality $\omega_1 \leq \mathfrak{b} \leq \mathfrak{b}(\mathcal{I}) \leq \mathfrak{c}$ (see [10, 11, 26]). We only need to prove $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{d} \leq \mathfrak{c}$. Suppose if possible $\mathfrak{d} < \mathfrak{b}(\mathcal{I})$. Let $\mathfrak{d} \leq |\mathcal{A}| < \mathfrak{b}(\mathcal{I})$. Then, by Theorem 3.6, $\psi(\mathcal{A})$ has the ASSZH property and by Lemma 3.2, $\psi(\mathcal{A})$ has the ASSM property. Then, by Theorem 3.3, $|\mathcal{A}| < \mathfrak{d}$, a contradiction. \square

Remark 1. $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, $\mathfrak{b} < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, $\mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{b}(\mathcal{I}) < \mathfrak{d} = \mathfrak{c}$ are all consistent with the axioms of (ZFC) (see [10, 11, 26]).

Lemma 3.7, Theorem 3.3, Theorem 3.4, Theorem 3.5 and Theorem 3.6 are summarized in the following figure. The cardinality with property in the diagram is the minimum cardinality of a set of reals that fails to have the corresponding property.



EXAMPLE. There exists a Tychonoff space having the ASSM property but does not have the ASSZH property.

Proof. Let $X = \psi(\mathcal{A}) = \mathbb{N} \cup \mathcal{A}$ be Isbell-Mrówka ψ -space, where \mathcal{A} is the AD family of infinite subsets of \mathbb{N} with $|\mathcal{A}| = \mathfrak{b}(\mathcal{I})$. Then, X has the ASSM property by Theorem 3.3. However, X does not have the ASSZH property by Theorem 3.6. \square

In Example 3, by Theorem 3.3, X has SSM property and by Theorem 3.6, X does not have the SSZH property. Thus, Example 3 can also work for the following example.

EXAMPLE. There exists a Tychonoff space having the SSM property but does not have the SSZH property.

EXAMPLE. There exists a Tychonoff space having the ASSZH property but does not have the ASSH property.

PROOF. Let $X = \psi(\mathcal{A}) = \mathbb{N} \cup \mathcal{A}$ be Isbell-Mrówka ψ -space, where \mathcal{A} is the AD family of infinite subsets of \mathbb{N} with $|\mathcal{A}| = \mathfrak{b}$. Then, by Theorem 3.6, X has the ASSZH property and by Theorem 3.4 X does not have the ASSH property. \square

Remark 2. Assuming $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, the space $X = \psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$. Then, by Theorem 3.6, X has the ASSZH property. This space shows that there exists a Tychonoff space having the ASSZH property X with $e(X) = \omega_1$, since \mathcal{A} is a discrete closed subset of X with $|\mathcal{A}| = \omega_1$. However, we do not know if there exists an example in (ZFC) showing that there exists a Tychonoff space having the ASSZH property with $e(X) \geq \mathfrak{c}$.

The proof of the following lemma is easy, thus omitted.

LEMMA 3.8. *Every space having the ASSZH property is absolutely star-Lindelöf.*

The following example shows that the converse of the above lemma is not true.

EXAMPLE. There exists a Tychonoff space having the absolutely star-Lindelöf property but does not have the ASSZH property.

PROOF. Let $X = \psi(\mathcal{A}) = \mathbb{N} \cup \mathcal{A}$, where \mathcal{A} is the AD family of infinite subsets of \mathbb{N} with $|\mathcal{A}| = \mathfrak{b}(\mathcal{I})$. Then, X does not have the ASSZH property by Theorem 3.6. Since \mathbb{N} is a countable dense subset of X , thus X has the absolutely star-Lindelöf property. \square

4. Preservation properties of the absolutely strongly star- \mathcal{I} -Hurewicz

In this section, we study the topological properties of the ASSZH property.

Remark 3. Assuming $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$. Then, space $X = \psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$ is ASSZH space. Since \mathcal{A} is a discrete closed subset of X with $|\mathcal{A}| = \omega_1$. This space shows that a closed subspace of a Tychonoff space having the ASSZH property need not have the ASSZH property.

Next, we give two stronger examples. The construction of the following example is from [25].

EXAMPLE. There exists a Tychonoff space having the ASSZH property having a regular-closed G_δ -subspace which does not have the ASSZH property.

PROOF. Let $S_1 = [0, \omega_1) \times [0, \omega]$ be the product of $[0, \omega_1)$ and $[0, \omega]$. Since $[0, \omega_1)$ is ACC, by [18, Theorem 1.8], then S_1 is ACC by [18, Theorem 2.3].

Hence S_1 has the ASSZH property. Let $S_2 = [0, \omega_1) \times [0, \omega_1]$ be the space X of Example 3. Then, S_2 does not have the ASSZH property. Let

$$\pi : [0, \omega_1) \times \{\omega\} \rightarrow [0, \omega_1) \times \{\omega_1\}$$

be a map defined by

$$\pi(\langle \alpha, \omega \rangle) = \langle \alpha, \omega_1 \rangle$$

for each $\alpha \in \omega_1$, and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle \alpha, \omega \rangle$ of S_1 with $\pi(\langle \alpha, \omega \rangle)$ of S_2 for every $\alpha < \omega_1$. Let $\phi : S_1 \oplus S_2 \rightarrow X$ be the quotient map. Then, $\phi(S_2)$ is a regular-closed subspace of X . For each $n \in \omega$, let

$$U_n = \phi\left(\left([0, \omega_1) \times [0, \omega_1]\right) \cup ([0, \omega_1) \times (n, \omega])\right).$$

Then, U_n is open in X and $\phi(S_2) = \bigcap_{n \in \omega} U_n$. Thus, $\phi(S_2)$ is a regular-closed G_δ -subspace of X . However, $\phi(S_2)$ does not have the ASSZH property, since it is homeomorphic to S_2 . Song [25, Example 2.9] shows that X is ACC. Thus X has the ASSZH property. \square

Clearly, the space X of Example 4 is not countably compact. The following example shows that a regular closed subspace of a Tychonoff countably compact having the ASSZH property may not have the ASSZH property.

EXAMPLE. There exists a Tychonoff countably compact space having the ASSZH property having a regular-closed subspace which does not have the ASSZH property.

PROOF. Let $S_1 = [0, \omega_1) \times [0, \omega_1]$. As we have mentioned in the proof of Example 4, this space is ACC. Hence S_1 has the ASSZH property. Let $S_2 = [0, \omega_1) \times [0, \omega_1]$ be the space X of Example 3. Then, S_2 is Tychonoff countably compact space which does not have the ASSZH property. Let

$$\pi : [0, \omega_1) \times \{0\} \rightarrow [0, \omega_1) \times \{\omega_1\}$$

be a map defined by

$$\pi(\langle \alpha, 0 \rangle) = \langle \alpha, \omega_1 \rangle$$

for each $\alpha \in \omega_1$, and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying

$$\langle \alpha, 0 \rangle \text{ of } S_1 \text{ with } \pi(\langle \alpha, 0 \rangle) \text{ of } S_2 \text{ for every } \alpha < \omega_1.$$

Then, X is Tychonoff countably compact by the construction of the topology of X . Let

$$\phi : S_1 \oplus S_2 \rightarrow X$$

be the quotient map. Then, $\phi(S_2)$ is a regular-closed subspace of X . However, $\phi(S_2)$ does not have the ASSZH property, since it is homeomorphic to S_2 . Similarly to the proof that X in Example 4 has the ASSZH property, it is not difficult to show that X has the ASSZH property. \square

Recall the Alexandorff duplicate $A(X)$ of a space X . The underlying set $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighbourhood of

$$\langle x, 0 \rangle \in X \times \{0\}$$

is a set of the form

$$(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\}),$$

where U is a neighbourhood of x in X . It is well-known that a Hausdorff space X is countably compact if and only if $A(X)$ is ACC (see [21, 28]). We give two examples to show that the result cannot be generalized to the space having the ASSZH property.

EXAMPLE. Assuming $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, there exists a Tychonoff space having the ASSZH property but $A(X)$ does not have the ASSZH property.

PROOF. Assuming $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, the space $X = \psi(\mathcal{A})$ with $|\mathcal{A}| = \omega_1$. Then, by Theorem 3.6, X has the ASSZH property. In fact, the set $\mathcal{A} \times \{1\}$ is open and closed subset of $A(X)$ with $|\mathcal{A} \times \{1\}| = \omega_1$, for each $A \in \mathcal{A}$, the point $\langle A, 1 \rangle$ is isolated in $A(X)$. Since ASSZH property is preserved under clopen subsets and $\mathcal{A} \times \{1\}$ does not have the ASSZH property. Thus, $A(X)$ does not have the ASSZH property. \square

EXAMPLE. There exists a Tychonoff space having the SSZH property which does not have the ASSZH property but $A(X)$ has the ASSZH property.

PROOF. Let $X = [0, \omega_1) \times [0, \omega_1]$ be the space X of Example 3. Then, X has the SSZH property which does not have the ASSZH property. Since X is countably compact, then $A(X)$ is ACC space (see [21, 28]), hence $A(X)$ has the ASSZH property. \square

THEOREM 4.1. *If X is a T_1 -space and $A(X)$ has the ASSZH property, then $e(X) < \omega_1$.*

PROOF. Suppose that $e(X) \geq \omega_1$. Then, there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence, $B \times \{1\}$ is an open and closed subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point. Since the ASSZH property is preserved under open and closed subset and $B \times \{1\}$ is not ASSZH. Thus, $A(X)$ does not have the ASSZH property. \square

PROBLEM 1. Does $A(X)$ of the space X having the ASSZH property with $e(X) < \omega_1$ have the ASSZH property?

The following example shows that the ASSZH property is not preserved under continuous mapping.

EXAMPLE. There exists a continuous mapping $f : X \rightarrow Y$ such that X has the ASSZH property but Y does not have the ASSZH property.

PROOF. In Example 4, we have already noticed that the space $[0, \omega_1) \times [0, \omega_1)$ is ACC. The space $X = ([0, \omega_1) \times [0, \omega_1)) \oplus [0, \omega_1)$ is ACC as the discrete sum of two ACC spaces. Hence X has the ASSZH property. Let $Y = [0, \omega_1) \times [0, \omega_1)$ be the space X of Example 3. Then, Y does not have the ASSZH property. Now, we have to find a continuous mapping from X to Y . Consider a mapping $f : X \rightarrow Y$ defined by $f(\langle \alpha, \beta \rangle) = \langle \alpha, \beta \rangle$ for each $\langle \alpha, \beta \rangle \in [0, \omega_1) \times [0, \omega_1)$ and $f(\alpha) = \langle \alpha, \omega_1 \rangle$ for each $\alpha \in [0, \omega_1)$. Then, f is a continuous one-to-one mapping, which completes the proof. \square

Recall that a continuous mapping $f : X \rightarrow Y$ is called varpseudoopen provided $\text{int}_Y f(U) \neq \emptyset$ for every nonempty open set U of X . In [18], it is proved that a continuous varpseudoopen image of an ACC space is ACC. Similarly, we can show the following.

THEOREM 4.2. *The ASSZH property is preserved under continuous varpseudoopen mappings.*

PROOF. Let $f : X \rightarrow Y$ be a continuous varpseudoopen onto mapping and let X have the ASSZH property. Suppose that $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of Y and D a dense subset of Y . For each $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}.$$

Then, $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X and $f^{-1}(D)$ a dense subset of X , since f is varpseudoopen. Since X has the ASSZH property, there exists a sequence $(E_n : n \in \mathbb{N})$ of finite subsets of $f^{-1}(D)$ such that and for each $x \in X$,

$$\{n \in \mathbb{N} : x \notin \text{St}(E_n, \mathcal{V}_n)\} \in \mathcal{I}.$$

Consequently,

$$\{n \in \mathbb{N} : x \in \text{St}(E_n, \mathcal{V}_n)\} \in \mathcal{F}(\mathcal{I}). \quad \text{for each } n \in \mathbb{N}.$$

Let $F_n = f(E_n)$. Then, $(F_n : n \in \mathbb{N})$ is a sequence of finite subsets of D . Let $y \in Y$. Then, $f(x) = y$ for some $x \in X$. We can easily verify that

$$\{n \in \mathbb{N} : x \in \text{St}(E_n, \mathcal{V}_n)\} \subset \{n \in \mathbb{N} : y \in \text{St}(F_n, \mathcal{U}_n)\}.$$

Therefore, Y has the ASSZH property. \square

Since an open map is varpseudopen, we have the following.

THEOREM 4.3. *Let X and Y be two spaces. If $X \times Y$ has the ASSZH property, then both X and Y have ASSZH properties.*

The following example shows that the converse of Theorem 4.3 need not be true even if X has the ASSZH property and Y is a compact space.

EXAMPLE. Consider the space $X = [0, \omega_1) \times [0, \omega_1]$ the product of $[0, \omega_1)$ and $[0, \omega_1]$ of Example 3 does not have the ASSZH property. The first factor is ACC by [18, Theorem 1.8,] hence the ASSZH property, and the second space is compact. Matveev showed that the product of a Hausdorff ACC space and a first countable compact space is ACC [18, Theorem 2.3]. However, we do not know that if product of the space having ASSZH property and a first countable compact space has the ASSZH property.

We show that the preimage of the ASSZH property under a closed 2-to-1 continuous map need not have the ASSZH property.

EXAMPLE. Assuming $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, there exists a closed 2-to-1 continuous map $f : X \rightarrow Y$ such that Y has the ASSZH property, but X does not have the ASSZH property.

PROOF. Let $Y = \psi(\mathcal{A}) = \omega \cup \mathcal{A}$ be the space of Example 4. Then, Y has the ASSZH property. Let X be the space $A(Y)$ of Example 4. Then, X does not have the ASSZH property. Let $f : X \rightarrow Y$ be the projection. Then, f is closed 2-to-1 continuous map. \square

EXAMPLE. The space $X = [0, \omega_1) \times [0, \omega_1]$ in Example 4 also shows that the preimage of a space having the ASSZH property under an open perfect map need not have the ASSZH property, since the projection map is an open perfect map.

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Sumit Singh
Manoj Bhardwaj
Department of Mathematics
University of Delhi
New Delhi-110007
INDIA
E-mail: sumitkumar405@gmail.com
manojmnj27@gmail.com

Brij K. Tyagi
Atma Ram Sanatan Dharma College
University of Delhi
New Delhi-110021
INDIA
E-mail: brijkishore.tyagi@gmail.com